Selling to Advised Buyers*

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Abstract

In many cases, buyers are not fully informed about their valuations and rely on the advice of biased experts. For example, the board of the bidder relies on the advice of managers when bidding for a target in a takeover contest. We study the design of sale mechanisms to such "advised buyers". In static mechanisms, such as first- and second-price auctions, advisors communicate a coarsening of information, and the revenue equivalence theorem holds. In contrast, in dynamic mechanisms, advisors can communicate information gradually as the auction proceeds, which leads to more efficient allocations. Whether this leads to higher revenues depends on the bias. If advisors are biased for overbidding, an ascending-price auction dominates static formats in both efficiency and expected revenues. If advisors are biased for underbidding, a descending-price auction dominates static mechanisms in efficiency but often results in lower revenues.

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1 Introduction

In many applications, agents that make purchase decisions have limited information about their valuations of the asset for sale. As a consequence, they rely on the advice of informed experts, who however often have misaligned preferences. Consider the following examples:

1. *A firm competing for a target in a takeover contest.* While the board of directors often has authority over submitting bids, the firm's managers are more informed about the valuation of the target. They, however, could be prone to overbidding because of career concerns and empire building preferences. A similar conflict of interest occurs if the expert is an investment banker.

2. *Bidding in spectrum auctions.* Telecommunication companies bidding in spectrum auctions have research teams preparing for the auction and advising the management and board on bidding. They can have misaligned incentives as winning the auction could give a positive signal of the team's competence.

3. *Suppliers competing in procurement.* When a construction company competes for a project in a procurement auction, managers that will work on it are privately informed about the cost, while the top management has authority over bidding. The informed managers can have a bias for overstating the cost.

4. *Realtors in real estate transactions.* A buyer of a house gets advice from a realtor about what offer to make. The realtor has private information about the value of the house but can be biased for overpaying.

We call such players "advised buyers" and study whether the presence of such advising relationships affects how the seller should design the sale process. We analyze this question both from the position of maximizing expected revenues, which is likely the goal if the designer is the seller, and from the position of allocative efficiency, which could be a relevant goal if the designer is the government.

We study a canonical setting in which the seller has an asset to auction among a number of potential buyers with independent private values. We depart from it in one aspect: Each potential buyer is a pair of a bidder (female) and her advisor (male), where the bidder controls bidding decisions (e.g., the firm's board) but has no information about her valuation, while the advisor (e.g., the firm's manager) knows the valuation but has a conflict of interest. Our initial focus is on the case in which advisors' bias is for overbidding, that is, given value v to the bidder, the advisor's maximum willingness to pay is $v + b$ with $b > 0$. We next consider the case of the bias for underbidding.

Prior to the bidder submitting an offer, the advisor communicates with the bidder via a game of cheap talk. If the sale process consists of a single round of bidding, there is only one round of communication. In contrast, if the sale process consists of multiple rounds, the advisor communicates with the bidder in each round of the auction. In this environment, communication and the design of the sale process interact. On one hand, communication from advisors affects bids and therefore efficiency and revenues of each auction format. On the other hand, the auction format affects how advisors communicate information to bidders.

We analyze equilibria of the model under the NITS ("no incentive to separate") condition adapted from Chen et al. (2008).[1] We first study static auctions. As one could expect from the classic game of cheap talk (Crawford and Sobel (1982)), communication takes a partition form: All types of the advisor are partitioned into intervals and types in each interval induce the same bid. Even though our game is not a special case of their problem, as payoffs are endogenous, the logic of Crawford and Sobel (1982) and Chen et al. (2008) applies. Communication strategies have a partition structure and the equilibrium with the highest number of partitions satisfies the NITS condition. We prove a version of the revenue equivalence theorem for static auctions. We focus on a large class of standard auctions with continuous payments introduced in Che and Gale (2006), including first-price, second-price, and all-pay auctions, and show that all static auctions in this class bring the same expected revenue and feature the same communication between bidders and advisors.

This conclusion changes drastically if the asset is sold via dynamic mechanisms. Consider the ascending-price (English) auction, in which the price continuously increases until only one bidder remains. From the position of a bidder and her advisor, the ascending-price auction is a stopping time problem: At what price level to drop out. At each price level, the advisor advises his bidder about whether to quit the auction now or not. We show that any equilibrium satisfying the dynamic counterpart of the NITS condition has the following structure. The advisor recommends to stay in the auction until the price reaches the advisor's maximum willingness to pay. In turn, the bidder follows the advisor's recommendation until the price reaches a high enough threshold, at which she drops out irrespectively of what the advisor says then. Thus, the advisors' types perfectly separate at the bottom of

[1]The NITS condition says that the payoff to the weakest type (the lowest valuation if advisors' bias is for overbidding) cannot be less than what he could achieve by credibly revealing himself. Intuitively, since message "I am the lowest type" goes extremely against the bias, it should be perceived credible by the bidder. We impose this condition in each round of the auction.

the distribution and pool at the top. Moreover, when the value is in the range of perfect information transmission, the bidder overbids: She exits the auction at a higher price than she would had she known her value at the start of the game. When the distribution satisfies a natural restriction, the NITS equilibrium is unique.

The intuition lies in the irreversibility of the running price in the auction: While a bidder can always bid until a price level higher than the current price, she cannot exit at a price lower than the current price. Informally, she can improve her offer but cannot renege on past offers. If the advisor is biased for overbidding, he recommends the bidder to continue bidding and sends the recommendation to quit only when the price reaches the advisor's indifference point, i.e., when the price exceeds the buyer's value by the amount of the bias. When the bidder gets such a recommendation, she infers that her valuation is below the running price and, hence, quits the auction immediately. When the bidder gets the recommendation to continue bidding, she trades off the value of the advisor's residual private information against the cost of possibly overpaying. The solution is to act on the advisor's recommendation unless the running price reaches a high enough threshold. Interestingly, this threshold can be infinite (i.e., there is full separation) if the distribution of values is unbounded and has fat tails.

We show that the ascending-price auction outperforms static auctions in both efficiency and expected revenues. The first result shows under rather general conditions the NITS equilibrium of the ascending-price auction is more efficient than any equilibrium of static auctions. While in static auctions communication has a partition structure, in the ascending-price auction advisors fully transmit their information up to a cut-off. This cut-off is higher than the cut-off in the highest partition in static auctions. Intuitively, in static auctions, the cut-off type is indifferent between pooling with higher types (and facing the risk of the bidder paying above the advisor's maximum willingness to pay) and with types in the second-highest partition (and facing the risk of losing to a rival also in the second-highest partition). This indifference implies that the bidder's best guess of its valuation when she learns that it is in the highest partition is strictly higher than the maximum willingness to pay of the cut-off type of the advisor. As a consequence, the advisor with type just above this cut-off could induce the bidder to wait until his most preferred price level in the ascending-price auction, implying that the cut-off there is higher.

The second result shows that if, in addition, the distribution is such that the virtual valuation is increasing, the expected revenues in the ascending-price auction are higher than in any NITS equilibrium of the second-price auction. This result may seem surprising, because the equilibrium exit price in the ascending-price auction can be higher for some or lower for other types than the equilibrium bid in the second-price auction. To understand it, think about the seller's auction design problem as selling to advisors directly, where communication between advisors and bidders puts restrictions on what the selling mechanism can be. According to Myerson (1981), the expected revenues equal the expected virtual valuation of the winning advisor less the expected payoff of the advisors with the lowest value. Since the ascending-price auction is more efficient, its expected virtual valuation of the winning advisor is higher. In addition, in the ascending-price auction, the lowest type of the advisor never wins, so his payoff is zero. At the same time, the NITS condition implies that his payoff in the second-price auction cannot be negative. Thus, the ascending-price auction generates higher revenues both because it is more efficient and because it leaves less rents to the lowest type.

Next, we consider the case in which advisors have a bias for underbidding, that is, given value v to the bidder, the advisor's maximum willingness to pay is $v + b$ with $b < 0$. In this case, the ascending-price auction loses its advantage over static formats. If the advisor follows the same strategy of recommending to quit when the running price reaches his maximum willingness to pay, the bidder has no incentive to follow this recommendation. Staying in the auction further is always an option, so the bidder would wait until the price reaches v.

One may conjecture that the decreasing-price (Dutch) auction, in which the running price continuously decreases until one bidder accepts it, dominates static auctions in this case. We show that the answer is "yes" for efficiency, but "no" for expected revenues. In this case, the descending-price auction has an equilibrium, which is conceptually similar to the equilibrium in the ascending-price auction with the overbidding bias: The advisor recommends to wait past the current price until it reaches his optimal bid in the auction, while the bidder follows his recommendation up to a certain price cut-off. Thus, there is full separation of high types and pooling of low types - the mirror image of what happens in the ascending-price auction with an overbidding bias. For the same reason, the descending-price auction is more efficient than static formats. However, it often results in lower expected revenues. The reason is that the separation of advisor's types in the descending-price auction comes at the cost of it occurring at lower prices, since advisors reveal their valuations with delay. Thus, the descending-bid auction often yields lower expected revenues, in contrast to the ascending-price auction always yielding higher expected revenues when the bias is for overbidding.

Our paper is related to two strands of the literature: auction design and communication of non-verifiable information (cheap talk). Our contribution to the auction theory literature is to study the design of auctions when bidders are advised by informed experts. A fundamental result in auction theory is the celebrated revenue equivalence theorem (Myerson (1981); Riley and Samuelson (1981)), generalized to arbitrary type distributions by Che and Gale (2006). In our setting, it holds for static mechanisms, but breaks down for dynamic mechanisms.[2] Our paper is related to studies of information acquisition by bidders and information design by the seller. In particular, Compte and Jehiel (2007) show that multiple-round auctions bring higher expected revenues than static counterparts because of more flexible information acquisition.[3] While this result is similar to ours when the bias is for overbidding, it follows from a very different argument, which relies on the asymmetry of bidders in information endowments and their knowledge of the number of remaining bidders in the auction. McAdams (2015) shows that multiple-round version of the second-price auction dominates the sealed-bid format when entry is costly. A unique normative implication of our model is that the choice between dynamic and static formats is quite different depending on the direction of the conflict of interest of advisors. Bergemann and Pesendorfer (2007), Eso and Szentes (2007), Chakraborty and Harbaugh (2010), and Bergemann and Wambach (2015) study design of information by the auctioneer. Our difference from this literature is in how bidders get information: from biased experts as opposed to the seller. Burkett (2015) studies a principal-agent relationship in auctions where the principal optimally constrains an agent with a budget and shows revenues equivalence of first- and second-price auctions.

Second, our paper is related to the literature on cheap talk, pioneered by Crawford and Sobel (1982). Because our results rely on the NITS condition, our paper is very related to Chen et al. (2008), who introduce it.[4] Cheap talk models usually have exogenous payoffs and timing of the game (typically, one round of communication). In contrast, the payoffs and the game itself are endogenous in our paper. In particular, by converting the mechanism from a single-round game to a stopping time game for bidders, the seller can make communication between bidders and advisors more efficient, which sometimes (but not always) leads to higher expected revenues. Thus, our paper builds on Grenadier et al. (2016) who study a cheap talk game in the context of an option exercise problem and show that, when the sender is biased

[2]Existing reasons for the failure of revenue equivalence include affiliation of values (Milgrom and Weber (1982)), bidder asymmetries (Maskin and Riley (2000)), and budget constraints (Che and Gale (1998, 2006); Pai and Vohra (2014)), among others.

[3]Other papers on information acquisition by bidders in auctions include Persico (2000), Bergemann and Välimäki (2002), Bergemann et al. (2009), Crémer et al. (2009), and Shi (2012).

[4]It is also related to Kartik (2009) and Chen (2011) who study perturbed versions of the classic cheap talk game with lying costs and behavioral players, respectively, since both variations can be used to motivate the NITS condition.

for delaying exercise, it leads to different equilibria than the static counterpart: separation up to a cut-off. Our contribution is to endogenize the design of the cheap talk game itself by having the seller designing the auction to maximize revenues.[5] A number of papers study cheap talk models with less related dynamic aspects of communication.[6]

Finally, several papers study other effects of cheap talk in auctions. Matthews and Postlewaite (1989) study pre-play communication in a two-person double auction. Ye (2007) and Quint and Hendricks (2016) study two-stage auctions, where the actual bidding is preceded by the indicative stage, which is a form of cheap talk between bidders and the seller. Kim and Kircher (2015) study how auctioneers with private reservation values compete for potential bidders by announcing cheap-talk messages. Several papers also study the role of cheap talk in non-auction trading environments.[7]

The structure of the paper is as follows. Section 2 introduces the model. Section 3 illustrates our main results in a simple two-bidder uniform example. Section 4 examines static auctions. Section 5 studies the ascending-bid auction when advisors have a bias for overbidding. Section 6 analyzes the case of advisors' bias for underbidding. Section 7 discusses possible extensions and gives a quantitative example. Section 8 concludes and Appendix gives proofs omitted in the text. Online Appendix contains proofs of the lemma and results of Section 7.

2 Model

Consider the standard setting with independent private values. There is a single indivisible asset for sale. Its value to the seller is normalized to zero. There are N potential buyers (bidders). The valuation of bidder i, v_i, is an i.i.d. draw from distribution with c.d.f. F and p.d.f. f. The distribution F has full support on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$ and satisfies $\int_{\underline{v}}^{\overline{v}} v dF(v) < \infty$. In the analysis, we will frequently refer to the distribution of valuation of the strongest opponent of a bidder. We denote by $\hat{v}$ the maximum of $N-1$ i.i.d. random variables distributed according to F and its c.d.f. by G: $G(\hat{v}) = F(\hat{v})^{N-1}$. We also use $F(a, b) = F(b) - F(a)$ to denote that a random variable distributed according to F falls in the interval $[a, b]$. Similarly, $G(a, b) = G(b) - G(a)$.

The novelty of our setup is that each bidder i does not know her valuation v_i, but consults

[5] Other differences include the ability to get more efficient communication for any sign of the bias by designing the game appropriately and the use of a weaker equilibrium selection criterion.

[6] See Sobel (1985); Morris (2001); Golosov et al. (2014); Ottaviani and Sørensen (2006a,b); Krishna and Morgan (2004); Aumann and Hart (2003).

[7] E.g., Levit (2014); Koessler and Skreta (2016); Inderst and Ottaviani (2013).

advisor i who does. Advisor i knows v_i, but has no information about v_j, $j \neq i$ except for their distribution F. While advisor i knows v_i, he is biased. Specifically, the payoffs from the auction are:

$$\text{Bidder } i \quad : \quad I_i v_i - p, \tag{1}$$

$$\text{Advisor } i \quad : \quad I_i (v_i + b) - p, \tag{2}$$

where I_i is the indicator variable that bidder i obtains the asset, p is the payment of bidder i to the seller, and b is the advisor's bias. Bias b is commonly known.[8] Our initial focus is on the bias of advisors for overbidding, $b > 0$, as it seems more relevant in applications. In Section 6, we also consider the case of $b < 0$, which shares several similarities with the case of $b > 0$, but also differs from it in a number of important aspects.

Formulation $(1) - (2)$ captures conflicts of interest described in the introduction. For example, consider a publicly traded firm bidding for a target. The board of the firm has formal authority over the bidding process, maximizes firm value, but does not know valuation v_i. Suppose that the CEO of the firm knows v_i, but is biased. Specifically, if the CEO owns fraction α of the stock of the company and gets a private benefit of B from acquiring the target and managing a larger company, his payoff is $\alpha(v_i - p) + B$. Normalizing this payoff by α and denoting $b = \frac{B}{\alpha}$, we obtain $(1) - (2)$.

In this paper, we compare how different selling mechanisms affect expected revenues and efficiency. We use the following definitions of standard auction formats:

1. **Second-price auction**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pays the second-highest bid.

2. **First-price auction**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pays her bid.

3. **Ascending-price (English) auction**. The seller continuously increases price p, which we refer to as the *running price,* starting from zero. Each bidder only observes the running price and the fact that the auction has not ended yet, and decides whether to continue participating or to *quit* the auction. Once a bidder quits, she cannot re-enter the auction. Once only one bidder remains, she wins and pays the running price.

[8]For many of our results it is sufficient to assume that b is commonly known by bidders and advisors, while the seller knows only the sign of the bias.

4. **Descending-price (Dutch) auction**. The seller continuously decreases price p, which we refer to as the *running price,* starting from a high enough level. Each bidder only observes the running price and the fact that the auction has not ended yet, and decides whether to *stop* the auction. The first bidder who stops the auction wins and pays the price at which she stopped the auction.

In all of these auction formats, if a tie occurs, the winner is drawn randomly from the set of tied bidders. We study a rich class of static auctions described in Section 4, but restrict attention to the ascending-price and descending-price auctions among dynamic mechanisms.

Communication between bidders and their advisors is modeled as a game of cheap talk. If the auction format is static (i.e., it consists of a single round of bidding), the timing of the game is as follows:

1. Advisor i sends a private message $\tilde{m}_i \in M$ to bidder i where M is some infinite set of messages.

2. Having observed message $\tilde{m}_i$, bidder i chooses what bid $\beta_i \in \mathbb{R}_+$ to submit.

3. Given all submitted bids $\beta_1, ..., \beta_N$, the asset is allocated and payments are made according to the rule of the auction.

We consider Perfect Bayesian Equilibria (PBE) of static auctions. Since all bidders are symmetric, we focus on symmetric PBEs in which all advisors play the same communication strategy $m : [\underline{v}, \overline{v}] \to M$ and all bidders play the same bidding strategy, which maps messages in M to distribution over bids.[9]

There is a multiplicity of equilibria in cheap talk games. To select among them, we impose the "no incentive to separate" (NITS) condition, adapted from Chen et al. (2008). When $b > 0$, call type $v_w \equiv \underline{v}$ the *weakest type* of advisor.[10] According to the NITS condition, the equilibrium payoff to the weakest type of the advisor cannot be below his payoff if he credibly revealed himself (and had the bidder best-respond to that information). Intuitively, when an advisor is biased for overbidding, every type of the advisor wants to convince the bidder to bid more than the bidder would bid if she knew her value. Thus, it is natural to assume that the recommendation to bid the lowest possible amount would be perceived as credible by the bidder. Chen et al. (2008) show that NITS can be justified by perturbations of the

[9]We use m to denote communication strategies and $\tilde{m}$ for messages in M.

[10]Similarly, when $b < 0$, $v_w \equiv \overline{v}$ is the weakest type of advisor.

cheap-talk game with non-strategic players or costs of lying.[11] We refer to an equilibrium as *babbling* if regardless of the message received, each bidder plays the same strategy.

If the auction format is dynamic (i.e., it consists of multiple rounds of bidding), the advisor sends a message to the bidder before each round of bidding. In ascending-price and descending-price auctions, we index rounds by corresponding running prices p. We assume that bidders and advisors only observe the running price p, but not the actions of other bidders. In the ascending-price or descending price auctions the history of the bidder i includes the current running price p and messages sent by advisor i up to round p.

A strategy of advisor i is a measurable mapping from the advisor's private information about the valuation v and a history into a message sent to bidder i after that history. A strategy of bidder i is a measurable mapping from a history and a current message into the action chosen by the bidder. A bidder's posterior belief process is a measurable mapping from a history into the distribution over $[\underline{v}, \overline{v}]$.

We will restrict attention to symmetric Perfect Bayesian equilibria in pure Markov strategies (PBEM) where the state consists of the auction round p and a bidder's posterior belief about her valuation v. Communication strategy $m(v, p, \mu)$ gives the message sent in round p when bidder's posterior is μ and the advisor's type is v. Bidding strategy $a(p, \tilde{\mu})$ gives the bidder's decision in round p to quit/stop the auction ($a_M = 1$) or continue ($a_M = 0$), when her beliefs are $\tilde{\mu}$ ($\tilde{\mu}$ is an updated version of μ after observing the advisor's last message). From now on, we refer to the equilibria we restrict attention to as simply *equilibria*.

For dynamic auctions, we require that the NITS condition holds in every round of the game. Specifically, let[12]

$$v_w(h) = \inf\{v | v \in supp(\mu(h))\}. \tag{3}$$

be the weakest remaining (according to the bidder's beliefs) type of the advisor after history h. Similarly to Chen et al. (2008), an equilibrium violates the dynamic version of NITS condition if after some history h, the advisor of type $v_w(h)$ is better off claiming that he is the weakest remaining type than playing his equilibrium strategy. To capture this condition, we require that any unexpected message is interpreted as a signal of the weakest type (then the advisor's sequential rationality implies that after any history, the equilibrium strategy is

[11]Notice that one way of imposing the NITS condition is to specify that there exists an off-equilibrium-path message, such that the bidder believes that it comes from the weakest type of advisor v_w. Since the preferences of players satisfy the single crossing condition, it is not restrictive that not only the weakest but any type of the advisor has the option to signal that the value is v_w.

[12]For $b < 0$, $v_w(h) = \sup\{v | v \in supp(\mu(h))\}$.

weakly preferred to signaling that you are the weakest type). Formally, the dynamic version of NITS that we impose is stated as follows:

Definition 1. *An equilibrium (m, a, μ) satisfies the NITS condition if the following holds. Consider any p-round history h in which the advisor deviates in round p' for the first time and sends $\tilde{m} \notin \bigcup_{v \in supp(\mu(h'))} m(v, p', \mu(h'))$ where h' is a truncation at round p' of history h. Then $\mu(h)$ assigns probability one to $v_w(h')$.*[13]

Several observations are in order. First, Definition 1 states that after the first unexpected message, the bidder assigns probability one to the weakest type in the round when the deviation happened and never updates her belief after that. Second, in dynamic auctions the weakest type can (and will) change as the auction progresses. Third, the condition in Definition 1 requires that any unsent message is perceived as a signal of the weakest type which is slightly stronger than assuming that the lowest type of advisor does not want to reveal itself in equilibrium.[14]

3 An Example: Uniform Distribution

We start the analysis by working out a simple example that illustrates the results of the paper: How dynamic auctions differ from static auctions and why the direction of the conflict of interest between bidders and advisors is important for the design of the sale process. In this example, there are two bidders ($N = 2$), each valuation is an i.i.d. draw from the uniform distribution over $[0, 10]$, and the advisors' bias is $b = 1$ (when the bias is for overbidding) or $b = -1$ (when the bias is for underbidding).

Overbidding bias ($b = 1$). First, consider the second-price auction. Because of the bias, the advisor cannot credibly communicate the valuation to the bidder, and the equilibrium must have a partition structure. Consider the conditions that characterize an equilibrium with K partitions, $[\omega_0, \omega_1]$, ..., $[\omega_{K-1}, \omega_K]$, with $\omega_0 = 0$ and $\omega_K = 10$. Given the advisor's

[13]We implicitly assume that the set of messages is rich enough so that there is always an "unused" message in any equilibrium.

[14]There are known technical difficulties in defining games in continuous time (see Simon and Stinchcombe (1989)). However, this problem of the outcome indeterminancy in continuous time does not arise in ascending- and descending-bid auctions in our model, because only the advisor can affect the evolution of posterior beliefs on which both sides condition their strategies. If the advisor deviates to an off-equilibrium message, the bidder assigns probability one to the weakest type in the round of the deviation, so future messages of the advisor become irrelevant. If the advisor deviates to a message sent by a different type, then such a deviation is not detected.

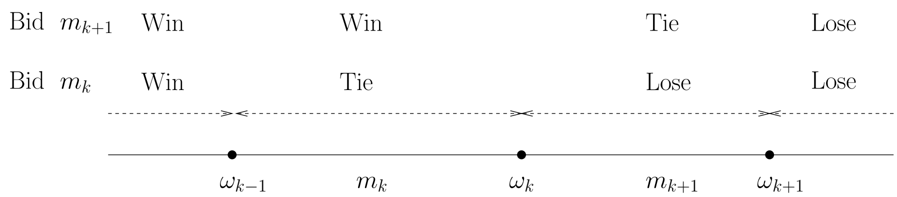


Figure 1: Thresholds in the partition equilibrium of the second-price auctions.

message that conveys that the valuation is in the k^{th} partition, the best response of the bidder is to bid the updated expected valuation, $m_k = (\omega_{k-1} + \omega_k)/2$. This bid is the winning bid with probability one, if the valuation of the rival bidder is below ω_{k-1}, with probability 50%, if it is between ω_{k-1} and ω_k, and with probability zero, if it is above ω_k (see Figure 1). By inducing the bidder to bid $(\omega_k + \omega_{k+1})/2$ instead of $(\omega_{k-1} + \omega_k)/2$, the advisor increases the probability of winning against types $[\omega_{k-1}, \omega_k]$ from 50% to one and against types $[\omega_k, \omega_{k+1}]$ from zero to 50%. Hence, for the cut-off type of the advisor ω_k, the additional payoff from a higher probability of winning against types $[\omega_{k-1}, \omega_k]$ must equal the cost from overpaying when the bidder wins against types $[\omega_k, \omega_{k+1}]$:

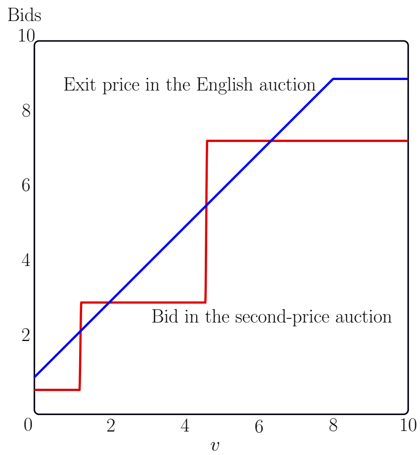
$$\frac{\omega_k - \omega_{k-1}}{10}\left(\omega_k + b - \frac{\omega_{k-1} + \omega_k}{2}\right) = \frac{\omega_{k+1} - \omega_k}{10}\left(\frac{\omega_k + \omega_{k+1}}{2} - \omega_k - b\right),\ k = 1, ..., N-1.$$

This indifference condition simplifies to

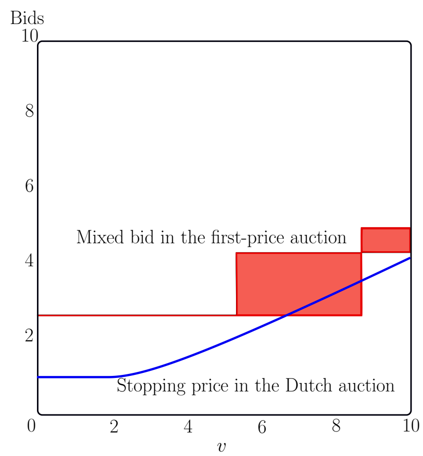
$$\omega_{k+1} = 2\omega_k - \omega_{k-1} + 2b,\ k = 1, ..., N-1.$$

When $b = 1$, the most informative equilibrium has three partitions, $\left[0, 1\frac{1}{3}\right]$, $\left[1\frac{1}{3}, 4\frac{2}{3}\right]$, and $\left[4\frac{2}{3}, 10\right]$. The corresponding bids are $\frac{2}{3}$, 3, and $7\frac{1}{3}$ (see Figure 2a). Since the lowest bid is below $b = 1$, this equilibrium satisfies the NITS condition: The weakest type of the advisor ($v = 0$) is better off inducing bid $\frac{2}{3}$ than communicating that $v = 0$. There exist two other equilibria: one with two partitions ($[0, 4]$ and $[4, 10]$) and the babbling equilibrium. Since the lowest bid (2 in the former case; 5 in the latter) exceeds $b = 1$, these equilibria violate the NITS condition. Indeed, the weakest type of the advisor ($v = 0$) is better off communicating that $v = 0$ and ensuring the payoff of zero.

Next, consider the ascending-price auction. Now a bidder faces a stopping time problem: At each price p, she decides whether to quit the auction or stay for a little longer. When $b = 1$, there exists the following equilibrium. Suppose that an advisor with type v plays the



(a) $b = 1$



(b) $b = -1$

Figure 2: Equilibrium bids in static and dynamic auctions. Horizontal axis depicts valuation v. The red region in panel (b) depicts the support of the bidder's mixed bidding strategy in the first-price auction.

threshold strategy of recommending to stay in the auction, if $p < v + 1$, and to quit once p hits $v + 1$ (see Figure 2a). Given this, what is the optimal strategy of the bidder? If she gets the recommendation to quit when the running price is $p \in [1, 11]$, she infers that her valuation is $v = p - 1$. Since p exceeds this valuation, the bidder finds it optimal to quit the auction immediately. If she has received recommendations to continue bidding, she trades off the value of waiting for more information against the possibility of overpaying for the asset. As the running price p increases, the support of bidder's beliefs, $[p - 1, 10]$, shrinks. Therefore, the best response of the bidder is to stay in the auction, as long as $p \leq \hat{p}$, given by

$$0 = \mathbb{E}\left[v|v \geq \hat{p} - 1\right] - \hat{p},$$

which implies $\hat{p} = 9$. Intuitively, $\hat{p} = 9$ is exactly the price at which the bidder is indifferent between winning the auction and getting the valuation of 9 on average (when the auction reaches this price, the bidder's posterior is that $v \in [8, 10]$) and quitting it.

This is the only equilibrium satisfying the NITS condition. To see this, consider why the equilibrium analogous to the equilibrium with three partitions in the second-price auction violates the NITS condition in the ascending-price auction. In this equilibrium, after the price passes $p = \frac{2}{3}$, the lowest remaining type is $v = 1\frac{1}{3}$. In this equilibrium, no bidder drops out after $p = \frac{2}{3}$ until p reaches 3. This implies that the advisor with type $v = 1\frac{1}{3}$ gets a negative expected payoff, since he wins with probability 50% at price $p = 3$, if the rival bidder's type is in $\left[1\frac{1}{3}, 4\frac{2}{3}\right]$. Therefore, the lowest type of the advisor $v = 1\frac{1}{3}$ is better off communicating that he is the lowest type at $p < 3$, since it would lead to the bidder quitting immediately. Hence, the equilibrium with three partitions does not satisfy the NITS condition. By the same logic, any equilibrium that satisfies the NITS condition has the property of separation up to a cut-off. The equilibrium with cut-off $\hat{p} = 9$ is the unique such equilibrium in this example.

As this argument shows and Figure 2a illustrates, the ascending-price and second-price auctions result in very different equilibrium outcomes. What does this imply for the comparison of revenues and efficiency? Clearly, the ascending-price auction is more efficient: Not only there is a separation of types up to $v = 8$, but the pooling interval $[8, 10]$ is contained in the pooling interval in the top partition in the second-price auction $\left[4\frac{2}{3}, 10\right]$. Indeed, the expected valuation of the winning bidder is $6\frac{49}{75}$ in the ascending-price auction and $6\frac{47}{135}$ in the second-price auction. Not only the ascending-price auction is more efficient, but it also generates higher expected revenues than the second-price auction: $4\frac{23}{75}$ versus $3\frac{88}{135}$. The comparison of revenues is not obvious at first glance, since one distribution of bids does not dominate the other (see Figure 2a). Nevertheless, higher expected revenues in the ascending-price auction is a general result.

Underbidding bias ($b = -1$). In this case, the most informative equilibrium in the second-price auction has partitions $\left[0, 5\frac{1}{3}\right]$, $\left[5\frac{1}{3}, 8\frac{2}{3}\right]$, and $\left[8\frac{2}{3}, 10\right]$. This is the unique equilibrium satisfying the NITS condition.[15]

Unlike with $b = 1$, the ascending-price auction does not have the equilibrium in which advisors separate themselves up to a cut-of. To see this, suppose that an advisor with type v plays the threshold strategy of recommending to stay in the auction, if $p < v - 1$, and to quit, otherwise. If the bidder gets the recommendation to quit at price $\tilde{p} \in (0, 9]$, she infers that her valuation is $\tilde{p} + 1$. Her best response is thus to stay in the auction until the price hits $\tilde{p} + 1$. Expecting that the bidder will not follow his advice, the advisor is better off deviating from his strategy. The asymmetry between cases $b = 1$ and $b = -1$ arises because the running price moves only in one direction.

However, the descending-price (Dutch) auction with $b = -1$ has similarities to the ascending-price auction with $b = 1$. In the descending-price auction, at each price p, the

[15]The equilibrium with two partitions ($[0, 6]$ and $[6, 10]$) violates the NITS condition, because the weakest type (now type $v = 10$) is better off credibly revealing himself. This is because the advisor with type $v = 10$ prefers to win when she faces a rival that bids 8. For the same reason, the babbling equilibrium violates NITS.

bidder chooses whether to accept it and buy the asset or to wait until a marginally lower price and risk losing the auction. Let us construct an equilibrium, which is similar to the equilibrium in the ascending-price auction for the case of $b = 1$. Suppose that the bidder does not stop the auction, unless the advisor recommends to do it or unless the price decreases to threshold $\underline{p}$. Given this, the optimal price at which the advisor with type v sends a recommendation to stop, $\sigma(v)$, satisfies:

$$\sigma(v) = \arg\max_{p \geq \underline{p}} (v - 1 - p)\, \sigma^{-1}(p), \tag{4}$$

which represents the familiar trade-off between a lower payment and a lower probability of winning. Let v^* denote the lowest type that recommends to stop the auction before $\underline{p}$. Pick v^* and $\underline{p}$ so that the bidder is indifferent between winning and losing when the price hits $\underline{p}$, given her belief at that point:

$$\mathbb{E}\left[v | v \leq v^*\right] = \underline{p} = v^* - 1. \tag{5}$$

Combining with (4), we obtain $v^* = 2$, $\underline{p} = 1$, and $\sigma(v) = \frac{v^2 - 2v + 4}{2v}$ (see Figure 2*b*). If a bidder receives a recommendation to stop the auction at price $\tilde{p} \in (1, 4.2)$, she infers that the valuation is $\sigma^{-1}(\tilde{p})$. Since $\tilde{p}$ is already below the bidder's optimal stopping point, she finds it optimal to stop the auction immediately.[16] If a bidder has received recommendations to continue staying in the auction, she trades off the value of waiting for more information against the possibility of losing the auction. As the running price p goes down, the bidder's posterior belief about the valuation, $[0, \sigma^{-1}(p)]$ shrinks, and her best response is to wait for the recommendation of the advisor until p gets too low, which happens to be $\underline{p} = 1$ in this example.

As in the bias for overbidding, the dynamic aspect of communication is crucial for the better information transmission. In particular, we show that the first-price auction, which without the conflict of interest is strategically equivalent to the Dutch auction, is equivalent in terms of information transmission and revenue to the second-price auction. More precisely, in the first price auction the advisor communicates the partition $\left[0, 5\frac{1}{3}\right]$, $\left[5\frac{1}{3}, 8\frac{2}{3}\right]$, $\left[8\frac{2}{3}, 10\right]$, and the bidder plays a mixed bidding strategy (see Figure 2*b*).

Can we conclude that the descending-price auction is more efficient and generates higher revenues than static auctions when $b = -1$, like we did with the ascending-price auction

[16]The fact that the advisor's optimal stopping price is below the bidder's optimal price follows from the single-crossing property of payoff function (4).

when $b = 1$? The answer to the first question is a "yes", but to the second one is a "no." The descending-bid auction is indeed more efficient than static auctions: Not only there is a separation of types in $[2, 10]$, but the pooling interval $[0, 2]$ is smaller than the pooling interval in the bottom partition in the first-price auction, $\left[0, 5\frac{1}{3}\right]$. Indeed, the expected valuation of the winning bidder is $7\frac{2}{25}$ in the descending-price auction and $6\frac{47}{135}$ in the first- and second-price auctions. However, the descending-price auction generates lower revenues than the first-price auction: approximately 2.7 versus $3\frac{88}{135}$. Thus, the first-price auction dominates if the goal of the designer is expected revenues, but the descending-price auction dominates if the goal is efficiency.

The opposite implications for expected revenues occur for the following reason. If advisors are biased for overbidding, the seller's goal of higher revenues is aligned with the bias of advisors. In contrast, if advisors are biased for underbidding, the bias goes in the opposite direction from the seller's goal of higher revenues.

4 Static Auctions

This section shows that the revenue equivalence theorem extends to the setting when the interests of bidders and advisors are not aligned ($b \neq 0$), if the auction is static. For a rich class of static auctions, we characterize equilibrium communication and show that there is necessarily an efficiency loss due to imperfect communication.

4.1 Revenue Equivalence

After a bidder gets message $\tilde{m}$ from her advisor, she updates her belief about her value and decides on the bid. By risk-neutrality, the bidder cares only about her posterior expected value, which we refer to as her *type* $\theta \equiv \mathbb{E}[v|\tilde{m}] \in [\underline{v}, \overline{v}]$. Let F_θ denote the distribution of a bidder's types, induced by equilibrium at the communication stage (by symmetry, F_θ is the same for all bidders).

We start by defining a class of standard auctions for which the revenue equivalence theorem holds for arbitrary distributions of values (Che and Gale (2006)):

Definition 2. *[Che and Gale, 2006] Call a static auction a* standard auction with continuous payments *if it satisfies the following conditions:*

1. *the highest bid wins and ties are broken randomly;*

2. *payment depends only on the bidder's own and the highest competing bids, i.e., bidder i pays $\tau_w(\beta_i, \beta_{m(i)})$, if she wins, and $\tau_l(\beta_i, \beta_{m(i)})$, if she loses, where $\beta_{m(i)} = \max_{j \neq i} \beta_j$;*

3. *$\tau_w(0,0) = \tau_l(0, \cdot) = 0$ and $\tau_k(\cdot, \beta_{m(i)})$ is continuous for $k = w, l$, in the relevant domain.*

This is a large class of auctions that includes many well-known formats, such as first-price, second-price, and all-pay auctions. The next theorem establishes revenue equivalence for auctions in this class in our model.

Theorem 1. *Suppose that $b \neq 0$ and there is a single round of communication. For any equilibrium in a standard auction with continuous payments there exists an equilibrium of the second-price auction that generates the same allocation, expected revenues, and distribution of bidders' expected values, F_θ, after the communication stage.*

Our main question is whether the choice of the auction format affects information transmission and through it expected revenues and efficiency. Theorem 1 tells us that it does not if one restricts attention to static auctions. For example, one does not get a better information transmission or higher revenues by switching between first- and second-price auctions.

The proof of Theorem 1 is based on two observations. First, Che and Gale (2006) establish a payoff equivalence for arbitrary distributions of bidders' values: for a fixed distribution of values F_θ, for any bidder's type θ, the expected probability of winning and expected payments are the same across standard auctions with continuous payments. Second, the advisor's decision what message to send depends only on how information conveyed through messages affects the probability of winning and expected payment. Since they are the same, the advisor's problem of choosing what message to send is also the same. Thus, if communication strategy m is an equilibrium in some standard auction with continuous payments, it is also an equilibrium in the second-price auction.

4.2 Characterization

Because of payoff equivalence established in Theorem 1, it is sufficient to study the second-price auction, which has a simple bidding equilibrium: each bidder bids her updated expected valuation of the asset. Given this, it is convenient to refer to messages as bid recommendations and denote equilibrium messages by conditional expected values $\mathbb{E}[v|\tilde{m}]$. The next theorem characterizes the set of symmetric equilibria of the communication game:

Theorem 2. *Suppose that $\overline{v} < \infty$ and $b \neq 0$. In any equilibrium, communication takes a partition form, in which types $v \in [\omega_{k-1}, \omega_k)$ send the same message and induce the same* *bid* $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. *In an equilibrium with* K *partitions, thresholds* $(\omega_k)_{k=0}^K$ *satisfy* $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, *and*

$$G(\omega_{k-1}, \omega_k)(1 - \Lambda_k)(\omega_k + b - m_k) = -G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}). \tag{6}$$

where

$$\Lambda_k = \frac{1}{G(\omega_{k-1}, \omega_k)} \left(\sum_{n=1}^{N-1} \binom{N-1}{n} \frac{F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n}}{n+1} \right) \tag{7}$$

is the probability of winning conditional on a tie at bid m_k.

Theorem 2 implies that in static auctions the conflict of interest results in coarse information transmission from the advisor to his bidder. Hence, ties arise with positive probability, and the asset is sometimes allocated inefficiently. Theorem 2 is a counter-part of Theorem 1 in Crawford and Sobel (1982) and relies on the same argument, but does not follow from it directly. The difference is that the payoffs in the communication game are endogenous: the action of a receiver is a bid, and its attractiveness depends on information transmission in other sender-receiver pairs. Eq. (6) is the condition that advisor with valuation ω_k is indifferent between sending messages that induce bids m_k and m_{k+1}. The left-hand side of (6) is the advisor's benefit from having the bidder bid m_{k+1} instead of m_k: it increases the probability of winning a tie from Λ_k to 1. The right-hand side of (6) is the cost of a higher bid: the bidder pays above the advisor's maximum willingness to pay, if the strongest rival also bids m_{k+1}.

Chen et al. (2008) show that in the standard cheap talk game, there always exist equilibria satisfying the NITS condition, and it selects equilibria that are sufficiently informative. This result also holds in our model:

Proposition 1. *Suppose* $b \neq 0$. *The equilibrium with the highest number of partitions satisfies the NITS condition.*

5 Ascending-Price Auction

This section solves for equilibria of the ascending-price auction when the bias is for overbidding ($b > 0$) and shows that it dominates static auctions from Section 4 in efficiency and expected revenues.

It turns out that it is sufficient to look for equilibria in which the advisor gives a real-time recommendation of the action ("quit" or "stay") to the bidder, both advisors' and bidders' strategies are of the threshold form, and bidders follow the recommendations of their advisors on equilibrium path. We refer to these equilibria as equilibria in online threshold strategies.

Definition 3. *An equilibrium in an ascending-price auction is* in online threshold strategies *if the strategies of each advisor and bidder satisfy:*

$$m(v,p,\mu)=\begin{cases}1, & \text{if } p\geq\hat{p}(v,\mu),\\ 0, & \text{if } p<\hat{p}(v,\mu),\end{cases} \tag{8}$$

$$a(p,\tilde{\mu})=\begin{cases}1, & \text{if } p\geq\bar{p}(\tilde{\mu}),\\ 0, & \text{if } p<\bar{p}(\tilde{\mu}),\end{cases} \tag{9}$$

for some $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$, where $\tilde{\mu}$ denotes the posterior belief of the bidder at price p, having observed her advisor's message in this round. Functions $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$ are such that on equilibrium path the bidder exits the auction the first time her advisor sends message $\tilde{m}=1$.

Intuitively, at any price p, the advisor sends a binary message to his bidder recommending to quit the auction immediately or stay in it, and on equilibrium path, the bidder follows the advisor's recommendation. The next lemma shows that the restriction to equilibria in online threshold strategies is without loss of generality:

Lemma 1. *For any equilibrium there is also an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path. For any equilibrium that satisfies NITS there is an equilibrium in online threshold strategies that satisfies NITS and results in the same bidding behavior on equilibrium path.*

The first statement is that any equilibrium with a general communication strategy has an equivalent in online threshold strategies. The proof is the manifestation of the sure-thing principle (Savage (1972)), stating that if an action is optimal for a decision-maker in every state, then it must be optimal if she does not know the state. Intuitively, since the advisor's information is only relevant for determining the price level at which the bidder quits the auction, any equilibrium quitting strategy can be achieved by the advisor delaying communication as much as possible, which occurs when she sends a recommendation to quit immediately when the price hits the level at which the bidder is supposed to quit. The second statement implies that the NITS condition is not stronger for equilibria in online threshold strategies than in general: If an equilibrium with a general communication strategy satisfies NITS, then an equivalent in online threshold strategies also satisfies NITS.

5.1 Characterization

This subsection shows that when the bias is for overbidding ($b > 0$), all equilibria of the ascending-bid auction satisfying NITS are in capped delegation strategies, defined as:

Definition 4. *Online threshold strategies in the ascending-bid auction are* capped delegation strategies *if for some v^*:*

- $\hat{p}(v, \mu) = \min\{v, v^*\} + b$, *i.e., the advisor of type $v \leq v^*$ starts recommending to quit the auction when the running price reaches his most preferred exit price;*
- *the bidder quits the auction if either the running price increases to $v^* + b$ or she receives message $\tilde{m} = 1$ from the advisor, whichever happens earlier.*

When players follow capped delegation strategies, advisor's types below v^* fully separate over the course of the auction, while types above v^* pool, since the bidder stops following recommendations when the running price reaches $v^* + b$. If the advisor were submitting the bids himself, he would stay until price $v + b$. Thus, even though the bidder makes bidding decisions herself, she essentially delegates it to the advisor with the restriction that he cannot stay in the auction beyond price $v^* + b$ ("cap"). The next theorem shows that *all* equilibria in the ascending-bid auction satisfying NITS are in capped delegation strategies.

Theorem 3. *Suppose that $b > 0$. Then, any equilibrium in the ascending-bid auction that satisfies the NITS condition is in capped delegation strategies with cutoff v^* satisfying:*

$$\begin{aligned} &\text{if } v^* \in (\underline{v}, \overline{v}), \text{ then } b = \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \underline{v}, \text{ then } b \geq \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \overline{v}, \text{ then } \overline{v} = \infty \text{ and } b \leq \lim_{s \to \infty} \mathbb{E}[v|v \geq s] - s. \end{aligned} \tag{10}$$

As we see from Theorems 2 and 3, equilibria in the ascending-bid and second-price auctions are different. The difference arises because of communication during the coarse of the auction. When the advisor recommends the bidder to quit the auction at the current price p, the bidder learns that her valuation is $p - b < p$ and exits immediately. As she gets recommendations to stay in the auction, she updates her belief that her valuation is not too low. Her decision whether to continue bidding trades off the benefit of waiting for more information against the cost of possibly overpaying. When the current running price is below $v^* + b$, the former factor dominates, so the bidder follows the advisor's recommendation to stay in the auction. However, when the price reaches a high enough level, $v^* + b$, the bidder learns that the valuation is in a narrow enough partition $[v^*, \bar{v}]$, so that it becomes optimal to quit the auction regardless of what the advisor recommends.[17] These equilibria are not possible in static auctions because of the commitment problem: The bidder would not follow the advisor's recommendation. The ascending-bid auction makes them possible by giving the advisor an opportunity to delay the recommendation to quit to the point when the bidder bids above her (unknown) valuation.

The ascending-bid auction also has equilibria that are not in capped delegation strategies. In particular, it has equilibria that are counterparts to equilibria of the second-price auction from Theorem 2. To construct them, we can specify that types in $[\omega_{k-1}, \omega_k)$, which send message m_k in the second-price auction, recommend that the bidder stays in the English auction until price m_k and quit after that. Theorem 3 implies that the NITS condition rules out these equilibria. Intuitively, at the start of the auction, the threshold type ω_k of the advisor is indifferent between the bidder exiting at prices m_k and m_{k+1}: m_{k+1} implies certain winning against types in $[\omega_{k-1}, \omega_k)$ but entails the risk of winning at a higher price m_{k+1} and paying above $\omega_k + b$ then. However, as the running price exceeds m_k, the bidder learns that the strongest rival bidder would bid at least $m_{k+1} > \omega_k + b$. At this stage he is better off inducing the bidder to quit immediately, which violates the NITS condition.

For a large class of distributions, the equilibrium satisfying the NITS condition is unique. To see when this is the case, consider the option value to the bidder of following the advisor's recommendation up to price $v^* + b$. Consider a bidder in round $p \in (\underline{v} + b, v^* + b)$ who has not received a recommendation to quit yet. From the fact that the auction reached this stage, the bidder infers that her valuation is in $[p - b, \bar{v}]$, and that there is at least one rival whose valuation is also in $[p - b, \bar{v}]$. Denoting the bidder's posterior probability that n rival bidders have valuations in $[p - b, \bar{v}]$ by $q_n(p)$ and the c.d.f. of the maximum of n i.i.d. random variables distributed according to F by $G_n(\cdot)$, the bidder's option value of following the advisor's recommendation up to price $v^* + b$ can be written as

$$V(p) = \int_{p-b}^{v^*} \frac{1 - F(s)}{1 - F(p-b)} \left(\mathbb{E}[v|v \geq s] - s - b\right) \left(\sum_{n=1}^{N-1} q_n(p)\, dG_n(s|s \geq p - b)\right). \tag{11}$$

[17] Clearly, the communication strategy is optimal for the advisor: It implements the advisor's unconstrained optimal bidding strategy of bidding up to $v + b$, if his type is low enough, and it is impossible to induce the bidder into bidding above $v^* + b$.

Intuitively, if the bidder wins when the strongest rival's valuation is $s < v^*$, she pays $s + b$ and gets, on average, $\mathbb{E}[v|v \geq s]$. The probability of this event is the probability that the strongest rival's valuation is s times the probability that the bidder's valuation is above s, corresponding to the last and the first terms, respectively.[18] From (11), we can see why v^* must satisfy $\mathbb{E}[v|v \geq v^*] = v^* + b$ when $v^* \in (\underline{v}, \bar{v})$. If $\mathbb{E}[v|v \geq v^*] < v^* + b$, then the bidder would prefer to exit the auction before price $v^* + b$, as the option value of waiting would be negative at a price just below $v^* + b$. Similarly, if $\mathbb{E}[v|v \geq v^*] > v^* + b$, then the bidder would get a positive payoff when she wins a tie at price $v^* + b$. Therefore, the bidder, whose advisor has not recommended to exit before price $v^* + b$, would prefer to wait a little beyond price $v^* + b$, since this would lead to a jump in the probability of winning to one.

To characterize v^*, we introduce the *mean residual lifetime* function $MRL(s) = \mathbb{E}[v|v \geq s] - s$, a well-studied function in industrial engineering and economics (Bagnoli and Bergstrom (2005)). It turns out that when either of the following conditions holds, the equilibrium is unique:

Assumption A. *$MRL(s)$ is strictly decreasing in s.*

Assumption B. *$MRL(s) > b$ for any $s \in [\underline{v}, \infty)$.*[19]

Decreasing $MRL(s)$ is a natural property. In industrial engineering, where $MRL(s)$ captures the expected time before a machine of age s breaks down, decreasing $MRL(s)$ means that the machine gets less durable as it ages. In our context, it means that winning at a higher price is worse news for the bidder than winning at a lower price. It holds for many distributions, such as Uniform, Normal, Logistic, Extreme Value, and many others. The next proposition shows that if the MRL is decreasing on a finite support or on an infinite support with a low enough limit, then the equilibrium is unique, and the pooling region is non-empty. This generalizes the example of Section 3 to a large class of distributions.

Proposition 2. *Suppose that $b > 0$, Assumption A holds, and either $\bar{v} < \infty$ or $\bar{v} = \infty$ and $\lim_{v\to\infty} MRL(v) < b$. Then, the unique equilibrium cut-off v^* satisfies $v^* < \bar{v}$. Moreover, the equilibrium is babbling if and only if $MRL(\underline{v}) \leq b$.*

Strict monotonicity of MRL implies that equation $MRL(v^*) = b$ has at most one solution. Furthermore, a strictly decreasing $MRL(\cdot)$ implies single-crossing: If cut-off type v^* satisfies $MRL(v^*) = b$, then the bidder's value of the option to wait for advisor's recommendation is strictly positive at any price prior to reaching this cut-off ($p < v^* + b$). This implies a unique

[18]Eq. (11) could also include the term, corresponding to the case of winning at a tie at price $v^* + b$. Since it equals zero by Theorem 3, we omit it.

[19]This condition can only hold if the support is infinite, $\bar{v} = \infty$.

equilibrium with non-empty separating and pooling intervals, if the bias is low. In contrast, if the bias is high, then the bidder's option value of waiting for the advisor's recommendation is never positive, so babbling occurs in this case.

The next proposition shows that if the distribution satisfies Assumption B, then the equilibrium satisfying NITS is also unique, but it features full separation:

Proposition 3. *Suppose that $b > 0$ and Assumption B holds. Then, the unique equilibrium cut-off v^* satisfies $v^* = \infty$. That is, the bidder always waits for the advisor's recommendation to quit the auction, which type v sends at price $v + b$.*

In this case, although the bidder has authority, she effectively fully delegates bidding to the advisor. Intuitively, Assumption B implies that no matter what the current price is, the bidder is always optimistic in the sense of holding a posterior that her expected valuation exceeds the current price. For example, if valuations are distributed according to Exponential distribution with parameter λ and $b < \frac{1}{\lambda}$, the unique equilibrium satisfying NITS will feature the bidder always following the advisor's recommendation.[20]

It is worth noting that the equilibrium in the English auction can be informative (and even stronger, fully separating) even if only babbling equilibrium exists in the second-price auction. As an example, consider $N = 2$ and Pareto distribution of valuations, $F(v) = 1 - \left(\frac{1}{v}\right)^2$ on $v \in [1, \infty)$. If $b \in \left(1, \frac{4}{3}\right)$, only the babbling equilibrium exists in the second-price auction, but the English auction has a fully separating equilibrium. Since $MRL\,(v) < b$ for low valuations v, winning is bad news for the bidder at the beginning of the auction. However, as the auction continues, the bidder eventually starts getting positive expected payoff from winning. Since the latter is incorporated in the bidder's option value of following the advisor's recommendation, waiting becomes optimal early in the auction too.

5.2 Auction Comparison

We next compare the ascending-bid and static formats in their efficiency and revenues. We will say that an equilibrium in one auction format is *(strictly) more efficient* than an equilibrium in another auction if the former results in a (strictly) higher expected valuation of the winning bidder. As the next theorem shows, the ascending-bid auction is more efficient than the second-price auction (and by Theorem 1, any standard static auction with continuous payments):

[20] Exponential distribution has a constant $MRL\,(s) = \frac{1}{\lambda}$ - the famous memoryless property.

Theorem 4. *Suppose that $b > 0$, and either Assumption A or B holds. Then the pooling region (if it is not empty) in the unique equilibrium satisfying NITS in the ascending-bid auction is contained in the top partition in any equilibrium of the second-price auction: $v^* \geq \omega_{K-1}$, with a strict inequality if $v^* > \underline{v}$, i.e., if there is no babbling in the ascending-bid auction. Therefore, the ascending-bid auction is more efficient.*

The source of inefficiency in auctions is the ties. Higher efficiency of the ascending-bid auction stems from superior information transmission. This result is clear when the equilibrium features full separation, since ascending-bid auction is efficient in this case. However, it is more nuanced when the ascending-bid auction is inefficient. It can be seen from the indifference condition (6) that determines partitions in the second-price auction. For advisor with type ω_{K-1} to be indifferent, the highest bid must exceed the maximum willingness to pay of the advisor with type ω_{K-1}: $\mathbb{E}\left[v|v \geq \omega_{K-1}\right] > \omega_{K-1} + b$, or, equivalently, $MRL\left(\omega_{K-1}\right) > b$. Hence, the bidder's option value of waiting is positive at price $\omega_{K-1} + b$. Consequently, types just above ω_{K-1} would recommend the bidder to stay in the ascending-bid auction at this price, and the bidder would follow the recommendation, implying a smaller pooling region and higher efficiency. An interesting feature of the ascending-bid auction is that communication in it does not depend on how many bidders there are. This is not the case in static auctions, since the number of bidders enters recursion (6) in a complicated way. While partitions in static auctions are affected by N, they never become finer than in the ascending-bid auction.

We next turn to the comparison of expected revenues. Let $\varphi(v) \equiv v+b-\frac{1-F(v)}{f(v)}$ denote the virtual valuation of advisor with type v. The next theorem shows that if the virtual valuation is strictly increasing, the ascending-bid auction generates higher expected revenues than the second-price auction:

Theorem 5. *Suppose that $b > 0$, $\varphi(\cdot)$ is strictly increasing, and either Assumption A or B holds. Then the unique equilibrium satisfying NITS in the ascending-price auction brings higher expected revenues than any equilibrium satisfying NITS in the second-price auction. It brings strictly higher expected revenues if $v^* > \underline{v}$, i.e., if there is no babbling in the ascending-bid auction.*

The result of Theorem 2 may seem surprising: It is a priori not clear if the ascending-bid auction should bring higher expected revenue. In the example in Section 3, the bids in the second-price and the ascending-bid auction are not clearly ordered. Moreover, Bergemann and Pesendorfer (2007) study the seller's problem of joint mechanism design (how to sell) and static information design (how much information about her valuation to disclose to the bidder) and show that the optimal information structure is represented by partitions. One crucial difference of our model is that a switch from the second-price auction to the ascending auction results not only in higher efficiency but also in biased bidding: The bidder with any valuation $v < v^*$ stays in the auction even after the price passes its maximum willingness to pay v, because she does not know it yet.[21]

The key idea of Theorem 5 is to view the seller's problem as the problem of selling directly to informed advisors, where communication between advisors and bidders puts a restriction on the set of mechanisms that can be implementable. By the envelope formula in Myerson (1981), we can write the seller's expected revenues as the expected virtual valuation of the winning advisor less the payoff of the lowest type:

$$\mathbb{E}\left[\sum_{i=1}^{N} \varphi\left(v_i\right) p_i\left(\boldsymbol{v}\right)\right] - NU_A\left(\underline{v}\right), \tag{12}$$

where $p_i\left(\boldsymbol{v}\right)$ is the probability that bidder i wins the auction if the types are $\boldsymbol{v} = (v_1, \ldots, v_N)$ and $U_A(\underline{v})$ is the expected payoff of type $\underline{v}$ of the advisor. In (12), the auction format determines $p_i\left(\cdot\right)$ and $U_A\left(\underline{v}\right)$. Higher efficiency of the ascending-price auction together with increasing virtual valuation implies the first term in (12) is higher in the ascending-price auction than in the second-price auction. The NITS condition guarantees that the expected payoff of the lowest type is non-negative in the second-price auction, while it is zero in the ascending-bid auction. Together, these two effects imply that the ascending-bid auction generates higher expected revenues.

While the seller's problem of maximizing over all possible mechanisms goes beyond the scope of the paper, we can say that the ascending-bid auction with an appropriate reserve price is the globally optimal mechanism when it features full separation (e.g., when Assumption B holds). The argument is as follows. We know from Myerson (1981) that if the seller were to sell directly to informed advisers, the ascending-bid auction with a reserve price $r = \varphi^{-1}\left(0\right) + b$ would achieve the highest expected revenues. Since the seller's problem of selling to bidders relying on the advice of informed advisors is a constrained problem of selling to advisors directly, the optimal mechanism in the former cannot generate higher expected revenues than the optimal mechanism in the latter. When there is full separation, the ascending-bid auction in which the seller sells to bidders relying on advisors is identical

[21] Other differences are that the information structure arising in the NITS equilibria of the second-price auction is typically suboptimal for the seller with full flexibility to design information that bidders get and that the second-price auction could be suboptimal with discrete types (see Che and Gale (2006)).

to selling to advisors directly, since in equilibrium bidders behave as if they fully delegate bidding to advisors.[22] We summarize this result in the following corollary:

Corollary 1. *Suppose that $b > 0$, $\varphi(v)$ is strictly increasing, and Assumption B holds. Then, the ascending-price auction with a reserve price $r = \varphi^{-1}(0) + b$ is optimal.*

In the somewhat unnatural case when neither Assumption A nor B are satisfied, the ascending-bid auction can have multiple equilibria satisfying NITS, which complicates the analysis. However, for the case of two bidders, we can generalize Theorems 1 and 2 for any equilibrium choice:

Theorem 6. *Suppose that $b > 0$ and $N = 2$. Then, the pooling region (if it is not empty) in any equilibrium satisfying NITS in the ascending-bid auction is finer than the top partition in any equilibrium of the second-price auction: $v^* \geq \omega_{K-1}$. If, in addition, $\varphi(v)$ is strictly increasing, then any equilibrium satisfying NITS in the ascending-bid auction brings higher expected revenues than any equilibrium satisfying NITS in the second-price auction. Both comparisons are strict if $v^* > \underline{v}$.*

5.3 Role of Magnitude of Advisors' Bias

Giving that the ascending-bid auction is attractive from both efficiency and revenues dimensions, it is interesting to explore how they depend on the magnitude of the advisors' bias. In particular, does the seller benefit from advisors more biased for overpaying? The next proposition sheds some light on this question:

Proposition 4. *Suppose that $b > 0$ and Assumption A holds. Then, in the unique equilibrium satisfying NITS:*

1. *The expected valuation of the winning bidder is decreasing in b.*
2. *The expected revenues are strictly increasing in b in the neighborhood of $b = 0$ and strictly decreasing in b in the neighborhood of $b = MRL(\underline{v})$.*
3. *For any $b > 0$, if $\bar{v} < \infty$, and for any $b > \lim_{v\to\infty} MRL(v)$, if $\bar{v} = \infty$, there exists $N(b)$ such that for all $N > N(b)$, the expected revenues strictly increase with a marginal decrease in b.*

[22] Proposition 3 can be easily modified to allow for a reservation price by simply assuming that the seller starts increasing price from the reservation price.

The first result of the proposition is that efficiency of the auction decreases with the advisors' bias. This is because a higher bias increases the size of the pooling region.

More interestingly, as the second result of the proposition shows, the effect of a bias on revenues is non-monotone. A higher bias has two opposite effects. On one hand, it leads to a more aggressive bidding when the valuation is in the separating region, $v < v^*(b)$, since the advisor recommends to quit the auction at a higher price. On the other hand, a higher bias leads to a less aggressive bidding when the valuation is in the pooling region, $v > v^*(b)$, since the bidder stops listening to the advisor's recommendation earlier. The former effect dominates when the size of the pooling region is small, which is the case when the bias is low, while the latter effect dominates when it is high.[23]

Finally, the last result of Proposition 4 implies that for any bias level, expected revenues decrease in the bias if the auction is sufficiently competitive. Intuitively, if the auction is very competitive, the valuations of the strongest two bidders are very likely to be in the pooling region, which implies that more aggressive bidding by high types is more important than more aggressive bidding by low types. Therefore, a lower bias increases expected revenues in sufficiently competitive auctions. Overall, our results suggest that the seller benefits from a higher bias if the bias is moderate and the auction is not too competitive.

6 Bias toward Underbidding

So far our focus has been on the case in which advisors are biased for overbidding. While this case is arguably more common in applications, there are examples of settings with biases for underbidding, such as the procurement auction example from the introduction.[24] We consider this case in this section. We show that, as in the case of $b > 0$, dynamic auction formats can attain higher efficiency than static auction by exploiting the irreversibility of the running price. However, unlike in the case of $b < 0$, higher efficiency often comes at the cost of lower revenues.

The analysis of static auctions holds regardless of the sign of b. Consider the ascending-bid auction. Since $b < 0$, all else equal, the advisor is now willing to quit the auction earlier than the bidder. Hence, equilibria in which types of the advisor are separated up to a cut-off do not exist if $b < 0$. Indeed, if the advisor with type $v < v^*$ followed the same strategy

[23]For illustration, in the example of Section 3, expected revenues are inverse U-shaped in b, reaching the maximum at $b \approx 3.54$.

[24]This bias can also be relevant in takeover contests if the management of a potential acquirer has the "quiet life" preference (Bertrand and Mullainathan (2003)): merging with the target requires additional private effort from managers of the acquirer.

of recommending that the bidder quits the auction at price $v+b$, the bidder would infer valuation v from the price at which the advisor recommended to quit the auction and would delay quitting until price $v > v + b$.

However, higher efficiency can be achieved by running a descending-price auction. By restricting bidders from submitting bids above the running price it creates a commitment device that a bidder will follow the advisor's recommendation: Since the advisor has a lower maximum willingness to pay by the bidder, when the advisor recommends to stop the auction at the current running price p, the bidder infers that the price is below her optimal stopping point, so she should definitely stop the auction now. Formally, let $MAI(s) \equiv s - \mathbb{E}[v|v \leq s]$ denote the *mean-advantage-over-inferiors* of a distribution. Many well-known distributions have strictly increasing MAI.[25] The next theorem constructs a semi-separating equilibrium of a descending-bid auction, which is similar to equilibria of the ascending-bid auction in the case of $b > 0$:

Theorem 7. *Suppose that $\bar{v} < \infty$, $b \in (-MAI(\bar{v}), 0)$, and $MAI(\cdot)$ is strictly increasing. Let v^* be the unique solution to*

$$\mathbb{E}[v|v \leq v^*] = v^* + b. \tag{13}$$

There exists an equilibrium of the descending-price auction, characterized by $\{\sigma(\cdot), v^*\}$ as follows. The advisor of type $v > v^*$ sends message "stay" until the running price p reaches $\sigma(v) \equiv b + \mathbb{E}[\max\{\hat{v}, v^*\}|\hat{v} \leq v]$ and sends message "stop" then. The advisor of type $v \leq v^*$ sends message "stay" until the running price p reaches $\sigma(v^*)$ and sends message "stop" then. The bidder stops the auction after she receives message "stop" from the bidder or when the running price p reaches $\sigma(v^*)$, whichever happens earlier.

As with the bias for overbidding and the ascending-bid auction, the communication strategies here differ significantly from those in static auctions and involve full separation up to a cut-off. The difference is that here high types separate, while low types pool, which is the opposite from the equilibria in Section 5. Function $\sigma(v)$ for types in the separating region $v > v^*$ is the equilibrium bidding strategy in the auction if bids were submitted directly by advisors. As in the case $b > 0$, different communication implies higher efficiency:

Theorem 8. *Suppose that $\bar{v} < \infty$, $b \in (-MAI(\bar{v}), 0)$, and $MAI(\cdot)$ is strictly increasing. The pooling region $[\underline{v}, v^*]$ in the equilibrium of the descending-price auction from Theorem 7*

[25]In particular, all distributions with a strictly log-concave c.d.f. have increasing $MAI(s)$. See Bagnoli and Bergstrom (2005) for related results.

is contained in the bottom partition in any equilibrium of the second-price auction: $v^* < \omega_1$. Therefore, it is more efficient than any equilibrium of the second-price auction.

While the efficiency comparison of the descending-bid auction with static auctions when $b < 0$ is similar to the comparison of the ascending-bid auction with static auctions when $b > 0$, there is a major difference when it comes to the comparison of expected revenues. Informally, higher efficiency in the descending-bid auction occurs at the cost of bidders bidding less aggressively, since they end up stopping the auction below the prices they would stop it if they knew their valuations. Formally, as in (12), we can break down expected revenues into two parts: the expected virtual valuation of the winning advisor minus the payoff of the lowest type. Because of higher efficiency, the first part is higher in the descending-price auction, provided that the virtual valuation is increasing. However, the payoff of the lowest type of the advisor is higher in the descending-price auction. Indeed, since the pooling region in the descending-price auction is smaller than the bottom partition in the second-price auction, the bidder with the lowest valuation $\underline{v}$ wins with a lower probability and pays a lower price in the descending-price auction than in the second-price auction. Therefore, the descending-price auction can result in lower expected revenues, which was the case in the example in Section 3. The next proposition shows that this revenues ranking holds generally when there are two bidders:

Proposition 5. *Suppose that $b \in (-MAI(\bar{v}), 0)$, $\bar{v} < \infty$, $N = 2$, and $MAI(\cdot)$ is strictly increasing. Then, any equilibrium of the second-price auction satisfying NITS brings strictly higher expected revenues than the equilibrium of the descending-price auction from Theorem 7.*

7 Discussion

In this section, we offer further discussion of the results. First, we assess the quantitative implications of our analysis, applying the model to auctions of companies. Second, we discuss commitment and possible generalizations.

7.1 Quantitative Example: Auctions of Companies

Consider a quantitative application of the model to auctions of companies. Suppose that each bidder i is a firm, consisting of the board and the manager. The board has control over bids but has no information about firm's valuation of the target v_i. The manager knows v_i, but has a bias $b > 0$ for overpayment.

To get a plausible value of b, we use the following argument. Empirical evidence suggests that the compensation of top executives is increasing in the absolute size of the firm. This dependence leads to their bias for overpaying for the target. On the other hand, overpaying for the target results in the destruction of firm value and ultimately in a poor performance of the acquirer's stock price. Since the wealth of top managers is sensitive to their company's stock price, there is a limit to which they are willing to overpay for the target. Bias b is the point at which the positive effect on compensation of higher firm size is exactly offset by the negative effect on compensation due to firm value destruction. Using CEO compensation regressions from Harford and Li (2007) and the characteristics of the typical deal from Betton et al. (2008), we estimate the overpayment bias b to be 9.2% of the value of the target under its current ownership.[26] For example, if the value of target under current ownership is \$1 billion and the value of the target to the acquirer is \$1.4 billion, the maximum willingness to pay the CEO is \$1.492 billion.

For distribution of valuations, we use estimates from Gorbenko and Malenko (2014). We normalize the value of the target under its current management to one. Using data on bids and assuming lognormal distribution, Gorbenko and Malenko (2014) estimate that the valuations of strategic bidders are distributed with parameters $\mu = 0.167$ and $\sigma = 0.258$. We use this distribution, truncated at one, for the distribution of valuations in our numerical example. We assume that there are $N = 4$ bidders.

The results are presented in Table 1. In the ascending-price auction, the unique equilibrium satisfying NITS features full separation. This is because the lognormal distribution has fat tails and b is low enough. In a static (for concreteness, second-price) auction, the most informative equilibrium has three partitions, $[1, 1.11]$, $[1.11, 1.38]$, and $[1.38, \infty]$. The corresponding expected valuations are 1.06, 1.24, and 1.64. The comparison of expected revenues is rather striking. The expected takeover premium is 48% in the ascending-price auction, which is 23% higher than the expected takeover premium in the second-price auction (21%). The comparison of efficiency is less significant: The expected valuation of the winning bidder is 1.65 in the ascending-price auction, but 1.57 in the static auction. As the comparison of expected bidders' payoffs illustrates, an increase in revenues largely occurs because of the more aggressive bidding.

[26]See Online Appendix for the details. Alternatively, one could infer the bias from estimates of private benefits of control (e.g., Dyck and Zingales (2004)).

	Ascending-price	Static	Ratio
Exp. Revenues	1.48	1.21	1.23
Exp. Valuation of Winner	1.65	1.57	1.05
Exp. Payoff of Bidder	0.04	0.09	0.46

Table 1: Expected revenue and efficiency of ascending-price and static auctions.

7.2 Generalization

Like most of the auction theory, we have assumed that bidders are symmetric. However, the equilibrium characterization of the ascending-bid auction can be generalized to the case in which bidders have different distribution of valuations F_i and/or have advisors with different biases $b_i \geq 0$. Indeed, since the argument of Theorem 3 does not rely on symmetry, it is also valid if the distribution of competing bids is exogenous or if it is generated by an equilibrium play of bidders and advisors with different biases and/or distributions of valuations. In particular, if the mean residual lifetime function $MRL_i(\cdot)$ of each distribution F_i is strictly decreasing, all cut-off types v_i^*, if interior, are characterized by $MRL_i(v_i^*) = b_i$. Similarly, the characterization does not rely on bidders and advisors knowing the bias of advisors of other bidders. Second, the analysis of the ascending-bid auction does not change with the introduction of a reserve price, which effectively truncates the distribution of valuations. Finally, since the result that the ascending-bid auction generates higher expected revenues than static auctions holds for any $b > 0$, the seller does not need to know the bias exactly to choose between these mechanisms: it is sufficient to know that it is positive.

This generality of the ascending-bid auction contrasts with equilibria in static auctions, which will be affected in a complicated way by asymmetries among bidders, a reserve price, or bidders having limited knowledge about advisors of competing bidders. For example, if bidders are asymmetric, the partitions of types differ for different bidders and pinned down by a complex system of recursive equations. The simplicity and robustness of the ascending-bid auction is an appealing property, in contrast with complexity of static auctions.

It is worth noting that dynamic auction formats have advantages over static formats when selling to advised buyers only when there are competing bidders. If there is only one buyer and the virtual valuation of the advisor $\varphi(\cdot)$ is strictly increasing, and the bias is not too high, $b < MRL(\varphi^{-1}(0))$, the optimal mechanism is to post a price $\varphi^{-1}(0) + b$. This is an optimal mechanism if the seller sells directly to the advisor. When $b < MRL(\varphi^{-1}(0))$, the buyer follows the advisor's recommendation. Hence, it is also an optimal mechanism if the seller sells to an advised buyer. Intuitively, when there is only one buyer, coarse information is sufficient to implement the optimal allocation, so there is no advantage of using dynamic mechanisms. In contrast, when there are competing buyers, the seller benefits from extracting finer information about their valuations.

7.3 Commitment

We have assumed that the bidder cannot commit to bidding strategies. In the ascending-bid auction with advisors biased for overbidding, bidder behave as if they delegate bidding to advisors with caps on bids. Interestingly, this is also what we would observe if each bidder could commit to bidding strategies. In other words, the irreversibility of the price in the ascending-bid auction gives commitment power to a bidder for free.

Formally, consider the following *auction with contracts.* At the initial date each bidder i simultaneously commits to a contract that maps each report of the advisor of valuation $w_i \in [\underline{v}, \bar{v}]$ to exit price in the ascending-bid auction (or bid in the second-price auction) $\hat{p}_i(w_i)$. Then, advisors communicate their information, and bidders bid in the auction abiding to their contracts. The optimal contract of bidder i maximizes her expected payoff subject to providing the advisor with incentives to report the valuation truthfully, $w_i = v_i$, taking as given contracts of other bidders, $\hat{p}_j(w_j)$, $j \neq i$. A symmetric equilibrium in this game is a contract $p^*(w)$ that satisfies the property that a bidder finds it optimal, given that she expects all other bidders to offer it. The next proposition shows that the equilibrium in the ascending-bid auction, in which bidders rely on cheap talk communication with their biased advisors, is also an equilibrium in the auction with contracts:

Proposition 6. *Suppose that $b > 0$, $\bar{v} < \infty$, Assumption A holds, $f(\cdot)$ is differentiable, and $(\ln f(v))' \geq -\frac{1}{b}$ for all $v \in [\underline{v}, \bar{v}]$. Then, bidding strategies $p^*(w) = b + \min\{w, v^*\}$, where v^* is implicitly defined by $MRL(v^*) = b$, constitute an equilibrium of the ascending-bid or second-price auction with contracts.*

The proof follows from a general analysis of the delegation problem by Amador and Bagwell (2013).[27] Proposition 6 thus illustrates the underlying reason that leads to our results about the ascending-bid auction: The irreversibility of the running price in it gives commitment power to a bidder for free.

[27] Their analysis does not apply directly, as their concavity assumptions on payoff functions are satisfied in our case in the range $p \geq b + v^*$. In the proof, we apply their results to a perturbed version of our problem, which satisfies the concavity assumptions, and then show that the solutions to the original and perturbed problems coincide.

8 Conclusion

The goal of the paper is to understand how to sell assets when potential buyers rely on the advice of biased experts. We show that the revenue equivalence theorem holds in a large class of static auctions, including first-, second-price, and many other familiar formats. However, dynamic auctions, such as ascending- and descending-bid auctions can result in very different outcomes. Our main result is that when advisors have a bias for overbidding, the ascending-bid auction is, quite generally, more efficient and also results in higher revenues than static auctions. This is because by communicating his information later in the game, advisors are able to persuade their bidders to stay in the auction for longer than they would have had they known the same information in advance. In contrast, when advisors have a bias for underbidding, the ascending-bid auction loses this property. In this case, the descending-bid auction results in a higher efficiency. However, it often results in lower expected revenues than static auction, since higher efficiency comes at a cost of less aggressive bidding.

Our analysis points to several directions for future research. First, the analysis of bidder asymmetries, in particular in the biases of their advisors, is relevant in applications and can be fruitful. Second, since our focus is on the comparison of static and dynamic formats, we do not solve for the optimal mechanism, except for the special case of Assumption B. Solving for the optimal mechanism in the general case is thus an avenue for future research. We conjecture that the optimality of ascending-bid auction with an appropriate reserve price generalizes beyond Assumption B.

A Appendix

A.1 Proofs for Section 4

Proof of Theorem 1. Consider a standard static auction $\mathcal{A}$ with continuous payments and an equilibrium in it. Let $m_{\mathcal{A}} : [\underline{v}, \overline{v}] \mapsto M$ be the equilibrium communication strategy, $F_{\theta,\mathcal{A}}$ be the distribution of each bidder's types generated by $m_{\mathcal{A}}$, and $\beta_{\mathcal{A}} : \Theta_{\mathcal{A}} \mapsto \mathbb{R}_+$ be the equilibrium bidding strategy, where $\Theta_{\mathcal{A}}$ is the support of $F_{\theta,\mathcal{A}}$. Let $x(\theta)$ and $t(\theta)$ be type θ's equilibrium expected probability of winning and expected payment, resp.

We first use the results of Che and Gale (2006) to argue that if bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the equilibrium $\beta_{\mathcal{S}}$ in the second-price auction $\mathcal{S}$ implies the same expected probabilities of winning and payments $x(\theta)$ and $t(\theta)$. Since this result follows directly from Che and Gale (2006), we simply outline the argument. Lemma 2 in Che and Gale (2006) shows that a symmetric equilibrium of a standard auction with continuous payments admits an efficient allocation, i.e., for any realization of bidders' types (which in our case are drawn i.i.d. from $F_{\theta,\mathcal{A}}$), a bidder with the highest type wins the auction. This implies that function $x(\theta)$ is the same across such auctions. Proposition 1 in Che and Gale (2006) shows that for standard auctions with continuous payments their conditions (A1) and (A2) hold. Condition (A1) implies that their inequality (3) holds as equality. This in conjunction with the envelope condition for the bidder's payoff (their eq. (4)) and condition (A2) implies that function $t(\theta)$ is the same across standard auctions with continuous payments.

We next show that the communication strategy $m_{\mathcal{A}}$ is also an equilibrium communication strategy in the second-price auction. By contradiction, suppose that is is not. Then, there exists value v, such that the advisor is better off sending message m' instead of $m_{\mathcal{A}}(v)$. Let θ' denote the type generated by message m'. Since $x(\theta)$ and $t(\theta)$ are the same in both auctions, it must be that $(v+b)\,x(\theta') - t(\theta') > (v+b)\,x(\theta) - t(\theta)$, where θ' and θ denote the types generated by messages m' and $m_{\mathcal{A}}(v)$, respectively. However, this implies that the advisor must also be better off sending message m' instead of $m_{\mathcal{A}}(v)$ in auction $\mathcal{A}$. Hence, $m_{\mathcal{A}}$ is not an equilibrium communication strategy in auction $\mathcal{A}$, which is a contradiction. Hence, $m_{\mathcal{A}}$ is also an equilibrium communication strategy in the second-price auction.

Thus, we have constructed an equilibrium in the second-price auction with the same communication strategy $m_{\mathcal{A}}$ as in $\mathcal{A}$. Moreover, we have shown that given that bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the two auctions exhibit payoff equivalence (functions $x(\theta)$ and $t(\theta)$ are the same) and thus, yield the same expected revenues. Moreover, the two auctions allocate the asset to the bidder with the highest type θ.

□

Derivation of Λ_k. Define Λ_k as the probability of a bidder with bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$ winning a tie, conditional on the tie taking place at bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. Without loss of generality, we refer to this bidder as bidder N and to her rivals as bidders $i \in \{1, 2, ..., N-1\}$. Since ties are broken randomly,

$$\Lambda_k = \mathbb{E}\left[\frac{1}{\tilde{n}_k + 1} | \hat{v} \in [\omega_{k-1}, \omega_k)\right],$$

where $\tilde{n}_k = \sum_{i=1}^{N-1} 1\{v_i \in [\omega_{k-1}, \omega_k)\}$ is a random variable, denoting the number of rival bidders with the same bid m_k. Re-writing,

$$\begin{aligned}\Lambda_k &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr\left[\tilde{n}_k = n, \hat{v} \in [\omega_{k-1}, \omega_k)\right]}{G(\omega_{k-1}, \omega_k)} \\ &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr\left[\tilde{n}_k = n, \sum_{i=1}^{N-1} 1\{v_i < \omega_{k-1}\} = N-1-n\right]}{G(\omega_{k-1}, \omega_k)} \\ &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n}}{G(\omega_{k-1}, \omega_k)},\end{aligned}$$

which coincides with expression (7) in Theorem 2.

Proof of Theorem 2. Denote by $q(\tilde{\beta})$ and $t(\tilde{\beta})$ the expected probability of winning and the expected payment, respectively, for a bidder submitting bid $\tilde{\beta}$ in the second-price auction given that other players follow equilibrium strategies m and β. Let $Q = \{q(\tilde{\beta}), \tilde{\beta} \in \mathbb{R}_+\}$ and $t(q) = \min_{\tilde{\beta} \in \mathbb{R}_+ : q = q(\tilde{\beta})} t(\tilde{\beta})$. Then the bidder and the advisor play the cheap-talk game with the bidder's action $q \in Q$ and payoffs

$$\text{Bidder} \quad : \quad qv - t(q), \tag{14}$$

$$\text{Advisor} \quad : \quad q(v+b) - t(q). \tag{15}$$

We first show that the equilibrium has an interval partition form. Suppose there is $\tilde{Q} \subseteq Q$ such that $\min \tilde{Q} < \max \tilde{Q}$ and $\tilde{Q}$ is dense in $\overline{Q} = [\min \tilde{Q}, \max \tilde{Q}]$. Then there is an interval of advisor's types $\tilde{V}$ that fully separate and induce all q in $\tilde{Q}$. Let $q', q \in \tilde{Q}$ and $v', v \in \tilde{V}$ be such that in equilibrium, v chooses q, and v' chooses q'. Then $v' \geq \frac{t(q')-t(q)}{q'-q} \geq v$ when $v' > v$ and $v' \leq \frac{t(q)-t(q')}{q-q'} \leq v$ when $v' < v$. By letting v' converge to v, we get that for all $v \in \tilde{V}$, $t'(q(v)) = v$ where $q(v)$ is the equilibrium action chosen by bidder with value v. For $v \in \tilde{V}$, the advisor's marginal utility at $q(v)$ is $v + b - t'(q(v)) > 0$ and so, the advisor prefers to induce a higher action than $q(v)$ which is a contradiction to the sequential rationality of the communication strategy. Therefore, there is no $\tilde{Q} \subseteq Q$ such that $\min \tilde{Q} < \max \tilde{Q}$ and $\tilde{Q}$ is dense in $\overline{Q} = [\min \tilde{Q}, \max \tilde{Q}]$ and so, to characterize equilibria of the second-price auction, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k. Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bider. From submitting a message m_k, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m} | \hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k),$$

where the expected utility from bidding m_k when the other bidders submit bids below m_k and when some bidders tie with the bidder is captured by the first and second terms, resp. Here, Λ_k

gives the expected number of other bidders who tie on m_k and its expression is provided in the statement of the theorem. Analogously, from submitting a message m_{k+1}, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}).$$

Type ω_k should be indifferent between the two which gives the eq. (6). Thus, any PBE communication strategy can be described by a solution $(\omega_k)_{k=0}^K$ to recursion (6) where $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$. We next show that K is bounded from above by some $\bar{K} < \infty$.

Claim 1. *If $\omega_{k+1} = \omega_k$, then $k = 0$ when $b > 0$ and $k = K$ when $b < 0$.*

Proof: Suppose by contradiction that $b > 0$ and $\omega_{k+1} = \omega_k$ for some $0 < k \leq K$ (the argument for $b < 0$ and $0 \leq k < K$ is symmetric). This implies that $G(\omega_k, \omega_{k+1}) = 0$ and so, from (6), $G(\omega_{k-1}, \omega_k)(1 - \Lambda_{k-1})(\omega_k + b - m_k) = 0$ which in turn implies that $\omega_k + b = m_k$ or $\omega_{k-1} = \omega_k$. If $\omega_{k-1} < \omega_k$, then $m_k < \omega_k < \omega_k + b$ which is a contradiction. If $\omega_{k-1} = \omega_k$, then choose j so that $\omega_{k-j-1} < \omega_{k-j} = \cdots = \omega_{k-1} = \omega_k = \omega_{k+1}$ and the argument proceeds as in the case $\omega_{k-1} < \omega_k$. *q.e.d.*

Claim 2. *If $b > 0$, then there exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ and $0 < k \leq K$ or $\omega_{k+1} = \omega_k$ and $k = 0$. If $b < 0$, then there exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ and $0 \leq k < K$ or $\omega_{k+1} = \omega_k$ and $k = K$.*

Proof: Again we prove the claim for $b > 0$ and the similar argument applies for $b < 0$. Suppose $\omega_{k+1} - \omega_k > 0$ and so, $0 < k \leq K$ by Claim 1. Since $\omega_k + b - m_k > \omega_k + b - m_{k+1}$, it follows from (6) and $\omega_{k+1} - \omega_k > 0$ that

$$\omega_k + b \leq \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{16}$$

If to contradiction for any $\varepsilon > 0$, there were an equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq \omega_k + \varepsilon < \omega_k + b$ which would contradict (16) for $\varepsilon < b$. Thus, there is $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k \leq K$ *q.e.d.*

Claims 1 and 2 imply that there is an upper bound $\bar{K}$ on the number of partitions in the communication strategy.

□

Proof of Proposition 1. The proof is an adaptation of the proof of Proposition 1 from Chen et al. (2008) for our problem. It is useful to introduce the following notations:

$$\begin{aligned}
\Psi(\omega_{k-1}, \omega_k) = \quad & G(\omega_{k-1}, \omega_k)(1 - \Lambda_k) = \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1}, \\
\Phi(\omega_k, \omega_{k+1}) = \quad & G(\omega_k, \omega_{k+1})\Lambda_{k+1} = \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_k, \omega_{k+1})^n F(\omega_k)^{N-1-n} \frac{1}{n+1}.
\end{aligned}$$

Denote $m(\omega_{k-1},\omega_k)=\mathbb{E}[v|v\in(\omega_{k-1},\omega_k)]$ and

$$H(\omega_{k-1},\omega_k,\omega_{k+1})\equiv\Psi(\omega_{k-1},\omega_k)(\omega_k+b-m(\omega_{k-1},\omega_k))+\Psi(\omega_k,\omega_{k+1})(\omega_k+b-m(\omega_k,\omega_{k+1})). \quad (17)$$

Note that an equilibrium with K partitions is given by recursion $H(\omega_{k-1},\omega_k,\omega_{k+1})=0$, with $\omega_0=\underline{v}$ and $\omega_K=\overline{v}$. To prove the statement, we will show that if an equilibrium with K partitions, $\omega=(\omega_0,\omega_1,...,\omega_K)$, violates the NITS condition, then for all $k=1,...,K$, there exists a solution to recursion $H(\omega_{k-1},\omega_k,\omega_{k+1})=0$, with $k+1$ partitions, ω^k, that satisfies $\omega_0^k=\underline{v}$, $\omega_k^k>\omega_{k-1}$, and $\omega_{k+1}^k=\omega_k$. After this result is established, the statement of the proposition follows from the following argument. By contradiction, suppose that the most informative equilibrium (i.e., one with $\bar{K}$ partitions) violates the NITS condition. Applying the result above for $k=\bar{K}$, there must be a solution to $H(\omega_{k-1},\omega_k,\omega_{k+1})=0$ with $\bar{K}+1$ partitions satisfying boundary conditions $\omega_0^{\bar{K}}=\underline{v}$ and $\omega_{\bar{K}+1}^{\bar{K}}=\omega_{\bar{K}}=\bar{v}$. By Theorem 2, this is an equilibrium, which contradicts the statement that $\bar{K}$ is the highest number of equilibrium partitions.

We show the result by induction on k. For brevity, we consider only the case $b>0$ here (case $b<0$ is analogous). As an induction base, consider $k=1$. If the equilibrium with K partitions $\omega=(\omega_0,\omega_1,...,\omega_K)$ violates the NITS, it must be that $\underline{v}+b<m(\underline{v},\omega_1)$, and hence, $H(\underline{v},\underline{v},\omega_1)<0$. At the same time, $H(\underline{v},\omega_1,\omega_1)>0$, since $\omega_1>m(\underline{v},\omega_1)$ and $b>0$. By continuity, there exists $x\in(\underline{v},\omega_1)$ at which $H(\underline{v},x,\omega_1)=0$. Hence, the claim holds for $k=1$: $\omega^1=\left(\omega_0^1,\omega_1^1,\omega_2^1\right)=(\underline{v},x,\omega_1)$ solves $H(\omega_{k-1},\omega_k,\omega_{k+1})=0$ with $\omega_0^1=\underline{v}$, $\omega_1^1>\omega_0=\underline{v}$, and $\omega_2^1=\omega_1$. Consider the difference $H\left(\omega_k^k,\omega_k,\omega_{k+1}\right)-H(\omega_{k-1},\omega_k,\omega_{k+1})$:

$$\begin{aligned}
&\Psi\left(\omega_k^k,\omega_k\right)\left(\omega_k+b-m\left(\omega_k^k,\omega_k\right)\right)-\Psi(\omega_{k-1},\omega_k)(\omega_k+b-m(\omega_{k-1},\omega_k))\\
=&F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F\left(\omega_k^k,\omega_k\right)}{F(\omega_k)}\right)^n\left(\frac{F\left(\omega_k^k\right)}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)\left(\omega_k+b-m\left(\omega_k^k,\omega_k\right)\right)\\
&-F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F(\omega_{k-1},\omega_k)}{F(\omega_k)}\right)^n\left(\frac{F(\omega_{k-1})}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)(\omega_k+b-m(\omega_{k-1},\omega_k)).
\end{aligned}$$

Since $\omega_k^k>\omega_{k-1}$, we have two implications. First, $m\left(\omega_k^k,\omega_k\right)>m(\omega_{k-1},\omega_k)$, implying $\omega_k+b-m\left(\omega_k^k,\omega_k\right)<\omega_k+b-m(\omega_{k-1},\omega_k)$. Second, binomial distribution with success probability $\frac{F(\omega_{k-1},\omega_k)}{F(\omega_k)}$ dominates binomial distribution with success probability $\frac{F\left(\omega_k^k,\omega_k\right)}{F(\omega_k)}$ in the sense of first-order stochastic dominance, implying $\Psi\left(\omega_k^k,\omega_k\right)<\Psi(\omega_{k-1},\omega_k)$. Therefore, $H\left(\omega_k^k,\omega_k,\omega_{k+1}\right)-H(\omega_{k-1},\omega_k,\omega_{k+1})<0$. Since $H(\omega_{k-1},\omega_k,\omega_{k+1})=0$, we conclude that $H\left(\omega_k^k,\omega_k,\omega_{k+1}\right)<0$.

On the other hand, since $\omega_k>m\left(\omega_k^k,\omega_k\right)$ and $b>0$, we have $\omega_k+b>m\left(\omega_k^k,\omega_k\right)$, implying $H\left(\omega_k^k,\omega_k,\omega_k\right)>0$. This, $H\left(\omega_k^k,\omega_k,\omega_{k+1}\right)<0$, and continuity imply that there exists $x\in(\omega_k,\omega_{k+1})$ at which $H\left(\omega_k^k,\omega_k,x\right)=0$. Since $\omega_{k+1}^k=\omega_k$, the same x satisfies $H\left(\omega_k^k,\omega_{k+1}^k,x\right)=0$. That is, there exists a solution in which the $(k+1)$st partition ends at $\omega_{k+1}^k=\omega_k$ and the $(k+2)$nd

Figure 3: Illustration for the proof of Theorem 1.

partition ends at $x < \omega_{k+1}$. By continuity, there exists a solution to the recursion in which the $(k+2)$nd partition ends at any $\omega \in \left(\omega_{k+1}^k, \omega_{k+1}\right)$. By continuity, for one such ω, denoted ω_{k+1}^{k+1}, the $(k+2)$nd partition ends exactly at ω_{k+1}, i.e., $\omega_{k+2}^{k+1} = \omega_{k+1}$. Hence, there exists a solution to recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $k+2$ partitions, ω^{k+1}, that satisfies $\omega_0^{k+1} = \underline{v}$, $\omega_{k+1}^{k+1} > \omega_k$, and $\omega_{k+2}^{k+1} = \omega_{k+1}$. This completes the proof of the inductive step.

$\square$

A.2 Proofs for Section 5

See Online Appendix for the proof of Lemma 1 . Here we prove all other statements.

Proof of Theorem 3. By Lemma 1 (proven in Online Appendix), it is without loss of generality to focus on equilibria in online threshold strategies. In the proof of Lemma 1, we introduced function $\bar{\tau}(v)$, which denotes the equilibrium exit price of the bidder if the advisor's type is v. In an equilibrium in online threshold strategies, $\bar{\tau}(v)$ is also the first price at which the advisor with type v sends message "quit" to the bidder.

Any equilibrium generates partition Π of $[\underline{v}, \overline{v}]$ satisfying $\bar{\tau}(v) = \bar{\tau}(v')$ for any $v, v' \in \pi$ for any element $\pi \in \Pi$. As shown in Lemma 1, $\bar{\tau}(v)$ is weakly increasing, so any $\pi \in \Pi$ is an interval (possibly consisting of one element). We say that types in $\pi \in \Pi$ *pool* if $\bar{\tau}(v)$ is constant on $v \in \pi$, i.e., these types start sending message "quit" at the same price. We say that types in $[v', v'']$ *separate*, if $\bar{\tau}(v)$ is strictly increasing on $[v', v'']$, i.e., these types start sending message "quit" at different prices. Let Π^P and $\Pi^S \equiv [\underline{v}, \overline{v}] \backslash \Pi^P$ be the sets of all types that pool with some other type and that separate, respectively. Denote by $\partial \Pi^P$ the boundary of Π^P.

Babbling ($\bar{\tau}(v) = \mathbb{E}[v]\ \forall v$) is an equilibrium of the English auction, and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. This proves case $v^* = \underline{v}$ of the theorem. Hence, we can consider the case in which there is a non-trivial information transmission in equilibrium.

Claim 3. *For any $\pi, \pi' \in \Pi^P$, π and π' are not adjacent.*

Proof: By contradiction, suppose that there are two adjacent intervals of types, π and π', such that $\bar{\tau}(v) = p\ \forall v \in \pi$ and $\bar{\tau}(v) = p'\ \forall v \in \pi'$. Without loss of generality, $p' > p$. Consider the advisor with type $\tilde{v}$ on the boundary of π and π'. By continuity, the advisor with type $\tilde{v}$ is indifferent between his bidder quitting the auction at prices p and p'. The benefit of the latter is winning against types in π, while the cost is risking to win against types in π' and paying p'. The indifference of type $\tilde{v}$ implies that $p' > \tilde{v} + b$. Consider running price $\frac{p'+p}{2}$. Type $\tilde{v}$ is the weakest remaining type at this price. Since $p' > \tilde{v} + b$, following his equilibrium strategy of waiting to send recommendation $m = 1$ until price p' generates negative expected payoff to the advisor at this point. In contrast, claiming that he is the weakest remaining type at the current price of $\frac{p'+p}{2}$ will lead to the bidder quitting immediately, yielding the payoff of zero to the advisor. This contradicts the NITS condition. *q.e.d.*

We next show that whenever types within an interval separate, they start recommending to quit the auction at their most preferred time.

Claim 4. *If $\bar{\tau}(v)$ is strictly increasing on (v', v''), then $\bar{\tau}(v) = v + b$ for any $v \in (v', v'')$.*

Proof: By contradiction, suppose there is $v \in (v', v'')$ with $\bar{\tau}(v) \neq v+b$. Then, either $\bar{\tau}(v) > v+b$ or $\bar{\tau}(v) < v + b$. First, consider the former case. Since v is interior, there exists a subset of (v', v'') of types $v + \varepsilon > v$ with positive measure with $\bar{\tau}(v) > v + \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v + \varepsilon) > \bar{\tau}(v) > v + \varepsilon + b$. Therefore, any such type $v + \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v + \varepsilon)$: by doing this, the advisor ensures that the bidder does not win when the valuation of the strongest rival is in $(v, v + \varepsilon)$, in which case the bidder overpays relative to the advisor's maximum willingness to pay of $v + \varepsilon + b$. Hence, it cannot be that $\bar{\tau}(v) > v + b$. Second, consider the case $\bar{\tau}(v) < v + b$. Now, there exists a subset of (v', v'') of types $v - \varepsilon < v$ with positive measure with $\bar{\tau}(v) < v - \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v - \varepsilon) < \bar{\tau}(v) < v - \varepsilon + b$. Therefore, any such type $v - \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v - \varepsilon)$: by doing this, the advisor ensures that the bidder wins when the valuation of the strongest rival is in $(v - \varepsilon, v)$, in which case the advisor gets a positive payoff, since the bidder pays below the advisor's maximum willingness to pay. Therefore, it cannot be that $\bar{\tau}(v) < v + b$. We conclude that $\bar{\tau}(v) = v + b$. *q.e.d.*

Claim 5. *If $\Pi^P \neq \emptyset$ and $\Pi^S \neq \emptyset$, then Π^P contains a single interval $\pi^P = [v^*, \overline{v}]$, where $v^* > \underline{v}$.*

Proof: By contradiction, suppose that Π^P contains more than one interval or that it contains one interval that is to the left of Π^S. In the former case, Claim 3 implies that the intervals are not adjacent. Therefore, there is an interval $\pi \in \Pi^P$ that lies to the left of an interval in Π^S. Let $v \in \pi$ be the highest type in this partition. Since it must be indifferent between separation and pooling and $\bar{\tau}(v) = v + b$ in the separation region by Claim 2, we have $v + b = \bar{\tau}(w)$ for any $w \in \pi$. Therefore, $\bar{\tau}(w) > w + b$ for any $w \in \pi$, $w \neq v$. In particular, it holds for the lowest type in the partition, $w' = \min_{w \in \pi} w$. However, this violates the NITS condition. Indeed, consider running price $p = \frac{\bar{\tau}(w)+w'+b}{2}$. Type w' is the weakest remaining type of the advisor at this price. Since $p > w' + b$, following his equilibrium strategy of waiting until price $\bar{\tau}(w)$, $w \in \pi$ to send recommendation $m = 1$ generates negative expected payoff to the advisor at the current point. In contrast, claiming that he is the weakest remaining type at the current price will lead to the bidder quitting immediately, yielding the payoff of zero. Therefore, Π^P contains only one interval that lies to the right of Π^S, i.e., the interval is of the form $\pi^P = [v^*, \overline{v}]$ for some $v^* > \underline{v}$. *q.e.d.*

Claim 6. *Cut-off* v^* *satisfies* (10).

Proof: Case $v^* = \underline{v}$ (the babbling equilibrium) was covered above (before Claim 3). Consider case $v^* = \overline{v}$, i.e., $\Pi^P = \emptyset$. By Claim 4, $\bar{\tau}(v) = v + b$ for any $v \in (\underline{v}, \bar{v})$. If $\bar{v} < \infty$, the upper bound on the bidder's utility in round $p = v + b$ is $\overline{v} - (v + b) \overset{v \to \overline{v}}{\to} -b$, which contradicts the optimality of the bidder to follow the advisor's recommendation. Hence, it must be that $\bar{v} = \infty$. Next, by contradiction, suppose that $b > \lim_{s \to \infty} \mathbb{E}[v|v \geq s] - s$. By continuity, there is $\overline{s} < \infty$ such that $b > \mathbb{E}[v|v \geq s] - s$ for any $s > \overline{s}$. If the bidder wins in any round $p \geq s + b$, then her expected utility equals $\mathbb{E}[v|v \geq s] - s - b < 0$ and so, the value of following the advisor's recommendations is negative, which is a contradiction. Therefore, it must be that $b \leq \lim_{s \to \infty} \mathbb{E}[v|v \geq s] - s$.

Finally, consider case $v^* \in (\underline{v}, \bar{v})$. By contradiction, suppose that $v^* + b \neq \mathbb{E}[v|v \geq v^*]$. By indifference of type v^*, it must be that $\bar{\tau}(v) = v^* + b$ for any $v \in [v^*, \bar{v}]$. If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v)$, $v > v^*$ violates the incentive compatibility condition of the bidder. To see this, consider running price just below $v^* + b$. The equilibrium behavior prescribes the bidder to exit the auction in the next instant, which is below her maximum willingness to pay of $\mathbb{E}[v|v \geq v^*]$. By waiting a little beyond price $\bar{\tau}(v) = v^* + b$, the bidder ensures that she wins the auction with probability one and pays below her estimated valuation of $\mathbb{E}[v|v \geq v^*]$. Since this strategy results in a discontinuous upward jump in the expected utility of the bidder, she is better off deviating. Hence, it cannot be that $v^* + b < \mathbb{E}[v|v \geq v^*]$. If $v^* + b > \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v) = v^* + b$, $v \geq v^*$ violates the incentive compatibility condition of the bidder, because she would prefer to exit the auction slightly earlier. Consider the running price $p = v^* + b - \varepsilon$ for an infinitesimal positive ε and suppose that the bidder has got a sequence of recommendations $\tilde{m} = 0$. Her posterior belief is that the valuation is in the range $(v^* - \varepsilon, \bar{v}]$. Suppose that the bidder follows her equilibrium play. If $v \in (v^* - \varepsilon, v^*)$ and the bidder wins, she pays $v + b$ above her valuation v. If $v \in (v^*, \bar{v}]$ and the bidder wins, she pays $\bar{\tau}(v) = v^* + b$, which is, on average, above her valuation v ($\mathbb{E}[v|v \geq v^*]$). Since the bidder wins with positive probability, her expected payoff from following the equilibrium play is negative. In contrast, immediate exit yields zero expected payoff. Hence, the bidder is better off deviating and exiting the auction immediately. Therefore, $v^* + b = \mathbb{E}[v|v \geq v^*]$. *q.e.d.*

The proof of Theorem 3 follows from Claims 4, 5, and 6. □

Proof of Proposition 2. Since $MRL(\cdot)$ is strictly decreasing, equation $MRL(v) = b$ has at most one solution. First, consider case $MRL(\underline{v}) > b$. If $\bar{v} < \infty$, then $MRL(\bar{v}) = 0$. Since $MRL(\cdot)$ is strictly decreasing (and continuous, since the distribution has full support on $[\underline{v}, \bar{v}]$), there exists a unique solution $v \in (\underline{v}, \bar{v})$ to $MRL(v) = b$. Similarly, if $\bar{v} = \infty$, then, since $MRL(\cdot)$ is strictly decreasing and $\lim_{v \to \infty} MRL(v) < b$, there exists a unique solution $v \in (\underline{v}, \bar{v})$ to $MRL(v) = b$. Denote it by v^*. Since $MRL(\cdot)$ is strictly decreasing and $MRL(v^*) = b$, $s + b - \mathbb{E}[v|v \geq s]$ for any $s < v^*$. Therefore, the bidder's option value of waiting, $V(p)$ in (11), is strictly positive for all $p < v^* + b$. Hence, this cut-off indeed corresponds to an equilibrium. By Theorem 3, it cannot be that $v^* = \underline{v}$, since $MRL(\underline{v}) > b$, and $v^* = \bar{v}$, since either $\bar{v} < \infty$ or $\bar{v} = \infty$ and $b > \lim_{v\to\infty} MRL(v)$. Hence, the equilibrium is unique.

Second, consider case $MRL(\underline{v}) \leq b$. Since $MRL(\cdot)$ is strictly decreasing, $MRL(v) < b$ for any $v > \underline{v}$. Hence, by Theorem 3 the only candidate for equilibrium is $v^* = \underline{v}$, i.e., babbling. Clearly, it is an equilibrium. To see that it satisfies the NITS condition, note that the implied expected payoff to type $\underline{v}$ of the advisor is $\frac{1}{N}(b - MRL(\underline{v})) \geq 0$. Therefore, the NITS condition is satisfied. Hence, babbling is indeed the unique equilibrium in this case. □

Proof of Proposition 3. Since $MRL(v) > b$ for all $v \in [\underline{v}, \infty)$, by Theorem 3 the only candidate for equilibrium is $v^* = \infty$. It follows from (11) that whenever $MRL(v) > b$ for all $v \in [\underline{v}, \overline{v}]$, the bidder's option value of waiting for the advisor's recommendation is always positive. Hence, $v^* = \infty$ indeed corresponds to an equilibrium. Moreover, it satisfies the NITS condition, since any type of the advisor gets his unconstrained optimal bidding strategy. □

Proof of Theorem 4. Under Assumption A, the number of partitions K must be finite. Since ω_{K-1} satisfies eq. (6), $\omega_{K-1} + b - \mathbb{E}[v|v \geq \omega_{K-1}] < 0$ or $b < MRL(\omega_{K-1})$. On the other hand, $b = MRL(v^*)$. Since $MRL(\cdot)$ is strictly decreasing, $v^* > \omega_{K-1}$. Under Assumption B, the unique equilibrium under NITS of the English auction is fully separating ($v^* = \infty$), while any equilibrium in the second-price auction has a partition structure (by the same logic as in the proof of Theorem 2 for $\overline{v} < \infty$). □

Proof of Theorem 5 *and Corollary* 1. See the argument after the theorem in the main text. □

Proof of Theorem 6. We prove the first statement by contradiction. Suppose there exists an equilibrium satisfying NITS in the ascending-bid auction and an equilibrium in the second-price auction satisfying $v^* < \omega_{K-1}$. Then, there exists partition (ω_{k-1}, ω_k) such that $\omega_{k-1} \leq v^* < \omega_k$. Consider the indifference condition of type ω_k in the second-price auction. When $N = 2$, , $G(\omega_k, \omega_{k+1}) = F(\omega_k, \omega_{k+1})$, and $\Lambda_k = \Lambda_{k+1} = \frac{1}{2}$. Therefore, (6) can be simplified to

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} m_{k+1} = \mathbb{E}[v|v \in (\omega_{k-1}, \omega_{k+1})]. \tag{18}$$

Since $\omega_{k+1} \leq \bar{v}$ and $\omega_{k-1} \leq v^*$, the right-hand side of (18) satisfies:

$$\mathbb{E}[v|v \in (\omega_{k-1}, \omega_{k+1})] \leq \mathbb{E}[v|v \geq \omega_{k-1}] \leq \mathbb{E}[v|v \geq v^*].$$

On the other hand, since $v^* < \omega_k$, the left-hand side of (18) satisfies $\omega_k + b > v^* + b$. Therefore, v^* satisfies

$$v^* + b < \mathbb{E}[v|v \geq v^*],$$

which contradicts Theorem 3. Therefore, $v^* \geq \omega_{K-1}$. If $v^* > \underline{v}$, then $v^* > \omega_{K-1}$, since $v^* = \omega_{K-1}$ cannot be by contradiction. Indeed, in this case, eq. (18) implies $v^* + b = \mathbb{E}\left[v | v \geq \omega_{K-2}\right] < \mathbb{E}\left[v | v \geq v^*\right]$, which contradicts Theorem 3. The second statement of the theorem follows from the same argument as Theorem 5, since it relies only on higher efficiency of an equilibrium of the ascending-bid auction than an equilibrium of the second-price auction and on the NITS condition. □

Proof of Proposition 4. The first statement follows directly from eq. (10). In the range $b \geq MRL\left(\underline{v}\right)$, the unique equilibrium has $v^* = \underline{v}$, so the expected valuation of the winning bidder is $\mathbb{E}\left[v\right]$ and does not depend on b. In the range $b \in \left(\lim_{v \to \bar{v}} MRL\left(v\right), MRL\left(\underline{v}\right)\right)$, the unique equilibrium has $v^*\left(b\right) = MRL^{-1}\left(b\right) \in \left(\underline{v}, \bar{v}\right)$. Since $MRL\left(v\right)$ is strictly decreasing, so the expected valuation of the winning bidder is strictly decreasing in b. Finally, in the range $b \leq \lim_{v \to \bar{v}} MRL\left(v\right)$,[28] the unique equilibrium has $v^* = \infty$, so the expected valuation of the winning bidder does not depend on b.

Consider the second statement. Consider $b > 0$ in the neighborhood of $b = 0$. If $\bar{v} = \infty$ and $\lim_{v \to \infty} MRL\left(v\right) > 0$, we have $v^*\left(b\right) = \infty$, so the expected revenues are $b + \int_{\underline{v}}^{\infty} v dH\left(v\right)$, where $H\left(\cdot\right)$ is the c.d.f. of the second-order statistic of N i.i.d. random variables with c.d.f. $F\left(\cdot\right)$. Therefore, the expected revenues are strictly increasing in b in this case. If $\bar{v} < \infty$ or $\bar{v} = \infty$ and $\lim_{v \to \infty} MRL\left(v\right) = 0$, $v^*\left(b\right) \in \left(\underline{v}, \bar{v}\right)$, so the expected revenues can be written as

$$b + \int_{\underline{v}}^{v^*(b)} v dH\left(v\right) + \left(1 - H\left(v^*\left(b\right)\right)\right) v^*\left(b\right). \tag{19}$$

The derivative of (19) with respect to b equals $1 + \left(1 - H\left(v^*\left(b\right)\right)\right) \frac{dv^*}{db}$. Applying the implicit function theorem to $MRL\left(v^*\left(b\right)\right) = b$ yields

$$MRL'\left(v^*\left(b\right)\right) = \frac{f\left(v^*\left(b\right)\right)}{1 - F\left(v^*\left(b\right)\right)} MRL\left(v^*\left(b\right)\right) - 1.$$

Therefore, $\frac{dv^*}{db} = -\left(1 - b\frac{f(v^*(b))}{1-F(v^*(b))}\right)^{-1}$, which is negative by Assumption A. Hence, the derivative of (19) with respect to b is

$$\frac{H\left(v^*\left(b\right)\right) - b\frac{f(v^*(b))}{1-F(v^*(b))}}{1 - b\frac{f(v^*(b))}{1-F(v^*(b))}}. \tag{20}$$

When $b \to 0$, $v^*\left(b\right) \to \bar{v}$. Hence, the derivative in the limit equals one. Therefore, the expected revenues are strictly increasing in b in the neighborhood of $b = 0$. When $b \to MRL\left(\underline{v}\right)$, $v^*\left(b\right) \to \bar{v}$. Therefore, (20) converges to $-MRL\left(\underline{v}\right) f\left(\underline{v}\right) / \left(1 - MRL\left(\underline{v}\right) f\left(\underline{v}\right)\right)$, which is negative. Therefore, (20) is negative for a sufficiently high b, implying that the expected revenues are strictly decreasing in b in the neighborhood of $b = MRL\left(\underline{v}\right)$.

[28]Note that this implies $\bar{v} = \infty$, since $MRL\left(\bar{v}\right) = 0$ if $\bar{v} < \infty$.

Finally, consider the third statement. Notice that for any $v < \bar{v}$, $\lim_{N\to\infty} H(v) = 0$. Indeed, by definition of $H(\cdot)$, $H(v) = NF(v)^{N-1} - (N-1)F(v)^N$. Therefore,

$$\begin{aligned}\lim_{N\to\infty} H(v) &= \lim_{N\to\infty}\left((N-1)F(v)^N\right) \times \left(\lim_{N\to\infty} \frac{NF(v)^{N-1}}{(N-1)F(v)^N} - 1\right) \\ &= \frac{\lim_{N\to\infty} F(v)^N}{-\ln F(v)} \times \left(\frac{1}{F(v)} - 1\right) = 0\end{aligned}$$

for any $v < \bar{v}$, where we used l'Hospital's rule. Also, notice that for any $b > 0$, the cut-off type $v^*(b)$ does not depend on N. Therefore, for any $b > 0$, there exists $N(b)$ such that $H(v^*(b)) - \frac{f(v^*(b))}{1-F(v^*(b))}b < 0$ for all $N > N(b)$. Therefore, for any $b > 0$, the derivative of expected revenues in b, (20), is negative for any $N > N(b)$. □

A.3 Proofs for Section 6

Proof of Theorem 7. Observe that $MAI(\bar{v}) > -b$, $MAI(\underline{v}) = 0 < -b$, and $MAI(v)$ is strictly increasing. Therefore, eq. (13) has a unique solution $v^* \in (\underline{v}, \bar{v})$. To prove that these strategies form an equilibrium, we need to show that the advisor sends message "stop" at the optimal time given the strategy of the bidder to follow the advisor's recommendation until price $\sigma(v^*)$ and that the bidder finds it optimal to follow the advisor's recommendation until price $\sigma(v^*)$.

First, consider the advisor's problem:

$$\max_x (v + b - x)G(\sigma^{-1}(x)). \tag{21}$$

Taking the first-order condition and substituting the equilibrium condition that the maximum is reached at $x = \sigma(v)$, we obtain differential equation

$$g(v)(v + b) = (G(v)\sigma(v))'. \tag{22}$$

Solving (22) with the initial condition $\sigma(v^*) = v^* + b$ yields

$$\sigma(v) = \frac{G(v^*)}{G(v)}(v^* + b) + \frac{1}{G(v)}\int_{v^*}^{v} g(\hat{v})(\hat{v} + b)\, d\hat{v}. \tag{23}$$

It that $\sigma(v) = b + \mathbb{E}[\max\{v^*, \hat{v}\} | \hat{v} \le v]$. The maximized function in (21) has the single-crossing property in (v, σ). Hence, since type v^* prefers to "stop" at price $\sigma(v^*)$, no type $v < v^*$ is better off inducing the bidder to stop at a higher price. Thus, the strategy from the theorem is optimal for the advisor, given that he expects the bidder to follow the strategy from the theorem.

Second, consider the bidder's problem. Because of the single-crossing property of the maximized function (21) in (b, σ), if the bidder knew v, then she would prefer to stop the auction before the price reaches $\sigma(v)$. Therefore, if at price $p > \sigma(v^*)$ the bidder gets a recommendation from the advisor to stop the auction, she finds it optimal to stop immediately. To finish the proof, we show the bidder does not want to stop the auction earlier. Let $v_p \equiv \sigma^{-1}(p)$ for all $p > p^*$ denote the type of the advisor that recommends to stop the auction at price p. Let $\hat{v}$ denote the highest valuation of the $N-1$ rival bidders. Recall that it is distributed according to $G(\cdot)$. Then, the expected payoff of the bidder at price p from following the recommendation of the advisor until price $\sigma(v^*)$, given that the advisor has not recommended to stop the auction yet is:

$$\begin{aligned}&\mathbb{E}\left[(v-\sigma(v))\,1\{v>\hat{v}\}\,|\hat{v}<v_p, v<v_p\right] \qquad (24)\\ =&\mathbb{E}\left[(v-\sigma(v))\,1\{v>\hat{v}\}\,|v^*<v<v_p, \hat{v}<v_p\right]\frac{F(v_p)-F(v^*)}{F(v_p)}.\end{aligned}$$

We can re-write (23) as

$$\begin{aligned}\sigma(v) &= b+\mathbb{E}\left[\hat{v}|\hat{v}<v\right]+\frac{G(v^*)}{G(v)}\left(v^*-\mathbb{E}\left[\hat{v}|\hat{v}<v^*\right]\right)\\ &= b+v-\frac{1}{G(v)}\left(vG(v)-v^*G(v^*)-\int_{v^*}^{v}\hat{v}dG(\hat{v})\right)\\ &= b+v-\frac{1}{G(v)}\int_{v^*}^{v}G(\hat{v})\,d\hat{v}. \qquad (25)\end{aligned}$$

Plugging this expression into (24) yields

$$\begin{aligned}&\int_{v^*}^{v_p}\left(-bG(v)+\int_{v^*}^{v}G(\hat{v})\,d\hat{v}\right)\frac{dF(v)}{G(v_p)F(v_p)}\\ =&-b+\frac{1}{G(v_p)}\int_{v^*}^{v_p}G(\hat{v})\,d\hat{v}+b\frac{G(v^*)F(v^*)}{G(v_p)F(v_p)}+\frac{1}{G(v_p)F(v_p)}\int_{v^*}^{v_p}F(v)\left(bg(v)-G(v)\right)dv. \qquad (26)\end{aligned}$$

On the other hand, if the bidder deviates and stops the auction at price p, her expected utility will be

$$\mathbb{E}\left[v|v<v_p\right]-p=\mathbb{E}\left[v|v<v_p\right]-v_p-b+\frac{1}{G(v_p)}\int_{v^*}^{v_p}G(\hat{v})\,d\hat{v}, \qquad (27)$$

where again we used (25) to substitute for p. We need to show that (27) is less than (26):

$$bG(v^*)F(v^*)+\int_{v^*}^{v_p}F(v)\left(bg(v)-G(v)\right)dv-\left(\mathbb{E}\left[v|v<v_p\right]-v_p\right)G(v_p)F(v_p)\geq 0. \qquad (28)$$

Equivalently, using $G(v)=F^{N-1}(v)$ and $g(v)=(N-1)F^{N-2}(v)f(v)$, we can rewrite the left-hand side of (28) as

$$\phi(v_p)=bF^N(v^*)+\int_{v^*}^{v_p}\left(b(N-1)F^{N-1}(v)f(v)-F^N(v)\right)dv-F^{N-1}(v_p)\int_{\underline{v}}^{v_p}vdF(v)+v_pF^N(v_p).$$

Differentiating.

$$\frac{d}{dv_p}\phi(v_p) = (N-1) F^{N-1}(v_p) f(v_p) (b + MAI(v_p)),$$

which is positive for any $v_p > v^*$, since $MAI(v_p)$ is strictly increasing with $MAI(v^*) = -b$. Since $\phi(v^*) = 0$, this implies $\phi(v_p) > 0$ for any $v_p > v^*$. Thus, the strategy from the theorem is optimal for the bidder, given that she expects the advisor to follow the strategy from the theorem. □

Proof of Theorem 8. By contradiction, suppose that $\omega_1 \leq v^*$. By eq. (6), $MAI(\omega_1) > -b$. Since v^* satisfies $MAI(v^*) = -b$ and $MAI(\cdot)$ is strictly increasing, $MAI(v) \leq -b$ for any $v \leq v^*$, including $v = \omega_1$. This contradicts $MAI(\omega_1) > -b$. Therefore, $\omega_1 > v^*$. □

Proof of Proposition 5. First, we show that any equilibrium of the second-price auction satisfying NITS is non-babbling in this case. By contradiction, suppose that the babbling equilibrium satisfies NITS. Then, $\overline{v} + b \leq \mathbb{E}[v]$ - otherwise, type $\overline{v}$ would have incentives to reveal itself to the bidder, induce a bid just above $\mathbb{E}[v]$, win the auction with probability one, and get the payoff of $\overline{v}+b-\mathbb{E}[v] > 0$. Thus, $MAI(\bar{v}) \leq -b$. Since $MAI(\cdot)$ is strictly increasing, $MAI(v^*) < -b$, which contradicts (13).

Second, we prove the revenues ranking result. Consider partition $(\omega_k)_{k=1}^K$ induced by an equilibrium of the second-price auction satisfying NITS. Expected revenues from equilibrium in Theorem 7 of the descending-bid auction equals the expected revenues from the first-price auction, in which bidders know their valuations and they are given by $\kappa(v) = b + \max\{v^*, v\}$. By the standard revenue equivalence theorem, the expected revenue from the first- and second-price auctions are equal, so we can equivalently compare expected revenues from the second-price auction with cheap-talk communication with partitions $(\omega_k)_{k=1}^K$ (denote it auction $\mathcal{A}$) and from the second-price auction in which bidders know their values and they are given by $\kappa(v)$ (denote it auction $\mathcal{B}$).

Let $v^{(1)} \equiv \max\{v_1, v_2\}$ and $v^{(2)} \equiv \min\{v_1, v_2\}$ denote the maximum and minimum of the two valuations, respectively. Let k and j be integers such that $v^{(1)} \in [\omega_{k-1}, \omega_k)$ and $v^{(2)} \in [\omega_{j-1}, \omega_j)$. By definition, $j \leq k$. Hence, there can be four cases:

Case $1 < j < k$. Conditional on this event, expected revenues in auction $\mathcal{A}$ are $\mathbb{E}[v|v \in [\omega_{j-1}, \omega_j)]$. Since $v^* < \omega_1 < \omega_j$, expected revenues in auction $\mathcal{B}$ are $\mathbb{E}[v + b|v \in [\omega_{j-1}, \omega_j)] < \mathbb{E}[v|v \in [\omega_{j-1}, \omega_j)]$.

Case $1 < j = k$. Conditional on this event, expected revenues in auction $\mathcal{A}$ are $\mathbb{E}[v|v \in [\omega_{j-1}, \omega_j)]$. Expected revenues in auction $\mathcal{B}$ are $\mathbb{E}\left[v^{(2)} + b|v^{(1)} \in [\omega_{j-1}, \omega_j), v^{(2)} \in [\omega_{j-1}, \omega_j)\right]$. Since the expectation of a minimum of two random variables is below the mean, this is less than $\mathbb{E}[v + b|v \in [\omega_{j-1}, \omega_j)]$, which in turn less than $\mathbb{E}[v|v \in [\omega_{j-1}, \omega_j)]$.

Case $1 = j < k$. Conditional on this event, expected revenues in auction $\mathcal{A}$ are $\mathbb{E}[v|v \leq \omega_1]$.

Expected revenues in auction $\mathcal{B}$ equals

$$\begin{aligned}\mathbb{E}\left[\kappa\left(v\right)|v\leq\omega_1\right]&=\left(v^*+b\right)\frac{F\left(v^*\right)}{F\left(\omega_1\right)}+\mathbb{E}\left[v+b|v\in\left(v^*,\omega_1\right)\right]\left(1-\frac{F\left(v^*\right)}{F\left(\omega_1\right)}\right)\\&<\mathbb{E}\left[v|v\leq\omega_1\right]\frac{F\left(v^*\right)}{F\left(\omega_1\right)}+\mathbb{E}\left[v|v\in\left(v^*,\omega_1\right)\right]\left(1-\frac{F\left(v^*\right)}{F\left(\omega_1\right)}\right)\\&\leq\mathbb{E}\left[v|v\leq\omega_1\right],\end{aligned}\tag{29}$$

where we used $v^* < \omega_1$.

Case $1 = j = k$. Conditional on this event, expected revenues in auction $\mathcal{A}$ are $\mathbb{E}\left[v|v\leq\omega_1\right]$, while expected revenues in auction $\mathcal{B}$ are $\mathbb{E}\left[\kappa\left(v^{(2)}\right)|v^{(1)}\leq\omega_1,v^{(2)}\leq\omega_2\right]\leq\mathbb{E}\left[\kappa\left(v\right)|v\leq\omega_1\right]<\mathbb{E}\left[v|v\leq\omega_1\right]$, where the first inequality holds since the expectation of the minimum of two random variables cannot exceed the expectation of a random variable and the second inequality holds by (29).

Since auction $\mathcal{A}$ results in higher expected revenues than auction $\mathcal{B}$ in each of these four cases and they cover all possible events, auction $\mathcal{A}$ also results in higher expected revenues overall. □

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Online Appendix

This appendix is for online publication. It contains the proof of Lemma 1, the argument underlying our estimate of b, and the proof of Proposition 6.

Proof of Lemma 1. The proof of the first statement follows the argument of the proof of Lemma IA.2 in Grenadier et al. (2016). Specifically, for any pure-strategy PBEM we construct an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path.

Consider any pure-strategy equilibrium E with some strategies $\bar{m}(v, p, \mu)$ and $\bar{a}(p, \tilde{\mu})$. It implies an equilibrium exit price $\bar{\tau}(v)$, which is the price at which the bidder exits the auction, if the valuation is v, provided that the bidder and her advisor play the equilibrium strategies $\bar{m}(\cdot)$ and $\bar{a}(\cdot)$. Note that $\bar{\tau}(v)$ must be weakly increasing in v. To see this, suppose by contradiction that $\bar{\tau}(v_1) > \bar{\tau}(v_2)$ for some $v_1 \in [\underline{v}, \bar{v})$ and $v_2 \in (v_1, \bar{v}]$. Since the advisor's payoff from acquiring the asset at any price p is higher for type v_2 than for type v_1 ($v_2 + b - p > v_1 + b - p$), the advisor's continuation value from not exiting the auction at any price p cannot be lower for type v_2 than for type v_1. The payoff from exiting the auction at any current price p does not depend on the type and equals zero. Thus, $\bar{\tau}(v_2) \geq \bar{\tau}(v_1)$. Let $\varrho \equiv \{p : \exists v \in [\underline{v}, \bar{v}]$ such that $\bar{\tau}(v) = p\}$ be the set of prices at which the bidder exits the auction for some realization of v. It will be convenient to define $v_l(p) \equiv \inf\{v : \bar{\tau}(v) = p\}$ and $v_h(p) \equiv \sup\{v : \bar{\tau}(v) = p\}$ for any $p \in \varrho$. We extend the definition of $v_l(p)$ for any $p \notin \varrho$ by setting $v_l(p) \equiv \inf\{v : \bar{\tau}(v) \geq p\}$.

Consider an online threshold strategy of the advisor, $m(v, p, \mu)$, with $\hat{p}(v, \mu) = \bar{\tau}(v)$ and the following belief updating rule of the bidder. For any belief μ, price p, and message $\tilde{m}$ such that there is $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}$, belief μ is updated via the Bayes rule. Any other message $\tilde{m}$ (i.e., a message for which there is no $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}$) is treated as some message $\tilde{m}'$ for which there is some $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}'$, and belief μ is updated following message $\tilde{m}$ in the same way as following message $\tilde{m}'$.[29] Given this, the posterior belief of the bidder for any history h is as follows. A sequence of messages $m = 0$ for all prices $p' \leq p$ up to price p implies that the bidder's posterior belief is given by the prior distribution of valuations truncated from below at $v_l(p)$. A sequence of messages $m = 0$ for all prices $p' < p'' \in \varrho$ and message $m = 1$ at price $p'' \in \varrho$ and any history of messages after that results in the bidder's posterior belief given by the prior distribution of valuations truncated at $v_l(p'')$ from below and at $v_h(p'')$ from above. Any history involving off-equilibrium messages leads to the posterior belief equivalent to one of these two posterior beliefs by construction of the updating rule. Given this, consider an online threshold strategy of the bidder, $a(p, \tilde{\mu})$, with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \geq v_l(p)]$ for the posterior belief $\tilde{\mu}$ in the history of the first kind (i.e., when the advisor never recommended quitting at one of prices $p \in \varrho$ in the past), and with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \in [v_l(p''), v_h(p'')]]$ for the posterior belief $\tilde{\mu}$ in the history of the second type (i.e., when the advisor recommended to quit the auction at price $p'' \in \varrho$). Let E' denote a combination of these online threshold strategies of the advisor and the bidder and the belief updating rule. Below we show that E' is indeed an equilibrium and that it results in the same equilibrium exit price $\bar{\tau}(v)$ as equilibrium E.

For the collection of strategies and beliefs E' to be an equilibrium, we need to verify the incentive compatibility (IC) conditions of the advisor and the bidder.

1 - IC of the advisor. First, we verify that the advisor is not better off deviating from (8) with $\hat{p}(v, \mu) = \bar{\tau}(v)$. Because of the above definition of the off-path beliefs, it is sufficient to consider only deviations to $m \in \{0, 1\}$ at $p \in \varrho$. First, consider a deviation of type v to $m = 1$ at $p \in \varrho$ at which $p < \bar{\tau}(v)$. This deviation is equivalent to mimicking the communication strategy of type $v' : \bar{\tau}(v') = p$. Since mimicking the communication strategy of type v' is not profitable for

[29] Intuitively, according to this updating rule, the bidder effectively ignores unexpected messages. As a consequence, it is sufficient to consider only deviations to on-path (expected) messages. Because no deviation to an off-path message can be beneficial, we do not lose any equilibria by focusing on this belief updating rule.

type v in equilibrium E (otherwise, it would not be an equilibrium), it is also not profitable here. Second, consider a deviation of type v to $m = 0$ at $p = \bar{\tau}(v)$. Depending on her communication strategy at later prices, this deviation will result in exit at price $\bar{\tau}(v')$ for some $v' \geq v_h(\bar{\tau}(v))$. Hence, any such deviation is equivalent to mimicking the communication strategy of type v'. Since it is not profitable for type v in equilibrium E, it is also not profitable here.

2 - IC of the bidder after observing $m = 1$ at $p \in \varrho$ and $m = 0$ before. We argue that $\bar{p}(\tilde{\mu}) \leq p$ in this case, so the bidder's best response is to quit the auction immediately. Given this history, the bidder's posterior belief is that $v \in [v_l(p), v_h(p)]$. Because the bidder expects the advisor to follow (8) with $\hat{p}(v, \mu) = \bar{\tau}(v)$, she expects the advisor to send $m = 1$ at any later price. Since the bidder expects to not learn anything new about v, her optimal exit strategy is given by the expected valuation, i.e., $\mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. It follows that the bidder exits immediately if $p \geq \mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. Next, we show that $\bar{\tau}(p)$ is equilibrium E must satisfy this condition at any $p \in \varrho$. Since exiting at price p is optimal for the bidder for any realization $v \in [v_l(p), v_h(p)]$ of the valuation, it must be that $p \geq \mathbb{E}\left[v|\mathcal{H}_p^E\right]$ for any history $\mathcal{H}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_l(p), v_h(p)]$. It follows that $p \geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right]$, where $\mathbb{H}_p^E$ denotes the set of such histories. Using the law of iterated expectations and fact that the maximum of a random variable cannot be below its mean,

$$
\begin{aligned}
p &\geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right] \geq \mathbb{E}\left[\mathbb{E}\left[v|\mathcal{H}_p^E\right]|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] \\
&= \mathbb{E}\left[v|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] = \mathbb{E}[v|v \in [v_l(p), v_h(p)]].
\end{aligned}
$$

Therefore, when the bidder observes message $m = 1$ at $p \in \varrho$ for the first time, she finds it optimal to quit the auction immediately.

3 - IC of the bidder after observing a sequence of messages $m = 0$ up to price $p < \bar{\tau}(\bar{v})$. We argue that $\bar{p}(\tilde{\mu}) > p$ for any such history, i.e., it is optimal for the bidder to wait. Given this history, the bidder's posterior is that $v \in [v_h(p'), \bar{v}]$ for highest $p' \in \varrho$ satisfying $p' < p$. Consider equilibrium E and any history $\tilde{\mathcal{H}}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_h(p'), \bar{v}]$. Denote the set of such histories by $\tilde{\mathbb{H}}_p^E$. Since the bidder finds it optimal to wait, the payoff from waiting is weakly above the payoff from quitting the auction immediately (i.e., zero) for any such history $\tilde{\mathcal{H}}_p^E$. In strategy profile E' the bidder learns no less between price p and the exit price than in strategy profile E. Hence, the fact that waiting is optimal for any history $\tilde{\mathcal{H}}_p^E \in \tilde{\mathbb{H}}_p^E$ implies that waiting is also optimal when the bidder expects the advisor to follow (8) with $\hat{p}(v, \mu) = \bar{\tau}(v)$.

Therefore, the collection of strategies and beliefs E' is an equilibrium. Furthermore, on equilibrium path, the advisor with type v recommends to quit the auction when the price reaches $\bar{\tau}(v)$, and the bidder exits the auction immediately. Therefore, E' results in the same bidding behavior as E.

The second statement of the lemma can be proved by contradiction. Consider equilibrium E that satisfies NITS, and suppose that an equilibrium in online threshold strategies with the same bidding behavior violates NITS. Hence, there exists price p such that the advisor with type $v_l(p)$ is better off credibly revealing itself at price p than getting the expected (as of information at price p) payoff in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself then exceeds the time-0 expected payoff of the advisor of type $v_l(p)$ in equilibrium E'. Now, consider equilibrium E, and the strategy of the advisor of type $v_l(p)$ to send equilibrium message $\bar{m}(v, p', \mu)$ for all $p' < p$ and to credibly reveal itself at price p (by definition of $v_l(p)$, type $v_l(p) = \inf\{v | v \in supp(\mu(h))\}$ for any history induced by this message strategy up to price p). In equilibrium E, bidding behavior of other bidders is the same and the bidder's reaction to the advisor credibly revealing itself at price p is the same as in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from this strategy is the same as the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself at price p in equilibrium E', which is strictly higher than the time-0 equilibrium expected payoff of the advisor of type $v_l(p)$. Hence, equilibrium E also violates the NITS condition, which is a contradiction. □

Estimate of b. Since the market leverage ratio of the median target is 13.1% and the median ratio of the deal size to the acquirer's assets is 31%, the ratio of the deal size to the acquirer's equity for a typical deal is $31\% \times \frac{1}{0.869} = 35.67\%$. Since the median acquisition premium is 39%, the ratio of the pre-deal target's equity to the acquirer's equity is $35.67\% \times \frac{1}{1.39} = 25.66\%$. Assume that after the acquisition, the sales of the combined company increase by the same amount, i.e., by 25.66% in perpetuity. Using the estimate of Harford and Li (2007), this increase in sales leads to an increase in the acquirer's CEO compensation by $0.435 \times \log(1 + 0.2566) = 4.32\%$ every year. In addition, acquiring the target is associated with an increase in the CEO compensation of 3.7%, irrespectively of the increase in sales, in the year of the acquisition. Thus, the positive effect of acquiring the target on CEO compensation for a typical deal is 8.02% in the year of the deal and 4.32% in every subsequent year. Using the expected tenure of 6 years and discounting at 10%, the present value of the positive effect is 22.52% of the CEO's annual compensation. On the other hand, overbidding by b (normalizing the pre-acquisition value of the target's equity to one), reduces the acquirer's equity value by $b \times 0.2566$. Since the porftolio value of equity incentives is 9.5 times the CEO annual pay (Table II in Harford and Li (2007)), the negative effect on the CEO wealth is $9.5 \times b \times 25.66\%$ of the CEO's annual compensation. The estimate of overpayment bias b is thus $\frac{0.2252}{9.5 \times 0.2566} = 9.2\%$.

Proof of Proposition 6. Suppose that all bidders choose contracts $p^*(w) = b + \min\{w, v^*\}$. Fixing this, consider the remaining bidder (to whom we will refer as the bidder) optimizing over contracts

$p(w)$. Any her contract $p(w)$ can be equivalently written as a mapping from report w into the probability of winning $q(w)$ and the corresponding expected payment $t(q(w))$:

$$q(w) = \begin{cases} 1, & \text{if } p(w) > v^* + b, \\ q^*, & \text{if } p(w) = v^* + b, \\ G(p(w) - b) & \text{if } p(w) < v^* + b, \end{cases} \tag{30}$$

$$t(q) = \begin{cases} \mathbb{E}[\hat{v} + b | \hat{v} < v^*] \bar{q} + (v^* + b)(1 - \bar{q}), & \text{if } q = 1, \\ \mathbb{E}[\hat{v} + b | \hat{v} < v^*] \bar{q} + (v^* + b)(q^* - \bar{q}), & \text{if } q = q^*, \\ \mathbb{E}\left[\hat{v} + b | \hat{v} \leq G^{-1}(q)\right] q, & \text{if } q < \bar{q}, \end{cases} \tag{31}$$

where $\hat{v}$ denotes the highest valuation among $N - 1$ rival bidders, $\bar{q} = G(v^*)$, and $q^* = \bar{q} + \sum_{n=1}^{N-1} \binom{N-1}{n} \frac{(1-F(v^*))^n F(v^*)^{N-1-n}}{n+1}$. The payoffs of the bidder and the advisor are:

$$\text{Bidder}: \quad q(w) v - t(q(w)), \tag{32}$$

$$\text{Advisor}: \quad q(w)(v + b) - t(q(w)). \tag{33}$$

The bidder's optimal contract $q(w)$ maximizes her expected payoff subject to the incentive compatibility constraint that the advisor reports his type truthfully:

$$\begin{aligned} &\max_{q(\cdot) \in Q} \int_{\underline{v}}^{\bar{v}} (q(v) v - t(q(v))) \, dF(v) \\ &s.t. \;\; v \in \arg\max_{w \in [\underline{v}, \bar{v}]} \{q(w)(v + b) - t(q(w))\} \;\; \textit{for all } v, \end{aligned} \tag{34}$$

where Q is the set of all measurable functions from $[\underline{v}, \bar{v}]$ into $[0, \bar{q}) \cup \{q^*, 1\}$ and $\bar{q} = G(v^*)$. We want to show that $q^A(v) \equiv \begin{cases} G(v), & \text{if } v < v^* \\ q^*, & \text{if } v \geq v^* \end{cases}$, solves (34). We will show a slightly stronger statement that $q^A(\cdot)$ solves (34) in which set Q is replaced by the set of all measurable functions from into $[0, \bar{q}] \cup \{q^*, 1\}$, where $t(\bar{q}) = \lim_{q \uparrow \bar{q}} t(q)$. To do this, it will be helpful to show some properties of $t(\cdot)$:

Claim 7. $t(\cdot)$ is strictly increasing, strictly convex, twice differentiable on $[0, \bar{q})$, and

$$\frac{t(1) - t(q^*)}{1 - q^*} = \frac{t(1) - t(\bar{q})}{1 - \bar{q}} = v^* + b = \lim_{q \uparrow \bar{q}} t'(q). \tag{35}$$

Proof: In the range $q < \bar{q}$, $t(q) = \int_{\underline{v}}^{G^{-1}(q)} (\hat{v} + b) g(\hat{v}) \, d\hat{v}$. Differentiating and substituting the derivative of the inverse function, $t'(q) = G^{-1}(q) + b > 0$ and $t''(q) = \frac{1}{g(G^{-1}(q))} > 0$. We next show (35). By (10), $\mathbb{E}[v | v \geq v^*] = v^* + b$, i.e., the bidder with expected value $\mathbb{E}[v | v \geq v^*]$ is indifferent between winning and losing at price v^*+b. Therefore, $(\bar{q}, t(\bar{q}))$, $(q^*, t(q^*))$, and $(1, t(1))$ are on the same line with slope $\mathbb{E}[v|v \geq v^*] = v^*+b$. Finally, since $t'(q) = G^{-1}(q)+b$, $\lim_{q\uparrow\bar{q}} = v^*+b$. *q.e.d.*

Eq. (35) implies that the bidder's expected payoffs from contracts $q^A(v)$ and $q^B(v) = \begin{cases} G(v) & \text{if } v < v^* \\ \bar{q} & \text{if } v \geq v^* \end{cases}$ are the same. Also, by Claim 7, if q solves (34) and $q(v) = q^*$, $q(v') = 1$ for some v and v', then $\tilde{q}(v)$ defined as $\tilde{q}(v) = q^*$ for $v : q(v) = 1$ and $\tilde{q}(v) = q(v)$, otherwise, also solves (34). Therefore, it sufficies to show that q^B solves (34) in which Q is replaced by the set Q' of all measurable functions from $[\underline{v}, \bar{v}]$ into either $[0, \bar{q}] \cup \{q^*\}$ or $[0, \bar{q}] \cup \{1\}$. We will prove the statement for the former case. The other case follows by the analogous argument. Define the program B as (34) with Q' instead of Q. It follows from Claim 7 that while $t(\cdot)$ is strictly convex below $\overline{q}$, it cannot be extrapolated to a strictly convex function to the whole interval $[0, 1]$, as points $(\bar{q}, t(\bar{q}))$, $(q^*, t(q^*))$, and $(1, t(1))$ lie on the same line. Thus, we cannot directly apply the results of Amador and Bagwell (2013) to program B. Instead, we perturb the program B so that we can apply their result to this perturbed program and then relate the solution of the perturbed program to the solution of program B. We next show that q^B defined above is a solution to program B. Specifically, we perturb function $t(\cdot)$ on $[0, q^*]$ as $t_\varepsilon(q) = t(q) + \varepsilon \max\left\{0, (q-\bar{q})^3\right\}$. Consider the auxiliary program B_ε:

$$\max_{q(\cdot)\in Q''} \int_{\underline{v}}^{\bar{v}} \left(q(v)\,v - t_\varepsilon(q(v))\right) dF(v)$$
$$\text{s.t. } v \in \arg\max_{w\in[\underline{v},\bar{v}]} \left\{q(w)(v+b) - t_\varepsilon(q(w))\right\} \text{ for all } v,$$

where Q'' is the set of all measurable functions from $[\underline{v}, \bar{v}]$ to $[0, q^*]$. Unlike $t(\cdot)$, function $t_\varepsilon(\cdot)$ is strictly convex and twice differentiable on $[0, q^*]$, so we can apply Proposition 1 in Amador and Bagwell (2013) to get the following claim:

Claim 8. *$q^B(v)$ is a solution to program B_ε for any ε.*

Proof: In Table 2, we verify that program B_ε satisfies the conditions of Proposition 1 in Amador and Bagwell (2013). Therefore, q^B_ε solves it. *q.e.d.*

Finally, we show that q^B solves program B. By contradiction, suppose that this is not the case. Denote the solution by $\tilde{q}^B$. First, if the range of $\tilde{q}^B(v)$ does not contain q^*, then $\tilde{q}^B \in Q''$, which is a contradiction to Claim 8. Thus, we suppose that the range of $\tilde{q}_B(v)$ contains q^*. Let us perturb $\tilde{q}^B$ as follows. Denote by $\tilde{Q}$ the range of $\tilde{q}^B$ and let $\tilde{q}^B_\varepsilon(v) = \arg\max_{q\in\tilde{Q}} \{(v+b)\,q - t_\varepsilon(q)\}$. By construction, $\tilde{q}^B_\varepsilon$ satisfies the constraints of program B_ε. Moreover, the measure of types for which

Amador and Bagwell's notations and assumptions	Our counterparts
$\gamma \in \Gamma = \left[\underline{\gamma}, \bar{\gamma}\right]$ distributed according to F	$v \in [\underline{v}, \bar{v}]$ distributed according to F
$\pi \in \Pi = [0, \bar{\pi}]$	$q \in [0, q^*]$
$b(\pi)$	$bq - t_\varepsilon(q)$
Agent's payoff: $\gamma\pi + b(\pi)$	Advisor's payoff: $(v+b)\,q - t_\varepsilon(q)$
Principal's payoff: $w(\gamma, \pi)$	Bidder's payoff: $vq - t_\varepsilon(q)$
γ_L	$\underline{v}$
$\pi_f(\gamma) \in \arg\max_{\pi\in\Pi} \gamma\pi + b(\pi)$	$q_f(v) \in \arg\max_{q\in[0,q^*]} (v+b)\,q - t_\varepsilon(q)$ which implies $v + b = t'_\varepsilon(q_f(v))$
γ_H solves $\int_{\gamma_H}^{\overline{\gamma}} w_\pi(\gamma, \pi_f(\gamma_H))\,dF(\gamma) = 0$	v^* solves $\int_{v^*}^{\overline{v}} (v - v^* - b)\,dF(v) = 0$, or equivalently $MRL(v^*) = b$
$\kappa = \inf_{(\gamma,\pi)\in\Gamma\times\Pi} \left\{\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}\right\}$	$\kappa = \inf_{(v,q)\in[\underline{v},\bar{v}]\times[0,q^*]} \left\{\frac{\frac{\partial^2}{\partial q^2}(vq - t_\varepsilon(q))}{\frac{d^2}{dq^2}(bq - t_\varepsilon(q))}\right\} = 1$
Assumption 1	
w continuous in γ and π	by Claim 7
$w(\gamma, \cdot)$ concave and twice differentiable in π	
b is strictly concave and twice differentiable in π	
π_f is twice differentiable and $\pi'_f(\gamma) > 0$	by $q'_f(v) = \frac{1}{t''_\varepsilon(q_f(v))} > 0$ and f differentiable
w_π is continuous in γ	by Claim 7
Assumptions of Proposition 1 in Amador and Bagwell (2013)	
(c1) $\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ is nondecreasing	$F(v) - (v - t'_\varepsilon(q_f(v)))\,f(v) = F(v) + bf(v)$ is nondecreasing whenever $(\ln f(v))' \geq -\frac{1}{b}$
(c2) For all $\gamma \in [\gamma_H, \bar{\gamma}]$, $(\gamma - \gamma_H)\,\kappa \geq \int_{\gamma}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma_H))\,\frac{f(\tilde{\gamma})}{1-F(\gamma)}$	For all $v' \in [v^*, \bar{v}]$, $v' - v^* \geq \mathbb{E}[v \mid v \geq v'] - v^* - b$ by Assumption A
(c3') $\omega_\pi\left(\underline{\gamma}, \pi_f\left(\underline{\gamma}\right)\right) \leq 0$	$\underline{v} - t'_\varepsilon(q_f(\underline{v})) = -b < 0$

Table 2: The left column shows assumptions and conditions of Proposition 1 in Amador and Bagwell (2013). The right column shows the corresponding variables in our model and verifies the conditions.

$\tilde{q}_{\varepsilon}^{B}(v) \neq \tilde{q}^{B}(v)$ converges to zero as $\varepsilon \to 0$. Then, for some positive c_0, c_1, and c_2,

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \left(\tilde{q}^{B}(v)\, v - t\left(\tilde{q}^{B}(v)\right)\right) dF(v) &> \int_{\underline{v}}^{\overline{v}} \left(q^{B}(v)\, v - t\left(q^{B}(v)\right)\right) dF(v) + c_0 \\
&> \int_{\underline{v}}^{\overline{v}} \left(q^{B}(v)\, v - t_{\varepsilon}\left(q^{B}(v)\right)\right) dF(v) + c_0 \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{q}_{\varepsilon}^{B}(v)\, v - t_{\varepsilon}\left(\tilde{q}_{\varepsilon}^{B}(v)\right)\right) dF(v) + c_0 \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{q}_{\varepsilon}^{B}(v)\, v - t\left(\tilde{q}_{\varepsilon}^{B}(v)\right)\right) dF(v) + c_0 - c_1\varepsilon \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{q}^{B}(v)\, v - t\left(\tilde{q}^{B}(v)\right)\right) dF(v) + c_0 - c_1\varepsilon - c_2\varepsilon,
\end{aligned}$$

where the first inequality follows from the fact that $\mathbf{q}^{B} \in Q'$ and satisfies the constraints of program B but does not solve it; the second inequality follows from $t_{\varepsilon}(q) \geq t(q)$; the third inequality follows from the fact that q^{B} solves B_{ε}; and the last two inequalities follow from the construction of $t_{\varepsilon}(\cdot)$ and q_{ε}^{B}, respectively. By taking $\varepsilon \to 0$, we get a contradiction. □