Model Uncertainty, Ambiguity Aversion, and Market Participation

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Abstract

Ambiguity aversion is a leading explanation for the market nonparticipation puzzle. We show that a passive index fund that offers the 'risk-adjusted market portfolio' reinstates the puzzle. In equilibrium, investors participate via the fund in all asset markets, even if they do not know the fund's composition and view its payoffs as highly uncertain. This results from a new portfolio information separation theorem which applies in the absence of model uncertainty: in equilibrium each investor combines positions in a common, deterministic portfolio, a portfolio that depends upon the investor's private signals, and the riskfree asset. In equilibrium, risk premia satisfy the CAPM with the fund as the pricing portfolio. These conclusions are robust to investor unawareness (appropriately defined) of some traded assets. We conclude that other considerations, such as investors not understanding the concept of market equilibrium, are needed to explain the puzzle.

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1 Introduction

Nonparticipation in domestic public equity markets and home bias in international investing are two of the main stylized facts of household finance. Only a minority of relatively well-off individuals—those with $\$100,000$ in liquid assets—participate in the equity market (Mankiw and Zeldes 1991), and poorer investors tend to participate even less. Similarly, almost 20% of households at the 80$^{\text{th}}$ percentile of the wealth distribution own no public equity (Campbell 2006). According to more recent evidence in the 2013 Survey of Consumer Finances, less than 15% of U.S. households report owning stocks directly, and only about 50% of households own stocks either directly or indirectly through mutual funds or retirement accounts (Bricker et al. (2014)). As for home bias, investors very often fail to participate in foreign stock markets despite the benefits to diversification and international risk sharing (French and Poterba 1991; Tesar and Werner 1995; Lewis 1999).

The puzzle in both of these phenomena are the same: why don't investors fully participate in risky asset markets to improve diversification and risk sharing? As such, nonparticipation is a challenge to frictionless optimal portfolio theories (Campbell 2006). Market frictions such as information asymmetry and transaction costs do not seem to fully explain the puzzle.[1]

To address this puzzle, a major strand of research proposes that nonparticipation derives from *investor ambiguity aversion*. In this explanation, investors do not know perfectly certain parameters of the distribution of assets payoffs—they face 'model uncertainty.' In making investment decisions, ambiguity averse investors place heavy weight upon worst-case scenarios for these parameters. For example, several papers model financial markets in which some investors are subject to model uncertainty and have multiple-priors utility functions, and provide conditions under which investors do not participate in certain assets or asset classes (Bossaerts et al. (2010), Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), Easley and O'Hara (2010), Epstein and Schneider (2010), and Cao, Han, Hirshleifer, and Zhang (2011)).

This literature focuses on how ambiguity aversion affects direct holdings by investors in securities. It therefore does not address the question of whether introducing a passive

[1] Information asymmetry can explain why investors underweight certain assets, but not zero positions (Van Nieuwerburgh and Veldkamp 2009). Moderate transaction costs of participation also do not seem to explain why even wealthy households would fail to participate; as discussed by Gouskova, Juster, and Stafford (2004), they are not "the major consideration."

index fund run by a money manager who knows the uncertain parameters of the economy, but does not have any private signals about payoffs, might ameliorate the problem. This allows investors to participate in assets markets indirectly. Hence, in this paper, we refer to an investor as participating in the market of an asset if they do so either directly or via a fund.

Whether such a fund will increase participation is not obvious, because investors are uncertain about each of the assets it holds. As shown in previous literature, ambiguity about individual assets carries over to portfolios, resulting in pessimism about portfolio returns as well. So an ambiguity-averse investor who evaluates based on worst case scenarios may view an index fund as extremely risky, and hence to be avoided.

We provide a model that addresses the question of whether a low-cost index fund solve the problem of nonparticipation when investors differ in the payoff information signals they possess, in their knowledge of exogenous parameters of the model, and perhaps in other ways as well. We find that even though ambiguity carries over from individual stocks to portfolios, ambiguity-averse investors in equilibrium hold an appropriate passive index fund, though one that differs from the market portfolio. In consequence, ambiguity aversion alone does not explain the nonparticipation puzzle. In developing our analysis, we also provide a new separation theorem for optimal security holdings under asymmetric information, and a version of the CAPM that holds under ambiguity aversion and asymmetric information.

In our multi-asset rational expectations setting, investors with identical preferences may be ambiguity-averse, and receive diverse signals about the payoffs of different securities. The legal/financial technology for offering an index fund to investors may or may not be available. For each asset, a relevant source of risk, the volatility of supply shocks, is known only to a subset of investors. For any given asset, those who do not know this parameter are highly averse to the ambiguity deriving from this parameter.

When creating an index fund is feasible, index fund managers observe all the supply volatility parameters, but do not have any private information about asset payoffs. The index fund market is costless and perfectly competitive, so fund fees are zero and the fund is constructed to be maximally useful to investors. In equilibrium, there could be any number of identical funds, but for convenience we refer to 'the fund,' and to the cases when 'an index fund is available' or absent.

Ambiguity aversion in our model takes the form of an investor assuming a worst case scenario for any parameter the investor is uncertain about. Consistent with existing literature, we show that when there is no index fund, an ambiguity averse investor takes a zero position in any security that the investor is ambiguous about (i.e., whose supply volatility is unknown to the investor) and perceives as extremely risky. So to avoid model uncertainty the investor holds an undiversified portfolio. If the investor is ambiguous about all risky securities, the investor does not participate in the market at all.

Surprisingly, however, when there is an appropriate fund, an equilibrium exists in which all investors prefer to hold, as a common component of their portfolios, identical positions in the index fund. This fund offers a particular deterministic portfolio with positive positions in all traded assets, with weights being determined by the exogenous parameters (including the parameters that are unknown to certain subsets of investors). So generically investors participate in all asset markets. This is despite the fact that the fund subjects investors to substantial ambiguity by virtue of the ambiguity of the assets that it holds.

In general investors hold an additional investor-specific portfolio component that includes any asset about which the investor has a private signal (and about which the investor has no ambiguity). This component is acquired to exploit private information and to take advantage of any risk premia induced by realizations of the supply shocks.

Since the index fund is the common component of all investors' equilibrium holdings, it serves as the pricing portfolio for the CAPM risk-return relationship. This portfolio is optimally held, for example, by an investor who has no private signals about any stock. This optimality implies that the fund is mean-variance efficient. So in equilibrium the CAPM holds, despite parameter uncertainty, ambiguity aversion, and asymmetric information about asset payoffs.

To understand the intuition for these results, we need to describe the economic setting more specifically. There is a continuum of investors with strictly positive endowments of all risky assets. There are random supply shocks, and asset prices are set to clear the markets for all assets. For each asset, investors are divided into two groups. Members of one group receive conditionally independent private signals about the asset payoff and know the precision of the supply shock. Members of the other group neither receive any private signals about the asset payoff, nor know the precision of the supply shock. In particular, the uninformed investors' subjective prior about the precision of the supply shock includes the possibility of precisions that are arbitrarily close to zero. This implies that the assets may potentially be perceived as extremely risky.

Ambiguity averse investors choose optimal portfolio to maximize expected utility under worst-case assumptions for the values of the supply volatility parameters that they are uncertain about. So for any portfolio contemplated by an investor, expected utility is calculated contingent on the unknown parameters having values that minimize traditional CARA expected utility.

There is an index fund whose manager knows the supply shock precisions of all assets. The fund offers all investors a single portfolio which is a deterministic function of the exogenous parameters, including the supply volatilities of all the assets. Though investors who face model uncertainty do not know the exact weights of the portfolio, the function used for constructing the portfolio is common knowledge.

The key intuition for our results derives from a new portfolio information separation theorem which applies in simplest form in the setting with no model uncertainty. In this setting, there is a rational expectations equilibrium in which any investor's equilibrium portfolio consists of three components. The first is a common deterministic component which plays a role in our model somewhat similar to the market portfolio in the CAPM, but is distinct from the endowed market portfolio. We call this component the Risk-Adjusted Market Portfolio (*RAMP*).

Based on any private signals that an investor possesses, the investor holds an additional risky portfolio, which we call the information-based portfolio. An investor holds a non-zero position of an asset in this portfolio if and only if the investor receives private signals about the asset. Importantly, an informed investor can construct the investor's information-based portfolio without making any use of information extracted from asset prices—it depends only on the investor's own signals and the exogenous parameters the investor knows. Finally, investors allocate their remaining wealth to the riskfree asset.

Consider again a setting where investors are subject to model uncertainty and an index fund offers *RAMP*. The fund is able to offer *RAMP* as its manager knows all parameters of the financial markets. One share of the index fund represents one unit of *RAMP*. Consider the following proposed strategy profile: all investors hold exactly one share of the index fund, and each investor additionally holds her investor-specific information-based portfolio (a zero position if the investor has no private information). Given that all other investors behave as prescribed in this profile, no investor has an incentive to deviate.

Consider any particular investor, named Lucy. Given any vector of precisions of assets' supply shocks that is possible according to Lucy's subjective priors, she will be in a possible world without model uncertainty. In such a possible world, since all other investors are holding exactly one share of the index fund and their own information-based portfolios, they are effectively holding the same portfolios as they do in the rational expectations equilibrium. Hence, the market clearing condition implies that the pricing function is the same as the one in the rational expectations equilibrium. Therefore, if Lucy knew the parameter values that characterize this possible world, her optimal portfolio choice would consist of *RAMP* and her own information-based portfolio.

By holding the index fund and her own information-based portfolio, Lucy implements exactly such an investment strategy. In different possible worlds, *RAMP* and therefore the composition of the index fund differ, but Lucy's information-based portfolio does not. So even when Lucy is ambiguous about some or all assets, her optimal portfolio choice is to hold exactly one share of the index fund and her own information-based portfolio. Intuitively, the fund uses its knowledge to do precisely what Lucy would choose to do if she knew what the fund knows.

This argument for the optimality of investing in the fund is a powerful one, as it requires only that Lucy be time-consistent in her decision-making. Specifically, consider a setting in which the fund is replaced by Lucy herself. In other words, Lucy can decide today to delegate the construction of the fund to her later self, after she has learned the relevant parameter values. If Lucy is time-consistent, and since she lacks the information she needs today, Lucy will be willing to commit to such delegation. Doing so gives her a way of taking the positions that she herself would later choose to take based upon the relevant knowledge.

Returning to the model with an index fund, Lucy's willingness to hold it is an equilibrium outcome; the reasoning relies on her conjecture that all other investors hold the same fund (along with their own information-based portfolios). This highlights the fact that investors hold *RAMP* for risk-sharing reasons. Even though the fund may be very risky to an ambiguity averse investor, it pays for investors to share risk by trading to *RAMP* (along with the private information portfolio component). This point has nothing to do with diversification incentives; we show that such an equilibrium exists even if there is only one risky asset traded in the market. So the benefits of holding the index do not derive solely from individual optimization considerations (e.g., the risk benefits of diversification).

It may seem surprising that all investors take the same position in the index fund, even though their priors about the precisions of supply shocks have different supports.

Investors with different priors have different worst-case scenarios, and therefore differ in how risky they view the fund. However, owing to the portfolio information separation theorem, in equilibrium all investor agree that it is good to delegate their non-information-based investments to a fund that has access to the true values of the supply volatilities. *RAMP* is based upon those actual values.

The fact that all investors hold the index fund implies a version of the CAPM security market line. Suppose that Lucy is uninformed and ambiguous about all the assets. In equilibrium, her optimal overall portfolio is just *RAMP*, as held by the index fund. In any given possible world, Lucy has a standard CARA expected utility function. So in that world, her optimal portfolio, *RAMP*, is mean-variance efficient. It follows that for any given values for the exogenous parameters, including supply volatilities, *RAMP* is mean-variance efficient. So the CAPM security line applies to *RAMP* (or the index fund) as the pricing portfolio.

An alternative approach to explaining nonparticipation is based upon *investor unawareness* (Merton 1987; Easley and O'Hara 2004) instead of ambiguity aversion. We implement the idea of unawareness about an asset as meaning that the investor knows nothing about the asset's characteristics. So we say that an investor is unaware of an asset if the investor holds uniform uninformative priors about the asset's payoffs, its aggregate endowed supply, and the precision of its supply shock. In this setting, for an asset that the investor is unaware of, the investor has no prior information whatsoever about its characteristics, or even about the probability distribution of these characteristics. Indeed, the investor may not even know about the existence of a given asset by name, and we allow for the possibility that the investor has no idea how many unknown assets are available for trading.

Superficially, it might seem that this would make an asset that the investor is unaware of too risky to invest in, even indirectly through a fund. However, we show that our main intuition can be applied to this setting as well, with the modification that the possible worlds an investor could face can include worlds with different numbers of assets, and that the specification of such worlds depends on all the parameters that the investor has uninformative priors about, not just the supply volatility. The same time-consistency insight applies. An investor is happy to delegate to a fund which will choose on behalf of the investor the portfolio that the investor would have chosen if that investor had the relevant knowledge. So when a low-cost index fund is available, investor unawareness does not lead to limited participation.

Overall, these findings suggest that when index investing is feasible, investors' ambiguity aversion or unawareness taken in isolation do not solve the limited participation puzzle. There must be other causes. We discuss some possibilities in the conclusion. One of the most interesting, suggested by our model, is that a failure of ambiguity averse investors to understand the concept of equilibrium makes investing in a fund with ambiguous payoffs seem too risky.

2 A Model with Investor Ambiguity Aversion

There are two dates, date 0 and date 1. The economy is populated by a continuum of investors with measure one, who are indexed by i and uniformly distributed over $[0,1]$. All investors trade at date 0 and consume at date 1. Any investor i invests in a riskfree asset and $N \geq 2$ independent risky assets by herself. (In this section, we assume that the number N is common knowledge.) The riskfree asset pays r units, and risky asset n pays F_n units of the single consumption good. Taking the riskfree asset to be the numeraire, let P be the price vector of the risky assets and D_i be the vector of shares of the risky assets held by investor i. Investor i can hold an index fund that commits to offering a portfolio X, which is an N-dimension column vector with the n^{th} element being the shares of the n^{th} risky asset in X. Then, by holding d_i (a scalar) shares of the fund, investor i effectively holds the portfolio $d_i X$. Therefore, an investor i's risky assets holdings are $d_i X + D_i$.

Let $W_i = (w_{i1}, w_{i2}, \ldots, w_{iN})'$ be the endowed shareholdings of investor i, and let $W = \int_0^1 W_i \mathrm{d}i >> 0$ be the aggregate endowments of shares in the capital market. So any investor i's final wealth at date 1 is

$$\Pi_i = r\left[W_i' - \left(d_i X' + D_i'\right)\right] P + \left(d_i X' + D_i'\right) F, \tag{1}$$

where $F = (F_1, F_2, \ldots, F_N)'$. The first term in (1) is the return of investor i's investment in the riskfree asset, and the second term is the total return from her investments in risky assets.

We assume that all investors share a common uniform improper prior of F, and so no investor has prior information about any risky asset's payoff. Hence, any investor i's information consists of the equilibrium price vector and the realization of a private information signal S_i only. In particular, $S_i = F + \epsilon_i$, where F and ϵ_i are independent; and ϵ_i and ϵ_j are also independent. Each ϵ_i is normally distributed, with mean zero and precision matrix Ω_i. We assume that Ω_i is diagonal for all $i \in [0,1]$, and so investor i's private signal about asset n's payoff is uninformative about asset k's payoff.

We call investor i an informed investor of asset n if and only if the n^{th} diagonal entry of Ω_i is strictly positive. Let the n-dimension column vector λ summarize the measures of informed investors of each asset, with the n^{th} element being the measure of the informed investors of asset n; we assume $\lambda_n \in (0,1)$. Let $\text{Diag}(\lambda)$ be the diagonal matrix with the n^{th} diagonal entry being the n^{th} element of λ. We assume that any investor i is uninformed about at least one asset. We call an investor i with precision matrix $\Omega_i = 0$ an uninformed investor. We assume that $\gamma \in (0,1)$ fraction of investors are uninformed, so $\gamma \leq \min_j \left(1 - \lambda_j\right)$.

For simplicity, we assume that the private signals of all informed investors of asset n have the same precision $\kappa_n > 0$. Let Ω be the $N \times N$ diagonal matrix with the n^{th} diagonal entry being κ_n. Denote by Σ the matrix of the average precision of private signals, we have

$$\Sigma = \int_0^1 \Omega_i \mathrm{d}i = \Omega \text{Diag}(\lambda) \tag{2}$$

As is standard, the independence of the errors implies that in the economy as a whole signal errors average out, so that the equilibrium pricing function does not depend on the error realizations (though it does depend on their distribution).

There are random supplies of all risky assets. Let Z denote the random supply vector. We assume that Z is independent of F and of ϵ_i (for all $i \in [0,1]$). We further assume that Z is normally distributed with mean 0 and the precision matrix U. By independence of assets, U is diagonal and positive definite, with the n^{th} diagonal entry being τ_n.

Investors commonly know all parameters except U. Specifically, we assume that only informed investors of asset n know τ_n; any uninformed investor i of asset n will have her own subjective prior belief about τ_n with the support $\left(\underline{\tau}_n^i, \bar{\tau}_n^i\right)$, where $\bar{\tau}_n^i > \underline{\tau}_n^i \geq 0$. Denote by $\mathcal{U}_i$ the set of investor i's belief about U, and by U_i a typical element in $\mathcal{U}_i$. We allow different uninformed investors of a particular asset n to have different supports of their beliefs about τ_n. To establish the benchmark, we assume that for any uninformed investor i, $\underline{\tau}_n^i = 0$ for all n; that is, any uninformed investor believes that any asset's supply shock could be extremely volatile.

The index fund knows all the model parameters but does not observe any signals about assets' payoffs. The fund offers the portfolio X below to investors:

$$X = \left[I + \frac{1}{\rho^2} \left(\Sigma U\right)^{-1}\right]^{-1} W, \tag{3}$$

where ρ is investors' common risk tolerance coefficient. Importantly, the portfolio X does not include any signals of assets' payoffs. In addition, one share of the fund represents the effective asset holding X, and so for any given price vector P, one share of the fund is sold at the price $X'P$. While investors know that conditional on a U, the portfolio offered by the index fund is X, they do not know the true composition of X, unless they are informed about all assets.

All investors are risk averse, so when all model parameters are common knowledge, at date 0 their expected utility is CARA,

$$\mathbb{E}_i u(\Pi_i) = \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{4}$$

However, investor i may be subject to model uncertainty about the precisions of some assets' random supplies, and will choose an investment strategy (d_i, D_i) to maximize the infimum of her CARA utility. Formally, each investor i's decision problem is

$$\max_{d_i, D_i} \inf_{U_i \in \mathcal{U}_i} \mathbb{E}_i \left[-\exp\left(-\frac{\Pi_i}{\rho} \right) \right]. \tag{5}$$

We are interested in a linear rational expectations equilibrium defined as follows.

Definition 1 *A pricing vector P^* and a profile of all investors' risky assets holdings $\{d_i^*, D_i^*\}_{i\in[0,1]}$ constitute a rational expectations equilibrium, if*

1. *Given P^*, (d_i^*, D_i^*) solves investor i's maximization problem in equation* (5)*, for all $i \in [0,1]$;*

2. *P^* clears the market, that is,*

$$\int_0^1 (d_i^* X + D_i^*)\, \mathrm{d}i = W + Z, \quad \textit{for any realizations of } F \textit{ and } Z; \text{and} \tag{6}$$

3. *Both d_i^* and D_i^* are linear functions of P and S_i.*

3 Benchmark: No Index Fund

We next establish a benchmark for comparison by studying a model without the index fund. In such a setting, any investor i's investment strategies are constrained by $d_i = 0$. Since all asset realizations are independent, we can first focus on investor i's decision whether to hold an asset n that she is uninformed about.

Investor i is risk averse, so she will not hold any non-zero position of asset n, unless the distribution of asset n's payoff has a finite variance, conditional on her information. Investor i, however, has neither prior information nor private information about asset n's payoff. Hence, she estimates the payoff based only on the price, which partially aggregates informed investors' private information. The informativeness of price increases in the precision of the supply shock. When the supply shock has a zero precision, price becomes completely uninformative.

Investor i does not know the precision of asset n's supply shock. By assumption, investor i's subjective prior belief about τ_n has the support $\left(0, \bar{\tau}_n^i\right)$. Then, as she considers the worst case scenario in making the investment decision, investor i will consider the case that the true τ_n is very close to 0, since in such a case, asset n's price is almost uninformative.

Suppose that investor i holds a non-zero position of asset n. As the price becomes almost uninformative, the payoff variance conditional upon price diverges to infinity. So holding a non-zero position is extremely risky in the worst case scenario. To avoid this risk, investor i optimally chooses a zero position. Proposition 1 below summarizes the argument above.

Proposition 1 *If an investor i is uninformed about asset n, and $\underline{\tau}_n^i = 0$, then investor i will hold a zero position of asset n.*

Since all asset realizations are independent, investors can evaluate assets' conditional (on prices) expected return and variance one by one. Then, because an uninformed investor i has $\underline{\tau}_n^i = 0$ for all n, her belief about any asset's payoff has potentially extremely large conditional variance. Therefore, given any $D_i \neq 0$, the infimum of investor i's utility will be $-\infty$; so, $D_i \neq 0$ is strictly dominated by $D_i = 0$. That is, any uninformed investor i refrains from participating in any asset market. Since there are γ measure of uninformed investors, Corollary 1 shows that limited participation presents in this benchmark model without an index fund, consistent with the prediction in the literature.

Corollary 1 *In the model without an index fund, there are γ measure of investors who do not participate in risky assets' markets.*

4 Introducing an Index Fund Results in Full Participation

In this section, we show that in equilibrium an appropriate index fund induces all investors to participate in all asset' markets. We also show how investors will allocate their initial wealth among the fund, their direct holdings of risky assets, and the riskfree asset, even when investors do not know the exact composition of the index fund. We then argue that the full participation with an index fund follows from a portfolio information separation theorem that applies in financial markets without the index fund and without ambiguity aversion.

4.1 An Equilibrium with Full Participation

In the model with an index fund that offers the portfolio X, any investor i's investment strategy (d_i, D_i) leads to effective asset holdings $d_i X + D_i$. If investor i is uninformed, she does not know U and hence does not know the exact composition of X. However, all investors commonly know X as a function of U that is specified in equation (3).

The main result of our paper is presented in Proposition 2 below, which shows that in an equilibrium, all investors hold exactly one share of the index fund and thus participate in all asset markets. Hence, with an appropriately constructed index fund, even with ambiguity aversion, there is full (though intermediated) participation.

Proposition 2 *In the model with an index fund that commits to offering the portfolio X specified in equation* (3)*, there is an equilibrium in which*

1. *All investors will buy one share of the index fund, and so $d_i^* = 1$ for all $i \in [0,1]$;*
2. *Any investor i will hold an extra portfolio $\rho\Omega_i\left(S_i - rP\right)$; and*
3. *For any given F and Z, the equilibrium price is*

$$P = \frac{1}{r}\left[F - \frac{1}{\rho}\left(\Sigma + \rho^2 \Sigma U \Sigma\right)^{-1} W - \frac{1}{\rho}\Sigma^{-1} Z\right]. \tag{7}$$

The intuition of Proposition 2 arises from a new portfolio information separation theorem that applies in the setting without ambiguity aversion and the index fund. Since such an intuition is not straightforward, we discuss it in detail in Section 4.2. In the rest of this subsection, we discuss some properties of the equilibrium characterized in Proposition 2.

First, in equilibrium, uninformed investors are indifferent between holding the index fund and not participating in asset markets. When contemplating a position in the index fund, uninformed investors believe that the fund's holdings of all assets are very close to zero, because in the worst case scenario, the precisions of all assets' supply shocks are almost zero. Hence, by holding the fund, the infima of the uninformed investors' utilities are the same as the utility from not participating.

Nevertheless, the only reasonable conclusion is that an uninformed investor who is ambiguity averse and otherwise-rational holds the index fund (when other investors follow equilibrium behavior). In particular, since any uninformed investor i's subjective prior about the precision of any asset n's supply shock is $(0, \bar{\tau}_n^i)$, she knows for sure that $\tau_n > 0$. For any given τ_n, holding the index fund is strictly better than not participating. So while investor i has the same (infimum) utility ex ante, once τ_n is realized, she knows she will be *strictly* better off holding the index fund. So only holding the index fund is time-consistent.

Given this, it is not surprising that there are ways to express preferences that capture formally the fact that a time-consistent investor is not indifferent, even ex ante, as to whether to invest in the fund. This can be done by considering perturbations of the model. Consider a sequence of perturbed models in which all uninformed investors' priors about U have strictly positive lower bounds. When the perturbed lower bounds converge to zero, the perturbed models converge to our original model. In any of these perturbed models, strictly positive lower bounds of investors' priors about U imply that holding the index fund is investor i's unique best response to other investors' strategies in an equilibrium, as shown in the proof of Proposition 2. Hence, when investor i's prior knowledge switches a little bit, investor i strictly prefers to hold the index; then, by the revealed preference, investors would like to choose the index fund in the original model, given all other investors' strategies. Therefore, the equilibrium characterized in Proposition 2 is *near strict*, an equilibrium refinement concept defined by Fudenberg, Kreps, and Levine (1988).[2]

Second, while investors have heterogeneous priors about U and thus different beliefs about the fund's composition, they all hold exactly one share of the fund. Take two investors, Lucy and Martin, for an example. Lucy is uninformed and believes that τ_n

[2]Formally, a strategy profile σ is *near strict* in a game Γ if there exists a sequence of games $\{\Gamma^n\}$ and a sequence of strategy profiles $\{\sigma^n\}$, such that (i) $\lim_n \Gamma^n = \Gamma$; (ii) for each n, σ^n is a strict equilibrium of Γ^n; and (iii) $\lim_n \sigma^n = \sigma$. Here, a *strict equilibrium* is an equilibrium in which any investor's strategy is her unique best response to all other investors' strategies in the equilibrium.

(for any n) could be arbitrarily close to 0; Martin does not receive private information about asset payoffs either, but he knows the true precisions of all supply shocks. According to Proposition 2, both Lucy and Martin will hold one share of the index fund, but neither Lucy nor Martin holds any extra positions because they don't have any private information about assets' payoffs. Hence, Lucy and Martin are effectively holding the same portfolio. So differences in investors' holdings arise only from differences in their information signals, not from differences in their model uncertainty or ambiguity aversion.

Third, Proposition 2 shows the importance of risk sharing among investors in their optimal portfolio choices. Specifically, consider an investor who faces model uncertainty about a subset of traded assets, and views the return distributions as exogenous. Even if she can indirectly trade those assets through an index fund, it may not be optimal for her to do so, because she cannot calculate the fund's expected return and risk. Therefore, arguments based on the incentive of individuals to diversify do not, under radical ignorance, justify holding of the fund. In contrast, in our equilibrium setting, an investor optimally holds the fund, given her belief that other investors will also do so (together with their direct portfolios). Hence, she is willing to hold the fund too, which achieves the benefit of optimally sharing risk with other investors.

Proposition 2 more broadly suggests a new behavioral reason why investors may fail to diversify: because they do not understand the concept of equilibrium. If investors reason about possible portfolios based solely on partial equilibrium risk and return arguments, portfolios containing assets that investors are ambiguous about can seem extremely risky (or in the limiting case, infinitely risky). Proposition 2 shows that even ambiguity averse investors, if otherwise rational, will take into account equilibrium considerations and still hold such assets. But actual investors may not understand the equilibrium reasoning which underlies this result.

Instructors in finance know that it is hard for students (or even experts), to keep in mind equilibrium considerations. This is reflected in the portfolio advice given to investors in Cochrane (1999), which repeatedly emphasizes that even when investors are heterogeneous, the average investor must hold the market portfolio. This implies that when investors are rational, an investor should not deviate from that norm unless there is something special about the investor that makes such a choice especially appropriate for that investor and not others, who in aggregate must take the opposite position. For example, Cochrane points out that, counter to naive intuition, in a rational setting, the low expected returns of growth stocks does not make growth a bad deal, and the fact that market returns are predictable does not make market timing a good deal. Why is there such a need to emphasize these points, even for the relatively sophisticated audience that Cochrane was addressing? Because equilibrium considerations are not immediately intuitive; careful thought, training, and vigilance is required to avoid error.[3]

4.2 The Portfolio Information Separation Theorem

Proposition 2 is a surprising result. It is true that investors are willing to hold the fund because the fund knows the precisions of all assets' supply shocks. However, the fund's knowledge about the financial markets' parameters is not sufficient for Proposition 2. First, the fund does not have absolute informational superiority, because informed investors of an asset receive private signals about its payoff, which are not observed by the fund. Hence, for investors who are informed about some assets, they seem trading off their informational superiority about the assets they are informed about and the fund's superior knowledge about the parameters of assets they are uninformed about.

Second, for the strategy profile described in Proposition 2 to be an equilibrium, the fund has to offer the portfolio X, specified in equation (3). We verify that if the fund offers another portfolio

$$X' = \left[I + \frac{1}{\rho^2} \left(\Sigma U^{\frac{1}{4}} \right)^{-1} \right]^{-1} W,$$

which is also a function of U and will converge to 0 as U converges to zero, uninformed investors will not hold the fund and thus will refrain from participating in the financial markets. This is because when U converges to 0, X' converges to 0 much slower than U. But the conditional variance of holding any non-zero positions diverges to infinity at the same speed as U converges to 0. Hence, the risk of holding X' diverges to infinity as U converges to 0, implying that holding the index fund is extremely risky for uninformed investors in the worst case scenarios.

Hence, the intuition of Proposition 2 must go beyond the index fund's superior knowledge about the financial markets. In this section, we provide greater insight into

[3]In our setting, owing to asymmetric information, neither the uninformed nor the informed hold the market portfolio (though of course the 'average' investor must hold the market inclusive of supply shocks). A further point analogous to Cochrane's also applies with respect to *RAMP*. When an index fund is available, an ambiguity averse investor always holds *RAMP* as a portfolio component despite its severe apparent riskiness, because investors should only deviate from this holding if they have a special reason to do so (i.e., if they have private information).

this result based upon a new portfolio separation theorem for financial markets with asymmetric information, but without model uncertainty or funds.

We now modify the model described in Section 2 by assuming that U is common knowledge among all investors and that there is no index fund.[4] Then the model is a traditional rational expectations equilibrium model with multiple risky assets, analyzed by Admati (1985). Proposition 3 characterizes a linear rational expectations equilibrium and shows investors' optimal risky assets holding when all parameters are common knowledge.

Proposition 3 *In the model whose parameters are all common knowledge among investors, there exists an equilibrium with the pricing function*

$$P = B^{-1}\left[F - A - CZ\right], \tag{8}$$

where

$$\begin{aligned} A &= \frac{1}{\rho}\left[\rho^2(\Sigma U \Sigma) + \Sigma\right]^{-1} W && (9) \\ B &= rI && (10) \\ C &= \frac{1}{\rho}\Sigma^{-1}. && (11) \end{aligned}$$

Any investor i's risky asset holding is

$$D_i = \left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W + \rho\Omega_i\left(S_i - rP\right). \tag{12}$$

Owing to supply shocks, asset prices are not fully revealing, so information asymmetry persists in equilibrium and different investors have different asset holdings. An investor's asset holding is the sum of two components. The first term in equation (12),

$$\left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W$$

is the *risk-adjusted market portfolio* (*RAMP*), which is deterministic. *RAMP* differs from the ex-ante endowed market portfolio W, because it is also influenced by the informativeness of the equilibrium price, as reflected in the variance of supply shocks and signal

[4]The theorem we are about to state does not require the assumption of an uninformative prior. Hence, we prove a more general version of Proposition 3 in the appendix for the case of normal priors. Since both the equilibrium pricing function and investors' equilibrium holdings are continuous in the prior precisions of assets' payoffs, substituting zero prior precisions will lead to exact Proposition 3.

noise. Investors take the informativeness of asset prices into account when trading to share risks. When the supply shock to an asset becomes more volatile, or on average investors' private information of such an asset is less precise, the equilibrium price contains less precise information about this asset. This increases risk, which, other things equal, reduces investor holdings of this asset.

The second component of any investor's risky asset holding, the second term in (12), is what we call information-based portfolio. This position, $\rho\Omega_i \left(S_i - rP\right)$, consists of extra holdings in the securities about which the investor has information. Investor i holds such an extra position of an asset n if and only if the n^{th} diagonal entry of Ω_i is $\kappa_n > 0$. This suggests that any investor i holds direct positions of a risky asset because possessing an informative signal about such an asset reduces its conditional volatility (independent of the signal realization). Investor i's direct positions of a risky asset also come from her speculation, which is taken to exploit superior information. Different investors, even if they are informed about asset n, hold different speculative portfolios, because they receive heterogeneous private signals.

A critical feature of the first component of investors' risky asset holdings is that it is independent of any investor's private information. A critical feature of the second component of an investor's risky asset holdings is that it can be formed based only on the investor's own private information: it is independent of the information extracted from equilibrium price.

These two features lead to a new portfolio separation theorem under asymmetric information. This will turn out to be important for understanding investors' behavior when they are subject to ambiguity aversion and may hold the risky assets through an index fund, as analyzed in Subsection 4.1.

Theorem 1 *When the characteristics of all assets are common knowledge, equilibrium portfolios have three components: a deterministic risk-adjusted market portfolio (RAMP); an information-based portfolio based upon private information and equilibrium prices but no extraction of information from prices; and the riskfree asset.*

This allows forming an optimal portfolio in separate steps: (1) buy one share of *RAMP*; (2) buy the information-based portfolio using only private information and not the information extracted from price; and (3) put any left-over funds into the riskfree asset. This separation theorem derives from market equilibrium as well as optimization considerations. This differs from those (non-informational) separation theorems in the literature that are based solely on individual optimization arguments.[5]

RAMP is exactly the same as the portfolio X specified in equation (3). The index fund can provide such a portfolio because the index fund knows all the model parameters, and X does not include any investor's private information. Meanwhile, the information-based portfolio is exactly the same as the direct holdings of the risky assets in Proposition 2. To form the information-based portfolio, an investor does not need to extract information from the equilibrium price: she can treat the equilibrium prices as given parameters, and solve for the information-based portfolio from her CARA utility maximization problem as in a partial equilibrium model.

The Portfolio Information Separation Theorem provides the intuition of investors' equilibrium investment strategies in the setting with model uncertainties. Consider the model in which investors are uncertain about the precisions of some assets' supply shocks. For each possible world $U_i \in \mathcal{U}_i$ (or each 'admissible' world defined in Section 6.1), investor i can solve her optimal risky assets holdings, assuming that the equilibrium pricing function is the one in equation (7) with U being U_i. Importantly, because all other investors are holding one share of the fund and their own direct information-based portfolio, they are effectively holding the risky assets as in the world with U_i being common knowledge. Therefore, in the possible world U_i, the market clearing condition implies that the pricing function is the one specified in equation (7) with U being U_i. That is, investor i's belief about the pricing function is correct. So, she would like to hold the risky assets as in the world U_i. Such risky assets holdings can be implemented by holding one share of the index fund and her information-based portfolio, so investor i would like to use the investment strategy in Proposition 2. Furthermore, investor i is still uncertain about U, so holding the risk-adjusted market portfolio through holding one share of the fund is strictly preferred.

[5]It may seem puzzling that none of the three portfolio components depend on the information that an investor extracts from price. How then does this information enter into the investor's portfolio decision? The answer is that *RAMP* is optimal precisely because of the ability of investors to extract information from price. As mentioned before, *RAMP* is deterministic; it does not depend on the private signals. But the fact that *RAMP* is an optimal choice is true only because investors update their beliefs based on price. So the optimal portfolio choice is indeed influenced by such information extraction.

5 CAPM Pricing with an Index Fund

Propositions 2 and 3 indicate that the model with ambiguity aversion and the index fund has an equilibrium in which investors' effective risky assets holdings are exactly the same as in the rational expectations equilibrium in the setting without model uncertainties. Therefore, the index fund induces full participation even with ambiguity aversion. It can also reduce asset risk premia, because in the equilibrium, uninformed investors are sharing risks with informed ones.

Since the portfolio offered by the index fund is effectively *RAMP* in the setting without model uncertainty, to analyze the effect of the index fund on risk premia, we return to the setting without model uncertainty. In such a model, the supply shocks make the asset prices in equilibrium imperfectly revealing, and so in the equilibrium, there are information asymmetries among investors and different risky asset holdings. Hence, the setting is very different from the classic CAPM setting, which assumes identical beliefs and has the implication that all investors hold the same risky asset portfolio.

Since holding the market is equivalent to the CAPM pricing relation, it might seems that in our setting there would not be a way to identify a portfolio that prices all assets and is identifiable ex ante based upon publicly available information. Nevertheless, even with information asymmetry, we identify an efficient portfolio in the model and therefore an implementable version of the CAPM pricing relationship.

From Proposition 3, we know that in the setting without model uncertainties, investors hold the risk-adjusted market portfolio as a common component of their holdings. Therefore, it is natural to consider the risk-adjusted market portfolio, which is just X specified in equation (3), as a candidate for CAPM pricing. From equation (8), the equilibrium pricing function is

$$P = \frac{1}{r}\left[F - A - \frac{1}{\rho}\Sigma^{-1}Z\right], \tag{13}$$

where $A = \frac{1}{\rho}[\rho^2(\Sigma U \Sigma) + \Sigma]^{-1}W$.

Given any realized equilibrium price P, the volatility of asset payoffs derives from the supply shock only. Let $\text{diag}(P)$ be an $N \times N$ diagonal matrix, whose off-diagonal elements are all zero and whose n^{th} diagonal element is just the n^{th} element of the vector P. Generically, as no asset has a zero price, $\text{diag}(P)$ is invertible. Then, by the definition of $\text{diag}(P)$,

$$\text{diag}(P)^{-1}P = \mathbb{1}, \tag{14}$$

where $\mathbb{1} = (1, 1, \dots, 1)'$. From the equilibrium pricing (equation (13)), we have

$$\text{diag}(P)^{-1}\mathbb{E}(F) - r\mathbb{1} = \text{diag}(P)^{-1}A. \tag{15}$$

The LHS of equation (15) is just the vector of the risky assets' equilibrium risk premia.

Given a realized equilibrium price, the risk-adjusted market portfolio X has value $P'X$. Then the vector of the weights of risky assets in the risk-adjusted market portfolio is

$$\omega = \frac{1}{P'X}\text{diag}(P)X.$$

Hence, conditional on the price P, the difference between the expected return of *RAMP* and the riskfree rate is

$$\begin{aligned}
\mathbb{E}(R_X) - r &= \omega'\text{diag}(P)^{-1}\mathbb{E}(F) - r \\
&= \frac{1}{P'X}X'\text{diag}(P)\text{diag}(P)^{-1}(A + rP) - r \\
&= \frac{1}{P'X}X'A,
\end{aligned} \tag{16}$$

where the expectations are all conditional on the equilibrium price.

The variance of *RAMP* is

$$\mathbb{V}(R_X) = \mathbb{E}\left[\left(\omega'\text{diag}(P)^{-1}CZ\right)\left(\omega'\text{diag}(P)^{-1}CZ\right)'\right] = \left(\frac{1}{P'X}\right)^2 X'CU^{-1}CX, \tag{17}$$

and the covariance between all risky assets and *RAMP* is

$$Cov(R, R_X) = \frac{1}{P'X}\text{diag}(P)^{-1}CU^{-1}CX. \tag{18}$$

Let α be the CAPM alpha. From equations (15)-(18), and since $X = \rho(CU^{-1}C)^{-1}A$, we have the following Proposition.

Proposition 4 (Risk Premia with Supply Shocks) *In the model with all parameters being common knowledge, asset risk premia satisfy the CAPM where the relevant market portfolio for pricing is the risk-adjusted market portfolio.*

This result may seem surprising, since investors have heterogeneous asset holdings, and since the portfolios held by informed investors are not mean-variance efficient with respect to the public information set. Nevertheless, in equilibrium, there are no extra risk premia incremental to those predicted by the CAPM using *RAMP*.

The CAPM pricing relation using *RAMP* is equivalent to the assertion that *RAMP* is mean-variance efficient conditional only on asset prices. This efficiency can be seen from the utility maximization problem of an investor who is uninformed about all assets. Such an investor balances the expected returns and the risks of her holdings, and her information consists of the equilibrium price only. In equilibrium, such an investor holds *RAMP*, implying that *RAMP* is mean-variance efficient conditional only on equilibrium prices.

Privately informed investors also hold *RAMP* as a component of their portfolios; this is the piece that does not depend upon their private signals (except to the extent that their signals are incorporated into the publicly observable market price). In addition they have other asset holdings taking advantage of the greater safety of assets they have more information about, and for speculative reasons based upon their private information. *RAMP* is not mean-variance efficient with respect to their private information sets, but it is efficient with respect to the information set that contains only publicly available information.

The online appendix of Van Nieuwerburgh and Veldkamp (2010) provides a somewhat similar model setup and shows that a different version of the CAPM holds.[6] The result they derive uses as the market portfolio for CAPM pricing the ex-post total supply of the risky assets, the sum of the endowed risky assets and the random supply of risky assets ($W + Z$ in our model). Hence, they derive that the market portfolio is mean-variance efficient conditional on the average investor's information set. The version of the CAPM presented in Proposition 4 is significantly different in that the pricing portfolio is determined ex ante and that the risk premia are calculated condition only on the public information (the price).

In the model whose parameters are all common knowledge among investors, *RAMP* is a natural candidate for the CAPM pricing portfolio, because it is the common component in all investors' risky asset holdings. We show that *RAMP* is mean-variance efficient unconditional on any investor's private information. Therefore, the CAPM security market line relation holds without conditioning on private information, with respect to *RAMP*. One of the further contributions here is to establish that increasing information asymmetry, and its effect on investor participation, does not clearly predict

[6] Biais, Bossaerts, and Spatt (2010) derive a similar version of the CAPM in a dynamic model in which the market portfolio used for CAPM pricing is also the total ex-post supply of the risk assets, and the security market line holds *conditional* on the average investor's information set.

whether there will be an increase versus decrease in risk premium.

We are now in a position to see how an index fund affects asset risk premia. Proposition 2 shows that in the model where investors are uncertain about the precisions of asset supply shocks, they all hold one share of the index fund. The portfolio provided by the index fund is just the risk-adjusted market portfolio in the setting with all parameters commonly known. Therefore, the index fund makes assets' risk premia equal to those predicted by the CAPM, even if investors have heterogeneous information and are uncertain about different model parameters. Corollary 2 presents this even more surprising result.

Corollary 2 *In the model where investors are uncertain about the precisions of some assets' supply shocks, and an index fund offering portfolio X specified in equation* (3)*, asset risk premia are those predicted by the CAPM using X as the pricing portfolio.*

6 Extensions

To evaluate the robustness of our conclusions about investor participation, we now consider two possible model generalizations. First, investors may be unaware of certain traded assets, making it unattractive or infeasible for them to hold such assets. Previous literature considers this another important possible reason for limited participation. Second, investors may have heterogeneous risk tolerances. These cases also suggest further empirical implications.

6.1 Uncertainty about Other Parameters and Unawareness

Section 2, assumed that investors were uncertain about the precisions of assets' supply shocks, and maximized their CARA utilities based upon worst case scenarios. In addition, we assumed that the number of the risky assets is common knowledge. Hence, all investors know the existence of all assets, and can observe their prices.

Investors unawareness is an important alternative possible explanation for nonparticipation. Specifically, investors may not know certain traded risky assets, and so they do not observe such assets' prices. It is infeasible for investors to directly hold assets they are unaware of (Merton 1987; Easley and O'Hara 2004). For example, if an investor has never heard of FLIR Systems (an S&P 500 firm), it seems natural for the investor not to participate in this market.

Such a definition of investor unawareness is rather restricted. We relax the definition of investor unawareness to allow for extreme ignorance about some of the asset's characteristics, even when the investor can observe its price. This makes it physically possible (though not necessarily attractive) for an investor to hold an asset the investor is unaware of.

Formally, we say that investor i is unaware of asset n, if she holds a diffuse uniform prior about the precision of asset n's supply shock;[7] that is, $\tau_n \sim U(0, +\infty)$. Here, we allow investor i to know all characteristics of asset n (other than the precision of the supply shock) and observe asset n's price.

Since our focus is now on unawareness, not ambiguity aversion, instead of maximizing her CARA utility in the worst case scenario, investor i maximizes her 'average' CARA utility over the set of all possible precisions of asset n's supply shock. Proposition 5 below shows that in this setting, the investor will not hold asset n directly.

Proposition 5 *When there is no index fund, investors will not participate in the markets of assets they are unaware of.*

We now analyze whether an index fund that offers *RAMP* can lead to full participation in the setting with investors' unawareness. This is not a trivial question, since investors still need to assess the expected return and the risk of holding the index fund, when they allocate their initial wealth among the index fund, the risky assets they are aware of, and the riskfree asset.

We assume that any investor i believes that the number of all traded assets is equally likely to be any integer that is greater than or equal to the number of assets she is aware of. For an asset investor i is unaware of, investor i holds diffuse uniform priors about all its parameters she does not know.[8] These priors consist of a uniform uninformative prior about its endowment over the support $(0, +\infty)$, a uniform uninformative prior about the precision of the average private information about its payoff, over the support $(0, +\infty)$, and a uniform uninformative prior about the precision of the supply shock in its market over the support $(0, +\infty)$. Investors know that all random variables about assets' characteristics are independent.

[7] If we assume that the prior about τ_n is $\tau_n \sim U(0, \bar{\tau}_n)$ for some $\bar{\tau}_n \in \mathbb{R}_{++}$, Proposition 5 below still holds.

[8] The assumption of diffuse uniform priors is not necessary for Proposition 6 below. Indeed, from its proof, we can see that Proposition 6 holds for any subjective priors investors may have.

We further assume that all investors are aware of the riskfree asset, and that there is a fund that invests in the risky assets. Investors know the existence and name of the fund and are aware of its price, but are unaware of (have diffuse priors about) its return characteristics. So an investor who is unaware of some assets can have extremely poor information about the distribution of returns on this fund.

The fund manager observes the characteristics and prices of all assets and therefore is able to construct and offer to investors the portfolio X as specified in equation (3). It is common knowledge that this is the portfolio offered by the fund.

In such a setting, an investor i's investment strategy is a function mapping from her information set to positions in the index fund and the other assets. As we argue above, if investor i is unaware of asset n, and decides not to hold the index fund, then investor i will have a zero holding of asset n, because either it is infeasible for investor i to hold asset n, or holding asset n is infinitely risky to the investor.

We define an *admissible world* of an investor as the union of the set of assets she is informed about and a possible set of assets she is uninformed about or unaware of and hypothesized possible characteristics for these assets. Specifically, consider any investor i. We divide all traded assets into two groups, Γ_{i1} and Γ_{i2}. Suppose that investor i is informed about Γ_{i1} assets only, and so she knows all characteristics of Γ_{i1} assets. However, she is uninformed about or unaware of Γ_{i2} assets. In particular, she knows the existence of Γ'_{i2} assets and she can contemplate a possible set of assets Γ''_{i2}. Denote by $\widetilde{\Gamma}_{i2}$ the union of Γ'_{i2} and Γ''_{i2}. The combined asset set $\Gamma_{i1} \cup \widetilde{\Gamma}_{i2}$, together with an hypothesized vector of characteristics for each asset in $\widetilde{\Gamma}_{i2}$, constitutes an admissible world. The set $\widetilde{\Gamma}_{i2}$ is associated with a number $\widetilde{N} \geq \#\left(\Gamma'_{i2}\right)$ of assets in $\widetilde{\Gamma}_{i2}$. For each asset $n \in \widetilde{\Gamma}_{i2}$, the possible world specifies the specific parameters values characterizing asset n: the endowment $\widetilde{W_n} \geq 0$, the average precision of private information $\widetilde{\lambda_n}\widetilde{\kappa_n}$, and the precision of the supply shock $\widetilde{\tau_n}$.

We assume that conditional upon an admissible world, the investor has the CARA utility function. However, investors maximize their average CARA utilities over all admissible world, when making investment decisions.

Proposition 6 below shows that in such a general case, investors will hold exactly one share of the index fund, and thus participate in all assets' markets.

Proposition 6 *In the general model where investors are uncertain about several characteristics of the traded assets, including the number of assets:*

- *There exists an equilibrium in which all investors hold one share of the index fund and their own information-based portfolios.*
- *Asset prices and investors' effective risky assets holdings are identical to those in the model without any model uncertainty.*
- *Generically all investors take non-zero positions in all traded assets.*

From information separation, the portfolio constructed by the fund described in Proposition 6 is implementable using only public information. So if an index fund wants to provide investors with *RAMP*, it does not need to know the private signal of any investor. For investors, buying a fund share is the same as holding *RAMP*—the first component described by the information separation theorem. Therefore, intuitively, all investors are satisfied to buy fund shares, despite their extreme ignorance about the return distribution of the fund and its assets.

Since we have assumed a uniform uninformative prior for an investor on the number of assets of which he is not aware, one might suspect that this would interfere severely with the investor's attempt to speculate even on the assets the individual is aware of. However, investors do not need to know the number of assets traded in the market when forming their information-based portfolios. Consider for example any investor i. Denote by N_i the number of assets that she is informed about. For any given $\hat{N} \geq N_i$, except the $N_i \times N_i$ block Ω_i, all other blocks in the $\hat{N} \times \hat{N}$ matrix Ω_i are 0. So lack of knowledge about the number N does not affect investors' information-based trading.

6.2 Heterogeneous Risk Tolerances

In the model described in Section 2, investors share a same risk aversion coefficient ρ. Such an assumption leads to investors' homogeneous holdings of the index fund. Indeed, in the equilibrium characterized in Proposition 2, all investors hold one share of the index fund. However, it is conceivably that differences in risk tolerances, and investor unawareness of other investors' risk tolerances, could resurrect investors' heterogeneous holdings of the index fund.

We extend the model in Section 2 by assuming that any investor i ($i \in [0,1]$) has the risk aversion coefficient ρ_i. Here, ρ_i is a continuous function of i. Let

$$\overline{\rho} = \int_0^1 \rho_i \mathrm{d}i \quad \text{and} \quad \overline{\Sigma} = \int_0^1 \rho_i \Omega_i \mathrm{d}i.$$

Here, $\overline{\rho}$ is the average risk tolerance, and $\overline{\Sigma}$ is the average precision of investors' private information that is weighted by their risk tolerances. We assume that any investor i knows ρ_i, but she does not know the distribution of ρ_j and thus the average risk tolerance $\overline{\rho}$.

The index fund cannot evaluate each individual investor's risk tolerance, but it has accurate information about the distribution of investors' risk tolerances; hence, it knows $\overline{\rho}$ and $\overline{\Sigma}$. Then, the index fund offers the portfolio

$$\overline{X} = \left[\overline{\rho} + \left(U\overline{\Sigma}\right)^{-1}\right]^{-1} W. \tag{19}$$

to all investors. Proposition 7 shows that investors with different risk tolerances hold different numbers of shares of the index fund.

Proposition 7 *In the model with investors' heterogeneous risk tolerances, there exists an equilibrium in which any investor i with the risk tolerance ρ_i holds ρ_i shares of the index fund and her own information-based portfolio $\rho_i\Omega_i\left(S_i - rP\right)$.*

7 Concluding remarks

A leading explanation for nonparticipation puzzles is investor ambiguity aversion. This literature focuses on direct trading of assets by investors in the face of model uncertainty. We study here whether ambiguity aversion can still solve the puzzle when an appropriately designed passive index fund is available. We show that when there is an index fund that offers the 'risk-adjusted market portfolio' (*RAMP*), all investors prefer to hold the fund and thus participate in all asset markets, even if they do not know the index fund's composition.

This conclusion arises from applying a new portfolio information separation theorem which holds in a setting without model uncertainty. The information separation theorem asserts that any investor's equilibrium asset holding consists of a common, deterministic portfolio (*RAMP*), a portfolio that depends upon the investor's private signals, and the riskfree asset. The intuition comes from time-consistency. In the setting with model uncertainty, investors can contemplate a possible world with any given set of possible parameter values, known to all investors, and the resulting rational expectations equilibrium that would apply. In this possible world, the separation theorem would describe the optimal asset holdings in the possible world.

The index fund does observe the parameters (though no private information). So by delegating to the fund the job of holding *RAMP* (but not delegating the job of trading based upon any private information the investor might possess), the investor is having the fund choose exactly the portfolio that the investor would have chosen had she known what the fund knows. In the proposed equilibrium, each investor understands that all other investors are behaving likewise. It follows that conditional upon any possible world, the market clearing condition is satisfied; effective asset holdings and demand functions are exactly the same as in the corresponding rational expectations equilibrium under common knowledge of all parameter values. Therefore, in equilibrium, all investors hold the index fund, thereby participating (at least indirectly) in all asset markets.

We further show that in equilibrium, asset risk premia conditional only on public information satisfy the CAPM, with the index fund (i.e., *RAMP*) as the pricing portfolio. We further show that these conclusions are robust to investor unawareness of some assets, and to heterogeneous risk tolerances even with extreme investor ignorance about the distribution of the risk tolerances.

Given our finding that ambiguity aversion and unawareness do not, by themselves explain the limited market participation puzzle, an important question is: what does? One possibility is that currently existing index funds are not properly designed, and that there is a need to make available to investors funds that are based on*RAMP* rather than the market portfolio. Another is that there are heavy trading frictions, though as discussed in the introduction, it seems unlikely that this is the full explanation.

Some may argue that providing investors with more information can encourage market participation. However, if the source of nonparticipation is a psychological bias, this solution could make the problem worse. More information does *not* always debias decision makers, since extraneous information can be distracting or overwhelming. For example, providing extensive information about numerous assets could make investors feel less competent about evaluating their investments. This could exacerbate ambiguity aversion. Similarly, such information might push investors toward the use of simple judgement heuristics such as narrow framing, which is another leading possible explanation for nonparticipation.

Finally, investors may make psychological errors other than those that come just from ambiguity aversion. Our approach suggests a new possible explanation for the nonparticipation puzzle: that, in addition to ambiguity aversion, investors do not adequately understand the concept of equilibrium. The optimality for an investor of investing in *RAMP* relies on an equilibrium in which other investors are also doing so. Investors need to understand more than just that a broadly diversified portfolio reduces risk, because if they are ambiguity averse, such a portfolio may still be perceived too pessimistically to invest in. Instead, they need to understand that there is a risk-sharing benefit to investors all trading to the same portfolio, *RAMP* (along with an additional investor-specific portfolio to exploit private information). It is this knowledge of the market equilibrium that allows an investor in the model to understand that a fund that buys *RAMP* can serve a diverse set of investors in a powerful way: by trading to exactly the portfolio that each investor would select if that investor knew what the fund manager knows.

A Omitted Proofs

Proof of Proposition 1:

Because investor i is uniformed about asset n, by assumption, $\kappa_i = 0$. Hence, investor i's only information about the distribution of asset n's payoff is its price, which may partially aggregate informed investors' private signals. Suppose the uninformed investors' aggregate demand for asset n is $(1-\lambda_n)D(P_n)$. Since uninformed investors do not observe τ_n, $D(P_n)$ is not τ_n-measurable.

Given any P and any $\tau_n \in \left(0, \bar{\tau}_n^i\right)$, we derive investor i's expected utility conditional on P_n as follows. Suppose asset n's pricing function in a linear equilibrium is

$$F_n = a + bP_n + cz_n,$$

where a, b, and c are undetermined parameters. Since informed investors know τ_n, they can extract information from the price without any ambiguity. Therefore, any informed investor j's demand is

$$D_j = \rho\left[\kappa_n S_j + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)P_n - r\kappa_n P_n\right].$$

Then, the informed investors' aggregate demand will be

$$\lambda_n\rho\left[\kappa_n F_n + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)P_n - r\kappa_n P_n\right].$$

Then, the market clearing condition implies that

$$\lambda_n\rho\left[\kappa_n F_n + \frac{\tau_n}{c^2}a + \frac{\tau_n}{c^2}(b-r)P_n - r\kappa_n P_n\right] + (1-\lambda_n)D(P_n) = W_n + z_n.$$

Matching the coefficient of the market clearing condition and the pricing function, we have

$$\begin{aligned}
a &= \frac{W_n}{\lambda_n\kappa_n\rho} - \frac{\tau_n}{c^2\kappa_n}a \\
bP_n &= -\frac{(1-\lambda_n)D(P_n)}{\lambda_n\kappa_n\rho} - \frac{\tau_n}{c^2\kappa_n}(b-r)P_n - rP_n \\
c &= \frac{1}{\lambda_n\kappa_n\rho}
\end{aligned}$$

Therefore, for any given $\tau_n \in \left(0, \bar{\tau}_n^i\right)$, conditional on the price P_n, $|\mathbb{E}(F_n - rP_n|P_n)| < +\infty$. On the other hand, the variance of asset n's payoff conditional on P_n is

$$\mathbb{V}\left(F_n|P_n\right) = c^2\tau_n^{-1},$$

which diverges to $+\infty$ as τ_n goes to 0. Hence, any non-zero position D_i of asset n brings investor i a utility

$$-\exp\left(-\frac{1}{\rho}w_i r P_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(F_n - rP_n|P_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(F_n|P_n\right)\right], \tag{20}$$

which goes to $-\infty$ as τ_n goes to 0. Therefore, if investor i is uninformed about asset n, and $\underline{\tau}_n^i = 0$, investor i refrains from participating in the market of asset n.

Q.E.D.

Proof of Proposition 2:

We first verify that the market clearing condition holds. Each investor i's effective risky assets holding is

$$d_i^* X + D_i^* = \left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W + \rho\Omega_i\left(S_i - rP\right).$$

Then, using the pricing function (equation (7)), the aggregate demand can be calculated as

$$\begin{aligned}
&\int_0^1 \left(d_i^* X + D_i^*\right) \mathrm{d}i \\
= \;& \left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W + \rho\Sigma\left(F - rP\right) \\
= \;& \left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W + \rho\Sigma\left(\frac{1}{\rho}\left(\Sigma + \rho^2\Sigma U\Sigma\right)^{-1} W + \frac{1}{\rho}\Sigma^{-1}Z\right) \\
= \;& \left[I + \frac{1}{\rho^2}\left(\Sigma U\right)^{-1}\right]^{-1} W + \left[I + \rho^2\Sigma U\right]^{-1} W + Z \\
= \;& \left[I - \left(I + \rho^2\Sigma U\right)^{-1}\right] W + \left[I + \rho^2\Sigma U\right]^{-1} W + Z \\
= \;& W + Z.
\end{aligned}$$

Therefore, the market clears.

Now, for any investor i, we consider a general investment strategy $d_i X + D_i$. Denote by D_{in} investor i's direct holding of asset n. Suppose that investor i is informed about asset n. Then, the pricing function (7) implies that investor i's optimal holding of asset n is

$$\left[I + \frac{1}{\rho^2}\left(\lambda_n\kappa_n\tau_n\right)^{-1}\right]^{-1} w_n + \rho\kappa_n\left(S_{in} - rP_n\right) = X_n + \rho\kappa_n\left(S_{in} - rP_n\right).$$

Therefore, any combination of d_i and D_{in} such that

$$d_i X_n + D_{in} = X_n + \rho\kappa_n \left(S_{in} - rP_n\right)$$

can lead to the optimal holding of asset n for investor i.

Now, consider an asset n that investor i is uninformed about. For any given d_i and D_{in}, investor i is effectively holding a position $d_i X_n + D_{in}$ of asset n. Then, for any given τ_n Such a holding will bring invest i a utility

$$-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)\exp\left[-\frac{1}{\rho}\left(d_iX_n + D_{in}\right)\mathbb{E}\left(F_n - rP_n|P_n\right) + \frac{\left(d_iX_n + D_{in}\right)^2}{2\rho^2}\mathbb{V}\left(F_n|P_n\right)\right]. \tag{21}$$

There are two cases. In the first case where $\underline{\tau}_n^i = 0$, similarly to Proposition 1, if $D_{in} \neq 0$, the infimum of such a utility is $-\infty$, since $\mathbb{V}(F_n|P_n) \to +\infty$ as $\tau_n \to 0$. Therefore, $D_{in}^* = 0$. Next, substituting X_n into equation (21), the investor's utility given τ is

$$-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)\exp\left[\left(d_i - \frac{1}{2}d_i^2\right)\frac{\rho\tau_n\lambda_n^2\kappa_n^2 w_n^2}{\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]^2}\right]. \tag{22}$$

It follows from equation (22) that for any d_i, the infimum of the investor's utility is at most $-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)$. Since the investor can get the utility at least $-\exp\left(-\frac{1}{\rho}w_{in}rP_n\right)$ by employing the investment strategy $d_i^* = 1$, there is no profitable deviation.

In the second case, $\underline{\tau}_n^i > 0$. Since investor i does not know τ_n, d_i and D_{in} are not τ_n-measurable. For any given τ_n, we can solve d_i^* and D_{in}^* by the first order condition of the following maximization problem:

$$\max_{d_i, D_{in}} \left(d_iX + D_{in}\right)\frac{w_n}{\rho\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]} - \frac{\left(d_iX_n + D_{in}\right)^2}{2\rho}\frac{1}{\rho^2\lambda_n^2\kappa_n^2\tau_n}. \tag{23}$$

The second order condition of such a maximization problem holds, because the utility function in equation (23) is strictly concave.

Differentiating the utility function in equation (23) with respect to d_i, we get as one of the first-order conditions:

$$X_n\frac{w_n}{\rho\left[\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2\right]} - \frac{\left(d_iX_n + D_{in}\right)X_n}{\rho}\frac{1}{\rho^2\lambda_n^2\kappa_n^2\tau_n} = 0.$$

So,

$$d_iX_n + D_{in} = d_i\frac{\rho^2\tau_n\lambda_n^2\kappa_n^2}{\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2}w_n + D_{in} = \frac{\rho^2\tau_n\lambda_n^2\kappa_n^2}{\lambda_n\kappa_n + \rho^2\tau_n\lambda_n^2\kappa_n^2}w_n + D_{in}.$$

Then, $d_i^* = 1$ and $D_i^* = 0$, because they are not τ-measurable. Therefore, if an investor i is uninformed about asset n, she will hold exactly one share of the index fund and a zero position of asset n.

In sum, given the pricing function specified in equation (7), it is optimal for any investor i to choose the investment strategy $d_i^* = 1$ and

$$D_{in}^* = \begin{cases} 0, & \text{if she is uninformed about asset } n; \\ \rho\kappa_n \left(S_{in} - rP_n\right), & \text{if she is informed about asset } n. \end{cases}$$

Q.E.D.

Proof of Theorem 3:

Let's first prove a more general version of Proposition 3, when investors hold a common prior belief about F, $F \sim \mathcal{N}\left(\overline{F}, V\right)$. As is standard in the literature of rational expectations equilibrium, we consider the linear pricing function

$$F = A + BP + CZ, \quad \text{with } C \text{ nonsingular.} \tag{24}$$

If and only if B is nonsingular, equation (24) can be rearranged to

$$P = -B^{-1}A + B^{-1}F - B^{-1}CZ, \tag{25}$$

which solves for prices. Recall that $S_i = F + \epsilon_i$, so conditional on F, P and S_i are independent. Therefore, we can write down assets' payoffs' posterior means and posterior variances conditional on all information that are available to investor i as follows.

First consider investor i's belief about F conditional on P. Conditional on P, F is normally distributed with mean $A + BP$ and precision $[CU^{-1}C']^{-1}$. On the other hand, conditional on S_i, investor i's belief about F is also normally distributed, with mean S_i and precision Ω_i. Therefore, investor i's belief about F conditional on what the investor observes, P and S_i, is also normally distributed. The mean of the conditional distribution of F is the weighted average of the expectation conditional on the price P, the expectation conditional on investor i's private signal S_i, and the prior mean $\overline{F}$. Therefore, the conditional mean of F is

$$\left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right]^{-1} \left[(CU^{-1}C')^{-1}(A + BP) + \Omega_i S_i + V^{-1}\overline{F}\right]. \tag{26}$$

The precision of the conditional distribution of F is

$$(CU^{-1}C')^{-1} + \Omega_i + V^{-1}. \tag{27}$$

Then, from any investor i's first order condition, investor i's demand is

$$\begin{aligned}
D_i &= \rho\left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right] \\
&\quad \left\{\left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right]^{-1}\left[(CU^{-1}C')^{-1}(A+BP) + \Omega_i S_i + V^{-1}\overline{F}\right] - rP\right\} \\
&= \rho\left\{\left[(CU^{-1}C')^{-1}(A+BP) + \Omega_i S_i + V^{-1}\overline{F}\right] - \left[(CU^{-1}C')^{-1} + \Omega_i + V^{-1}\right] rP\right\} \\
&= \rho\left\{(CU^{-1}C')^{-1}(B - rI) - r\Omega_i - rV^{-1}\right\} P \\
&\quad + \rho\Omega_i S_i + \rho[(CU^{-1}C')^{-1}A + V^{-1}\overline{F}].
\end{aligned} \tag{28}$$

Integrating across all investors' demands gives the aggregated demand as

$$\begin{aligned}
\int_0^1 D_i \mathrm{d}i =& \rho\left\{(CU^{-1}C')^{-1}(B - rI) - r\left(\int_0^1 \Omega_i \mathrm{d}i\right) - rV^{-1}\right\} P \\
& + \rho\left(\int_0^1 \Omega_i S_i \mathrm{d}i\right) + \rho[(CU^{-1}C')^{-1}A + V^{-1}\overline{F}].
\end{aligned} \tag{29}$$

By equation (2), we have $\int_0^1 \Omega_i \mathrm{d}i = \Sigma$. Also, note that

$$\int_0^1 \Omega_i S_i \mathrm{d}i = \Sigma F.$$

Therefore, from the market clearing condition, we have

$$\int_0^1 D_i \mathrm{d}i = Z + W. \tag{30}$$

In an equilibrium, both equation (24) and equation (30) hold simultaneously for any realized F and Z, therefore, by matching coefficients in these two equations, we have

$$\rho\left[(CU^{-1}C')^{-1}A + V^{-1}\overline{F}\right] - W = -C^{-1}A \tag{31}$$

$$\rho\left[(CU^{-1}C')^{-1}(B - rI) - r\Sigma - rV^{-1}\right] = -C^{-1}B \tag{32}$$

$$\rho\Sigma = C^{-1} \tag{33}$$

Therefore, from equation (33), we have

$$C = \frac{1}{\rho}\Sigma^{-1}$$

Obviously, C is positive definite and symmetric. Then from equation (31), we have

$$[\rho^2(\Sigma U \Sigma) + \Sigma]A = \frac{1}{\rho}W - V^{-1}\overline{F}.$$

Because both $(\Sigma U \Sigma)$ and Σ are both positive definite, we have

$$A = [\rho^2(\Sigma U \Sigma) + \Sigma]^{-1}\left(\frac{1}{\rho}W - V^{-1}\overline{F}\right).$$

From equation (32), we have

$$[\rho^2(\Sigma U \Sigma) + \Sigma](B - rI) = rV^{-1}.$$

Again, because $[\rho^2(\Sigma U \Sigma) + \Sigma]$ is positive definite, we have

$$B = rI + r[\rho^2(\Sigma U \Sigma) + \Sigma]^{-1}V^{-1}.$$

Obviously, B is invertible. By substituting A, B, and C into equation (25), we solve the equilibrium pricing function.

Now, let's look at any investor i's holding. Substituting the coefficients into investor i's holding function (28), we have

$$D_i = \left(I + \frac{1}{\rho^2}(\Sigma U)^{-1}\right)^{-1} W + \rho\left[I + \rho^2 \Sigma U\right]^{-1} V^{-1}(\overline{F} - rP) + \rho\Omega_i(S_i - rP).$$

Finally, because the pricing function P and any investor i's demand function D_i are continuous in V^{-1}, we can substitute $V^{-1} = 0$ to get Proposition 3.

Q.E.D.

Proof of Proposition 4:

By equations (16), (17), and (18), we have

$$\begin{aligned} & \frac{\frac{1}{P'X}\mathrm{diag}(P)^{-1}CU^{-1}CX}{\left(\frac{1}{P'X}\right)^2 X'CU^{-1}CX}\frac{X'A}{P'X} \\ = \quad & \frac{\mathrm{diag}(P)^{-1}CU^{-1}CX}{X'CU^{-1}CX}X'A. \end{aligned}$$

This is the RHS of the Security Market Line relation. We want to show that this equals the difference between the risky assets' rates of return and the riskfree asset's rate of return, which is shown to be $\mathrm{diag}(P)^{-1}A$ from equation (15).

Then, we have

$$
\begin{aligned}
& \frac{\mathrm{diag}(P)^{-1}CU^{-1}CX}{X'CU^{-1}CX}X'A = \mathrm{diag}(P)^{-1}A \\
\Leftrightarrow\ & \mathrm{diag}(P)^{-1}CU^{-1}CXX'A = \mathrm{diag}(P)^{-1}AX'CUCX \\
\Leftrightarrow\ & CUCXX'A = AX'CUCX.
\end{aligned}
$$

The last equation holds because $X = \rho(CU^{-1}C)^{-1}A$ and $(CU^{-1}C)^{-1}$ is a symmetric matrix.

Q.E.D.

Proof of Proposition 5:

Because there is no index fund, $d_i = 0$. Consider $D_i \neq 0$ (any non-zero direct holding of asset n). For any given $\tau_n \in (0, +\infty)$, conditional on asset n's price P_n, investor i's utility is

$$
-\exp\left(-\frac{1}{\rho}w_i r P_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(F_n - rP_n|P_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(F_n|P_n\right)\right].
$$

Similarly to the proof of Proposition 1, $|\mathbb{E}(F_n - rP_n|P_n)|$ is bounded, and the variance of asset n's payoff conditional on P_n is

$$
\mathbb{V}\left(F_n|P_n\right) = c^2\tau_n^{-1},
$$

where $c = 1/(\lambda_n\kappa_n\rho)$ is independent of τ_n. Then, by Jensen's inequality, we have

$$
\begin{aligned}
& \lim_{h\to+\infty}\int_0^h -\frac{1}{h}\exp\left(-\frac{1}{\rho}w_i r P_n\right)\exp\left[-\frac{1}{\rho}D_i\mathbb{E}\left(F_n - rP_n|P_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(F_n|P_n\right)\right]\mathrm{d}\tau_n \\
\leq\ & -\exp\left(-\frac{1}{\rho}w_i r P_n\right)\exp\left[\lim_{h\to+\infty}\int_0^h \frac{1}{h}\left(-\frac{1}{\rho}D_i\mathbb{E}\left(F_n - rP_n|P_n\right) + \frac{D_i^2}{2\rho^2}\mathbb{V}\left(F_n|P_n\right)\right)\mathrm{d}\tau_n\right].
\end{aligned}
$$

The right-hand side of this inequality diverges to $-\infty$, implying that the left-hand side, which is the investor's average CARA utility, is also $-\infty$. Therefore, $D_i \neq 0$ is dominated by $D_i = 0$; hence, investors will not directly hold the assets they are unaware of.

Q.E.D.

Proof of Proposition 6:

Consider any investor i, who is aware of assets in Γ_{i1} but is uninformed about or unaware of assets in Γ_{i2}. Then, any of investor i's admissible world $\widetilde{\Gamma}$ consists of Γ_{i1} assets and possible $\widetilde{\Gamma}_{i2}$ assets; that is, $\widetilde{\Gamma} = \Gamma_{i1} \cup \widetilde{\Gamma}_{i2}$.

The strategy profile under consideration prescribes that all investors buy one share of the fund and hold their own information-based portfolios. Hence, in $\widetilde{\Gamma}$, by the information separation theorem, all other investors' portfolio choices are effectively the same as in equation (12), because the fund offers the risk-adjusted market portfolio in $\widetilde{\Gamma}$. Therefore, in $\widetilde{\Gamma}$, the pricing function will be the same as in equation (8). Then, for any given price vector, investor i's optimal portfolio choice will be the same as in equation (12) too. Such a portfolio choice can be implemented by holding the index fund and investor i's own information-based portfolio based only on her knowledge about Γ_{i1} assets. Therefore, in $\widetilde{\Gamma}$, it is optimal for investor i to hold the fund and her information-based portfolio, when all other investors do the same thing.

Since the admissible world $\widetilde{\Gamma}$ is constructed arbitrarily, the arguments above show that it is optimal for investor i to hold one share of the fund and her own information-based portfolio, when all other investors hold the fund and their own information-based portfolio. By similar arguments, when all other investors hold the fund and their own information-based portfolios, any investor will optimally hold the fund and her own information-based portfolio. Therefore, the strategy under consideration is an equilibrium.

Then, for any realized world, since all investors' effective holdings are exactly same as in equation (8), the market clearing condition implies that the equilibrium price function is same as in the case where all parameters are common knowledge. In addition, since all investors hold the fund who offers the risk-adjusted market portfolio, all investors will have strictly positive positions of all assets.

Q.E.D.

Proof of Proposition 7:

We first analyze the model in which investors have heterogeneous risk tolerances and all parameters are common knowledge. We again consider the linear pricing function as in equation (24),

$$F = A + BP + CZ, \quad \text{with } C \text{ nonsingular.}$$

Therefore, conditional on the price, assets' payoffs have the conditional distribution is

$$F|P \sim \mathcal{N}\left(A + BP, CU^{-1}C'\right).$$

An investor i gleans such information from the price. Therefore, an investor i's demand is

$$D_i = \rho_i \left[(CU^{-1}C')^{-1}(B - rI) - r\Omega_i\right] P + \rho_i \Omega_i S_i + \rho_i (CU^{-1}C')A. \tag{34}$$

Then, by integrating all investors' demands and equalizing the aggregate demand and the total supply (the aggregate endowments and the supply shocks), we can derive the pricing function

$$P = B^{-1}\left[F - A - CZ\right], \tag{35}$$

where

$$A = \left[\overline{\rho} + \left(U\overline{\Sigma}\right)^{-1}\right]^{-1} W \tag{36}$$

$$B = rI \tag{37}$$

$$C = \overline{\Sigma}^{-1}. \tag{38}$$

Any investor i's risky asset holding is

$$D_i = \rho_i \left[\overline{\rho} + \left(U\overline{\Sigma}\right)^{-1}\right]^{-1} W + \rho_i \Omega_i^{-1}\left(S_i - rP\right). \tag{39}$$

Because the index fund provides the portfolio $\overline{X}$ specified in equation (19), Equation (39) can be rewritten as

$$D_i = \rho_i \overline{X} + \rho_i \Omega_i^{-1}\left(S_i - rP\right). \tag{40}$$

Then, when investors are uncertain about some parameters and thus are subject to ambiguity aversions, they still want to hold the index fund. In particular, investor i first buys ρ_i shares of an index fund and then use her own private information to form the information-based portfolio $\rho_i \Omega_i^{-1}\left(S_i - rP\right)$. Finally, investor i invests the rest of her endowments in the riskfree asset.

Q.E.D.

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