Rational Quantitative Trading in Efficient Markets*

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Abstract

We present a model of financial markets where quantitative trading emerges endogenously as an automated price-contingent strategy under human discretion. Price-contingent trading has been argued to be at odds with (semi-strong) market efficiency. In contrast, we show that price-contingent trading is the profitable equilibrium strategy of a large rational agent whose trading strategy, price-contingent or fundamentals-based, is their source of private information. Even when uninformed about fundamentals he will trade non-zero quantities whose direction—trend-following or contrarian—depends on the magnitude of the order flow in a non-monotonic manner. One additional implication of our model is that future order flow is predictable even if returns are not.

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1 Introduction

It is well recognized by financial market participants and academic research that a large part of trading in financial markets is driven by price-contingent strategies, i.e., strategies that are based on past asset price movements rather than fundamental analysis. Importantly, among the major players in implementing price-contingent strategies are the funds specialized in quantitative trading. Price-contingent trading can be trend-following (e.g., buying after prices have gone up), contrarian (e.g., buying after prices have gone down), or can take more complex forms where the direction of price-contingent trading varies across time horizons, magnitudes of past price changes, instruments and trade structures (see Section 3 in Narang (2013) for a characterization of data driven quantitative strategies).

The empirical evidence on the performance of price-contingent trading documents a striking difference between retail investors and large quantitative funds. In fact, the evidence shows that individual and retail investors lose from pursuing trend-following or contrarian strategies.[1] These findings have motivated a large literature to argue that trend-following or contrarian trading must stem from behavioral biases, imperfect or bounded rationality, non-standard preferences, or institutional frictions (e.g., Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), Hong and Stein (1999); see Shleifer (2000) and Barberis and Thaler (2003) for surveys).

By contrast, large institutions appear to systematically profit by price-contingent trading.[2] As a result, the unprecedented growth of quantitative trading by sophisticated large financial institutions (see e.g., Osler (2003) and Hendershott, Jones, and Menkveld (2011)) poses a challenge to a purely behavioral view of price-contingent trading. The reason is that—unlike retail investors—these large institutions appear to systematically profit by price-contingent trading, while in the behavioral models reviewed above price-contingent traders incur systematic financial losses.

Our objective in this paper is to develop a micro-founded model of trading by rational uninformed agents and to use this model to understand the mechanics and the profitability of quantitative price-contingent trading by large financial institutions. While maintaining standard assumptions of (semi-strong) market efficiency and preferences, we demonstrate that price-contingent trading is the equilibrium strategy for large rational traders, who systemat-

[1] Contrarian trading leads to portfolio losses, as shown for individuals in Odean (1998) and Barber and Odean (2000). Trend-following trading leads to portfolio losses, as shown for mutual funds in Grinblatt, Titman, and Wermers (1995), Carhart (1997), and Coval and Stafford (2007) and for pension funds in Lakonishok, Shleifer, and Vishny (1992). See also Nofsinger and Sias (1999) and Griffin, Harris, and Topaloglu (2003).

[2] For example, Commodity Trading Advisors (CTAs) profit from trend-following strategies in futures markets (e.g., see Clenow (2013), and Baltas and Kosowski (2014)); and various institutions profit from contrarian strategies in equities (e.g., see Lehmann (1990), and Jegadeesh (1990)).

ically profit from trading in response to past price changes. Furthermore, we show that in general the direction of optimal price-contingent trading depends on the magnitude of past order flows (or equivalently, the magnitude of past price changes) in a non-monotonic manner, i.e., it tends to be trend-following small order flows and contrarian following large order flows. To the best of our knowledge, such non-monotonicity is a novel result and represents our main contribution. We discuss the related literature in Section 2 and the empirical implications that are unique to our model in Section 5.

Before addressing the drivers of trend-following and contrarian trading strategies, we should highlight two key assumptions that are important for explaining why quantitative trading is a rational equilibrium strategy in our setting. First, we assume that a quantitative trader has market impact, i.e., his trades move prices. Second, there is a positive probability that the quantitative trader may have fundamental information, while it is not known with certainty by the rest of the market whether or not the trader has such information. The first assumption must hold by definition when the traders are "large". The second assumption is realistic because quantitative trading nearly always involves discretion and human supervision, which implies that a quantitative trader can and will override the trading algorithm should he become aware of some fundamental information. In fact, practitioners' accounts of quantitative trading (e.g., Narang (2013)) describe specific examples of quants overriding their trading algorithms upon learning fundamental information, and in general repeatedly highlight such discretion to be crucial to understand quantitative trading by professional investors.

We present a stylized setting with one risky asset and two trading dates, and we model quantitative trading by assuming that there is a large risk-neutral trader, called P, who may be directly informed about the fundamental value (type I) or not informed (type U) with some probability. Trader P knows his own type and takes into account the market impact of his order on the expected market price. As standard in this literature, we impose that prices are set such that the market is semi-strong efficient by introducing another risk-neutral agent, the Market (often called the "market maker" in the rational expectations literature)[3] who observes the aggregate order flow and sets prices such that he breaks even in expectation. Crucially, the Market does not observe P's type. Instead, he knows the prior probability of P being type I or U, and updates his beliefs about the asset value and P's type based on the order flows.

As a simple example, consider the case when P is the only strategic large trader in the market, and the rest of the aggregate order flow comes from noise traders whose trading volume is independent from other random variables, serially uncorrelated and drawn from the normal distribution.[4] It is intuitive in this setting that the equilibrium price is an increasing function

[3]We call this agent "the Market", as the same outcome would be obtained if prices are set in a competitive market populated by a large number of small uniformed traders.

[4]We introduce noise traders as a standard assumption to guarantee that order flows are not fully revealing.

of date 1 order flow, simply because if P is informed (type I), P's orders are driven by his fundamental information. Crucially, even if not informed (type U), P still knows his own type and therefore infers that any deviation of prices from the prior mean must be driven entirely by noise trading, implying that he can and does benefit from pursuing a contrarian strategy (at date 2) in a rational expectations equilibrium.

This simple example gives an intuition why P's market impact and uncertainty about P's type are both needed for quantitative trading to be an equilibrium strategy. If P's past trades did not have market impact, knowledge of his own past trades would not suffice to generate superior information. This explains why it is relatively more difficult for retail investors to benefit from quantitative trading compared to larger institutions. Also, if P's type was public information, it would also be impossible for P to earn abnormal returns, because the Market and P would have exactly the same information (see Easley and O'Hara 1991 for a similar argument for the case where P's type is known).

In our baseline model, we consider a richer setting where there are other informed traders in the market. To account for this possibility, we introduce another large trader, called K, who is also large and risk-neutral, but unlike P, is always informed about the fundamental value. Distribution of the fundamental value of the asset is also important because it determines what informed traders can do in equilibrium. To obtain all our effects about the direction of P's trading, there must be, at a minimum, three possibilities – fundamental information reflecting positive or negative news, as well as confirming that the fundamental is (close to) the prior mean (i.e., news confirming the "base-case scenario"). Our baseline model therefore uses a symmetric three-point distribution that covers distributions of very different shapes (from U-shaped to hump-shaped).[5]

We show that at large past order flows the optimal price-contingent strategy is contrarian. The reason is similar to that in the simple example above. It always holds that two large informed traders will jointly trade more than one. Hence, in the state in which P is uninformed, the Market, upon observing a large positive aggregate order flow, tends to optimally set prices too high compared to uninformed P's best guess. Consequently, by knowing his own type (U), and observing a large past order flow, P knows better than the Market that the past price changes observed at date 1 most likely reflect noise trading rather than fundamental

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Furthermore, predictability of noise trading (e.g., through serial correlation) would immediately create a reason for any rational agent to profit by trading against the noise traders' demand. We abstract from such considerations, and consider noise traders as a large number of retail investors each trading for idiosyncratic reasons outside the model. As a result, the normal distribution of noise trades follows from the central limit theorem. In Section 4.3 we discuss the robustness of our findings to relaxing the assumption of normal noise trading and consider the wide class of log-concave distributions.

[5] As we allow for any probability for the middle outcome, our setting also covers the special case of a Bernoulli prior, which is less realistic for most assets as it eliminates the possibility that private information confirms the "base case scenario".

information.

The intuition for trend-following trading at small order flows is just the reverse. When observing a small aggregate order flow, the Market rationally believes that the order flow is most likely generated by noise trading rather than informed trading. As a result, rational equilibrium prices are too insensitive to order flow compared to uninformed P's best estimate, and the optimal quantitative trading strategy is trend-following. It is important to highlight that such trend following incentive would be absent if we eliminated the possibility that informed traders may have chosen not to trade (or to trade little), because they have learned that the fundamental value is close to the prior.[6]

We study the robustness of our results to a variety of extensions, including studying the robustness of our results to other distributions of the fundamental, and we show that the main drivers or contrarian and trend-following incentives apply for a wide set of other prior distributions, including continuous ones.

For clarity of exposition, our baseline setting also assumes that informed traders only trade at date 1 - after all, quantitative strategies are likely to react faster to public signals than purely fundamental-based strategies. Nevertheless, we show that our results are robust to allowing all rational (and forward-looking) traders to trade in all periods - we obtain exactly the same qualitative predictions. It is also straightforward to extend our results to allow for more informed traders and/or more trading rounds.

At the most basic level, our theory rationalizes why quantitative trading is profitable on average, over and above standard remunerations for risk, as it is better able to chase information than the rest of the market. It also delivers an additional theoretical result that the order flow is predictable from past information even if returns are not. Order flow predictability is consistent with the evidence of Biais, Hillion, and Spatt (1995), Ellul, Holden, Jain, and Jennings (2007), and Lillo and Farmer (2004). Furthermore, as our main novel prediction is about the non-monotonic equilibrium trading strategies, our model also suggests that this non-monotonicity carries over to the direction of order flow predictability, which provides further testable implications (see Section 5).

Our theory also highlights the importance of equilibrium forces that typical practitioners' accounts (such as Narang (2013)) do not consider. While practitioners emphasize the realistic contemporaneous costs of market impact[7], which we incorporate, a perhaps less intuitive finding of our model is that the same costly market impact is also a source of subsequent superior private information.

[6]For this exact reason, we show that a Bernoulli prior implies that only the contrarian trading incentives described above survive in the equilibrium.

[7]E.g., see Kissell (2014). Anecdotal evidence even suggests that hedge funds may decline opportunities to raise funds under management because a larger trading volume would imply too much market impact.

It should also be emphasized that out model is most insightful for quantitative trading strategies at intermediate, relatively short time horizons, i.e., daily, weekly, and/or monthly frequencies. While the forces we highlight may be present also at high frequency level, we recognize that at very high frequencies a potentially larger part of quantitative trading takes the form of market making. By contrast, at very long horizons the Market would eventually learn P's type and asset prices would converge to reflect only fundamental information.

The paper proceeds as follows. Section 2 discusses some of the related literature. Section 3 outlines the baseline model and presents the main results. Section 4 considers extensions. Section 5 discusses the empirical implications, and Section 6 concludes.

2 Literature

The broad literature on asset pricing and learning in micro-founded financial markets is surveyed in Brunnermeier (2001) and Vives (2008), among others. Our work relates to the part of the literature that studies trading in markets with asymmetric information. Our results on the profitability of rational price-contingent trading require that informed traders be large, i.e., that their trades have market impact. We develop our model in a setting that generalizes the Kyle (1985) framework.[8] Our model shows that rational traders with market impact and superior information about their own type can learn from prices better than average market participants or market makers. Another related strand of the literature studies whether past prices contain useful information for a rational trader (e.g., Grossman and Stiglitz (1980), Brown and Jennings (1989)). However, in these papers there are no profits from uninformed trading in excess of the risk premium. It is worth noting that this is because in these models uninformed traders do not have market impact and the number/share of such traders is known to all market participants.

Our paper also relates to the literature on stock price manipulation, that is, the idea that rational traders may have an incentive to trade against their private information. Provided manipulation is followed by some (exogenously assumed) price-contingent trading, short run losses can be more than offset by long term gains (see Kyle and Viswanathan (2008) for a review). Somewhat closer to our work, Chakraborty and Yilmaz (2004a, 2004b) study the incentives of an informed trader when there is uncertainty about whether such trader is informed, or is a noise trader instead. If this trader turns out to be informed, he may choose to disregard his information and trade randomly, in order to build a reputation as a noise trader. In their model, uninformed traders are assumed to always act as noise traders and are never strategic

[8]Similar implications could be obtained in a Glosten and Milgrom (1985) framework in which trades arrive probabilistically and market makers observe individual trades. See also Back and Baruch (2004).

and rational. Therefore, Chakraborty and Yilmaz do not analyze the trading incentives of rational agents when they are uninformed, which is our main focus.

Goldstein and Guembel (2008) show that if stock prices affect real activity then a form of trade-based manipulation such as short-sales by uninformed speculators can be profitable insofar as it causes firms to cancel positive NPV projects, and justifies ex post the "gamble" for a lower firm value. In their setting, both uninformed trading and successful stock price manipulation stem from the feedback effect between stock prices and real activity. By contrast, in our paper there is price-contingent trading but no manipulation. Therefore, our results demonstrate that price-contingent trading does not make uninformed investors the inevitable prey of (potentially informed) speculators.[9]

Our paper is also related to the literature that studies the consequences of introducing uncertainty about the types of traders in models with asymmetric or dispersed information.[10] One notable strand of this literature focuses on rational herding (see Avery and Zemski (1998), Park and Sabourian (2011), and Chamley (2004) for a review). Unlike our setting in which traders never disregard their private information, these models characterize conditions under which, when information precision is uncertain, rational traders ignore their noisy private signal and follow the actions of other traders instead.[11]

A more recent strand of this literature studies static and dynamic competitive rational expectations models with uncertainty about traders' types (Gao, Song, and Wang (2013), Banerjee and Green (2014)), or a Kyle (1985) model with one strategic agent and uncertainty about her information (Odders-White and Ready (2008), Li (2013) and Back, Crotty and Li (2016). In particular, Odders-White and Ready (2008) consider a single-period Kyle model in which a trader may or may not be informed; and Li (2013) studies a continuous time model in which a trader may or may not be informed. These papers consider uncertainty about the existence of "informational events", but does not consider the consequences of the distribution of "informational events" when they exist. The latter is needed to derive our main non-monotonicity results.[12] Back, Crotty and Li (2015) adopts a setting similar to Li (2013) and is mostly con-

[9] Allen and Gale (1992) also study a setting with manipulation but without price-contingent trading.

[10] Uncertainty about the number of informed traders also shows up in static noisy REE models where agents face short-sale or borrowing constraints, and uninformed traders are uncertain whether the constraint is binding or not (e.g., Yuan (2005)).

[11] One strand of this herding literature has focused on understanding the stock market crash of October 1987 (Grossman (1988), Gennotte and Leland (1990), Jacklin, Kleidon, and Pflederer (1992) and Romer (1993)). A common theme of these papers is that market participants are assumed to have strongly underestimated the extent of portfolio insurance—i.e., trend-following trading—which in turn is assumed to be exogenous. Our focus instead is on deriving endogenously price-contingent trading, and characterize conditions under which it is trend-following as opposed to contrarian.

[12] In particular, these papers consider a setting that is similar to our special case in the Supplementary Appendix in which there is one strategic trader P who may be informed or not. Uncertainty about the presence of informed traders can give rise to contrarian strategies, but not to the full spectrum of rational quantitative

cerned with the question of whether the (adverse selection) risk of informed trading (labeled PIN) can explain the spread between actively traded and inactively traded securities, following Easley, Kiefer, O'Hara, and Paperman (1996). Unlike us, these papers are concerned with explaining the price impact of trades, measured as the market maker's price sensitivity to order imbalance, with empirical proxies for the probability of informed trading.

Finally, we should note that our model is most appropriate to understand quantitative strategies that trade daily or weekly, so that there is both some probability that trading reflects information, and some benefit from a relatively fast execution in response to changes in market prices. It is less appropriate for the millisecond environment in which high frequency traders may benefit from momentary imbalances between supply and demand. With this in mind, our paper is also related to a few recent papers that focus on the speed advantage of quantitative traders. Clark-Joseph (2013) studies a partial equilibrium model in which prices are exogenous, and finds empirical support for the idea that high-frequency traders learn from their own trades better than the rest of the market, very closely related to the ideas developed in our model. Biais, Foucault, and Moinas (2013) consider the decision of a financial institution to invest in a high-speed trading technology and derive conditions under which such investment is excessive from a social welfare standpoint; and Pagnotta and Philippon (2012) consider trading exchanges competing on speed to attract future trading activity. Unlike us, these papers do not focus on price-contingent trading.

3 The Model

We consider a stylized setting with one risky asset that is traded at dates 1 and 2. The fundamental value, θ, is realized at date 3, and can take the following three values:

$$\theta = \begin{cases} -\bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \\ 0 & \text{wpr. } \gamma \\ \bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \end{cases}, \tag{1}$$

where $0 \leq \gamma < 1$ and $\bar{\theta} > 0$. Such three-point prior provides a simple representation of distributions of many different shapes (i.e., U-shaped, hump-shaped, uniform), while the parameter

strategies that include trend-following strategies and non-monotonic strategies. In contrast, our main setting highlights that in order to understand the drivers of all these strategies, one needs to consider not only whether there is an "informational event" (whether there are any informed traders), but also "the distribution of these informational events" (how likely are informed traders to learn information that is close to the prior mean as opposed to indicating a substantial change in fundamentals). In contrast to this literature, we emphasise the difference between "no news" because no-one is informed and "no-news" because informed investors learn that there has not been any substantial change in fundamentals relative to the prior.

γ is a measure of how much mass is in the centre of the distribution compared to the tails. Furthermore, a three state environment reflects the common approach adopted by fundamental analysts, who typically identify the "base", "best" and "worst" case scenario and provide an assessment of the likelihood of these cases, captured by parameter γ. In addition, considering three-states has the convenient feature of presenting (informed) strategic traders with a sufficiently rich set of actions to choose from - not just to buy or sell, but also not to trade or to trade little.[13] In Section 4.2 we extend our analysis to consider other priors.

As our goal is to understand the incentives of large traders who know that their trades have a market impact, we adopt a setting in the spirit of Kyle (1985). We maintain the assumption that large strategic risk-neutral traders and non-strategic noise traders submit market orders before knowing the execution price, and that the equilibrium prices are set by a hypothetical agent, the Market, who observes the total order flow and implements the market efficiency condition. Namely, he sets period t price,

$$p_t = \mathbb{E}\left[\theta|\Omega_t^M\right], \tag{2}$$

where Ω_t^M is the information set available to the Market in $t \in \{1, 2\}$, which includes all publicly available information such as the current and past order flows. Denoting the date t total order flow with y_t, it holds that $\Omega_1^M = \{y_1\}$ and $\Omega_2^M = \{y_1, y_2\}$.

To analyze quantitative trading by a large strategic investors, we introduce a risk-neutral trader, P. This trader sets up his quantitative trading strategy before date 1 without knowing the fundamental value θ. Such trading strategy can only depend on publicly observed variables, i.e., prices and order flow, and on the parameters of the distributions. However, as we have emphasized, quantitative trading involves the possibility of human discretion - P can alter his strategy should he learn direct fundamental information. This implies that P may either trade based of his knowledge of the fundamental or based on his original strategy. To formalize this idea, we assume that P knows his own type

$$R = \begin{cases} I & \text{if } P \text{ is "informed" (i.e., knows } \theta\text{)} \\ U & \text{if } P \text{ is "uninformed" (i.e., does not know } \theta\text{)} \end{cases},$$

where that probability that P is informed is $\Pr(I) = \eta$, where $0 < \eta < 1$.[14]. This is crucial as P

[13] Naturally, the special case with $\gamma = 0$ corresponds to a Bernoulli prior, which is of potential independent interest as settings with two possible outcomes are rather common in the literature. In general, however, by eliminating the "base" case from the set of possible outcomes, a Bernoulli prior does not capture many realistic environments, including the possibility that privately informed traders choose not to trade.

[14] We also assume that η is not arbitrarily close to one. This realistically avoids a situation in which P is uniformed, but the Market is convinced that he is informed and is very reluctant to update his beliefs about P's type.

knowing his own type is, *de facto*, a source of private information should he remain uninformed about θ.[15] In fact, while the Market knows η, it does not know its realization, that is, it does not know P's type (R). We further assume that R is independent of the fundamental and noise trading.

While we give an illustrative example where P is the only trader able to learn the fundamental value (see Supplementary Appendix), our baseline setting aims to capture quantitative trading in an environment in which P's potential fundamental information is not the only source of fundamental information relevant to the market. In fact, in reality there are typically also other sophisticated traders, who have comparative advantage on fundamentals-based strategies. To capture this idea, we assume that in addition to P, there is a large risk-neutral trader, K, who is rational, strategic, and always informed about the fundamental. Trader K is therefore equivalent to the insider in Kyle (1985).

To stack the cards against P, we assume that K knows P's type.[16] This assumption highlights that what is crucial for our argument is that the Market does not know P's type, irrespective of other sophisticated traders knowing it or not. In our main setting we also assume that fundamentals-based traders are "slower" than quantitative traders whose trades simply react to changes in observable variables. Formally, we assume that K and informed P can only trade at date 1, while uninformed P can trade any time. While both realistic and convenient to illustrate our results in a clean manner, this "speed" assumption turns out to be unimportant for our qualitative findings, as we find in Section 4.1 where we study the case in which informed K and P can trade at date 1 and 2.

We denote the market order by trader $J \in \{K, P\}$ in state $R \in \{I, U\}$ at date $t \in \{1, 2\}$ as h_t^{RJ}. If both traders are informed, $R = I$, then trader J solves

$$\max_{h_1^{IJ}} \pi_1^{IJ} = \mathbb{E}\left[h_1^{IJ}\left(\theta - p_1\right)|\theta, I\right], \tag{3}$$

where $J \in \{K, P\}$. If only K is informed about the fundamental, $R = U$, then K solves

$$\max_{h_1^{UK}} \pi_1^{UK} = \mathbb{E}\left[h_1^{UK}\left(\theta - p_1\right)|\theta, U\right], \tag{4}$$

[15] An alternative interpretation of this specification is that investors (i.e, the Market) do not know for sure whether a large fund is pursuing a price-contingent or a fundamental-based strategy regarding a particular asset at a given point in time. Strategies pursued by sophisticated Hedge Funds are rarely concentrated to only one approach and these institutions are known to be highly secretive about their portfolio.

[16] This assumption makes it ex ante harder for P to develop an information advantage as P never has more information than K.

and P solves[17]

$$\max_{h_1^{UP}} \pi_1^{UP} = \mathbb{E}\left[h_1^{UP}(\theta - p_1) + h_2^{UP}(\theta - p_2)\,|U\right] \tag{5}$$
$$\max_{h_2^{UP}} \pi_2^{UP} = \mathbb{E}\left[h_2^{UP}(\theta - p_2)\,|y_1, U\right].$$

The total order flow is

$$y_1 = h_1^{RK} + h_1^{RP} + s_1 \text{ for } R = \{I, U\} \tag{6}$$
$$y_2 = \begin{cases} s_2 \text{ if } R = I \\ h_2^{UP} + s_2 \text{ if } R = U, \end{cases}$$

where s_t is date $t \in \{1, 2\}$ demand by noise traders.[18] We assume that noise traders demand is drawn from a normal distribution with mean zero and variance σ_s^2, serially uncorrelated and independent of fundamental and state. We denote the probability density function with $\varphi_s(s_t)$ for $t \in \{1, 2\}$. While being a standard assumption in this literature, there is also a natural economic argument for this choice of distribution. As we interpret noise trading as the total demand by a large number of small traders who trade for idiosyncratic reasons unrelated to the fundamental (such as liquidity shocks, private values, etc.), the normality of the distribution of noise trading follows directly from the central limit theorem.

Technically, a useful property of the normal distribution is that it is strictly log-concave, allowing us to use some general properties of log-concave functions.[19] Log-concavity of noise trading also guarantees some desirable properties of the model, and we discuss generality further in Section 4.3.[20]

Further, while in settings with known types the total order flow provides noisy information about the fundamental, θ, it also reveals noisy information about P's type. By the law of total

[17] We condition P's expectation on the order flow (instead of the price or both), because date 1 order flow is always at least as informative as date 1 price. Provided that price is monotonic in the order flow, the two have the same information content.

[18] As usual, the presence of noise traders is needed to avoid the Grossman and Stiglitz's (1980) paradox about the impossibility of a fully revealing price in equilibrium.

[19] A function $f(\mathbf{x})$ (where x is a n-component vector) is log-concave if $\ln(f(\mathbf{x}))$ is concave. In the univariate and differentiable case, the following are equivalent: 1) $\partial^2 \ln(f(x))/\partial x \partial x < 0$, 2) $f'(x)/f(x)$ is decreasing in x, 3) $f''(x) f(x) - (f'(x))^2 < 0$. It is easy to verify that the normal distribution, $\varphi_s(.)$, is logconcave.

[20] Many other well known distributions are log-concave and symmetric. Notable examples include the beta (with parameters $\alpha = \beta > 1$) and the truncated normal. See Bagnoli and Bergstrom (2005) for an overview and further examples of log-concave densities.

expectations, the Market efficiency condition (2) can be expanded as

$$p_1 = \mathbb{E}[\theta|y_1] = Q_1\mathbb{E}[\theta|y_1, I] + (1 - Q_1)\mathbb{E}[\theta|y_1, U] \tag{7}$$
$$p_2 = \mathbb{E}[\theta|y_1, y_2] = Q_2\mathbb{E}[\theta|y_1, y_2, I] + (1 - Q_2)\mathbb{E}[\theta|y_1, y_2, U],$$

where $Q_1 \equiv \Pr(I|y_1)$ and $Q_2 \equiv \Pr(I|y_1, y_2)$ are the probabilities of P being informed conditional on the observed total order flows. We also use notation $p_1(y_1)$, $p_2(y_2)$, $Q_1(y_1)$ and $Q_2(y_2)$ to express these prices and probabilities as functions of contemporaneous order flows.

To summarize the setup, the timing of events is as follows:

- **date 0** - Nature draws $R \in \{I, U\}$ and θ. K and P learn R. If $R = I$, then both K and P learn θ. If $R = U$, only K learns θ.
- **date 1** - K, P, and noise traders submit market orders before knowing the price. The Market observes total order flow and sets the price p_1 based on the market efficiency condition (2).
- **date 2** - Noise traders submit market orders. If $R = U$, then P also submits a market order before knowing the price. The Market observes total order flow and sets the price p_2 based on the market efficiency condition (2).
- **date 3** - uncertainty resolves and P and K consume profits given the realization of θ.

As standard in the literature we focus on equilibria in pure strategies by K and P, and we proceed by backward induction.

3.1 Date 2 problem

Assume $\eta > 0$ and notice that all date 1 quantities, $\mathbb{E}[\theta|y_1, R]$, p_1 and $Q_1 = \Pr(I|y_1)$ can only depend on y_1 and are therefore known to P and the Market before date 2. Date 2 problem is only interesting if there is a difference between P's and the Markets expectations about the fundamental ($\mathbb{E}[\theta|y_1, U] \neq p_1$ or equivalently $\mathbb{E}[\theta|y_1, U] \neq \mathbb{E}[\theta|y_1, I]$) and the Market has not fully learned P's type ($Q_1 > 0$). For now, we conjecture that this is the case. We verify it later when analyzing the date 1 problem.

As there is no informed trading at date 2, it holds that conditional on a given state $R \in \{I, U\}$ and y_1, the date 2 order flow only depends on θ through y_1, which is already incorporated in prices and expectations and therefore $\mathbb{E}[\theta|y_1, y_2, R] = \mathbb{E}[\theta|y_1, R]$. Using (7) we obtain

$$p_2 = p_1 + \frac{(Q_1 - Q_2)}{Q_1}(\mathbb{E}[\theta|y_1, U] - p_1). \tag{8}$$

Clearly prices change between date 1 and 2 only if $Q_2 \neq Q_1$, which implies that they only change if the Market updates its beliefs about P's type after observing date 2 order flow. Provided that the true state is $R = U$, the Market updates in the "correct" direction if $Q_2 < Q_1$. In such case prices increase (decrease) if $\mathbb{E}[\theta|y_1, U] > (<) \, p_1$. Using (8), we can restate P's problem (5) as

$$\begin{aligned} \max_{h_2^{UP}} \pi_2^{UP} &= h_2^{UP} \mathbb{E}[Q_2|y_1, U] \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1} = \\ &= h_2^{UP} \left(\int_{-\infty}^{\infty} Q_2 \left(h_2^{UP} + s_2 \right) \varphi_s(s_2) \, ds_2 \right) \frac{(\mathbb{E}[\theta|y_1, U] - p_1)}{Q_1}. \end{aligned} \tag{9}$$

We can make some immediate observations. Suppose that $\mathbb{E}[\theta|y_1, U] > p_1$, i.e., uninformed P expects the fundamental to be higher than date 1 price. On the one hand, P can profit from trading any positive quantity. Ignoring the effect of his trade on $Q_2(.)$ would make him to want to buy an infinitely large quantity of the asset at date 2. On the other hand, the term $\mathbb{E}[Q_2|y_1, U]$ captures the expected updating of P's type by the Market. Because Q_2 depends on date 2 order flow, P knows that his trade will affect the Markets' beliefs about his type. Since these beliefs directly affect p_2, one would expect the traditional trade-off between transaction size and information disclosure to be present, namely, that increasing order size increases the extent of potential profits for the informed trader but also reveals to the Market the trader's private information, which implies that prices will move in the opposite direction and limit the informed trader's profits. To establish this formally we need to investigate further the properties of Q_2.

As we focus on pure strategies and uninformed P's trading strategy, we can see that the beliefs of the Market are characterized by the quantity it expects P to trade. Thus, consider that the market expects P to trade some quantity $\bar{h}_2$, whereby $\bar{h}_2$ can take any value in $\mathbb{R}$.[21] Then, from (6) $y_2 = \bar{h}_2 + s_2$ if $R = U$ and $y_2 = s_2$ if $R = I$, we can derive Q_2 by using Bayes' rule, as

$$Q_2 = \frac{Q_1 f(y_2|y_1, I)}{Q_1 f(y_2|y_1, I) + (1 - Q_1) f(y_2|y_1, U)} = \frac{Q_1}{Q_1 + (1 - Q_1) r(y_2)}, \tag{10}$$

where

$$r(y_2) \equiv \frac{\varphi_s \left(y_2 - \bar{h}_2 \right)}{\varphi_s(y_2)} \tag{11}$$

is the likelihood ratio and we used the fact that conditional on the state R the date 2 order flow is normally distributed with density $\varphi_s(.)$.

Lemma 1.1 *The following properties hold for* $Q_2 = \Pr(I|y_1, y_2)$

[21] Naturally, in equilibrium it must hold that the Market's expectations and P's optimal trade are internally consistent, i.e., $h_2^{UP} = \bar{h}_2$.

1. Q_2 *is decreasing (increasing) in* y_2 *for any* $\bar{h}_2 > (<)\, 0$.

2. *If* $\bar{h}_2 > 0$ *then* $Q_2 > (<)\, Q_1$ *for any* $y_2 < (>)\, \frac{\bar{h}_2}{2}$. *If* $\bar{h}_2 < 0$ *then* $Q_2 > (<)\, Q_1$ *for any* $y_2 > (<)\, \frac{\bar{h}_2}{2}$.

3. $Q_2\left(0\right) = Q_1 \cdot \left(Q_1 + \left(1 - Q_1\right) \frac{\varphi_s\left(\bar{h}_2\right)}{\varphi_s(0)}\right)^{-1} = Q_1 \cdot \left(Q_1 + \left(1 - Q_1\right) \exp\left(-\frac{\left(\bar{h}_2\right)^2}{2\sigma_s^2}\right)\right)^{-1}$,

4. *If* $\bar{h}_2 > (<)\, 0$ *then* $\lim\limits_{y_2 \to \infty} Q_2\left(y_2\right) = 0\, (= 1)$ *and* $\lim\limits_{y_2 \to -\infty} Q_2\left(y_2\right) = 1\, (= 0)$.

5. $Q_2\left(y_2\right)$ *is a log-concave function.*

Proof. Part 1: Differentiating and simplifying we obtain $\partial Q_2 / \partial y_2 = -Q_2^2 \left(1/Q_1 - 1\right) r'\left(y_2\right)$. Lemma A.1 in Appendix A shows that log-concavity of φ_s implies the monotone likelihood ratio property (MLRP), i.e., $r'\left(y_2\right) > (<)\, 0$ for any $\bar{h}_2 > (<)\, 0$. This is because $\varphi_s\left(\tilde{y}_2 - \bar{h}_2\right) / \varphi_s\left(\tilde{y}_2\right) > (<)\, \varphi_s\left(y_2 - \bar{h}_2\right) / \varphi_s\left(y_2\right)$ for any $\tilde{y}_2 > 0$ and $\bar{h}_2 > (<)\, 0$. Parts 2-4 are straightforward from (10), (11) and the expression for the normal density. Part 5: Taking logs and differentiating, we obtain that $\frac{\partial^2 \ln(Q_2)}{\partial y_2 \partial y_2} = -\frac{(1-Q_1)^2\left[(1/Q_1 - 1) r''(y_2) + r''(y_2) r(y_2) - (r'(y_2))^2\right]}{\left(Q_1 + (1-Q_1) r(y_2)\right)^2}$. It is sufficient to show that the likelihood ratio (11) is (at least weakly) log-convex.[22] Indeed from using the normal density in (11) we find that $\ln\left(r\left(y_2\right)\right)$ is linear in y_2 and therefore weakly convex. ■

Part 1 of Lemma 1.1 implies that the Market updates its beliefs about P's type (the state R) in a "sensible" manner. For example, if the Market believes that trader P in state $R = U$ trades a finite and positive quantity, then observing a higher order flow always leads the Market to assign a lower probability on P being informed. This also confirms that P indeed faces a meaningful trade-off - the presence of a profit opportunity gives P incentives to trade, but trading too aggressively will reduce P's expected profit as he expects the Market to assign a higher probability on him being uninformed and to adjust the price accordingly. It is worth emphasizing that such realistic trade-off is always guaranteed because the likelihood ratio (11) is monotone (for a similar argument, see also Milgrom (1981)).[23]

While Bayesian updating itself guarantees that the Market updates its beliefs in the correct direction on average, we can see from part 2 of Lemma 1.1 that depending on the realized date 2 order flow, the Market can update the probability that P is informed, Q_2, in the "correct" or "incorrect" direction. This is because the total order flow includes a random noise trading component. Namely, if the realized order flow is relatively small (i.e., less than half of the volume

[22] $r\left(y_2\right)$ is log-convex if $\ln\left(r\left(y_2\right)\right)$ is convex. Equivalently, it must hold that $r''\left(y_2\right) r\left(y_2\right) - \left(r'\left(y_2\right)\right)^2 \geq 0$. This, together with $r\left(y_2\right) > 0$ also implies that $r''\left(y_2\right) > 0$.

[23] The monotone likelihood ratio property holds for the whole family of log-concave distributions, to which the normal belongs (see Lemma A.1 in Appendix A).

that the Market expects uninformed P to trade) or has an opposite sign to P's expected trade, then the Market updates in the "correct" direction if the state is $R = I$ and in the "incorrect" direction if the state is $R = U$. It is also immediate from parts 2-4 of Lemma 1.1 that the Market never learns P's type perfectly for finite order flows. Therefore, despite some learning about P's type, it is clear from (9) that P would always earn positive profits from trading any finite quantity that has the same sign as the difference $\mathbb{E}[\theta|y_1, U] - p_1$.

Part 4 of Lemma 1.1 confirms that the Market's learning about P's type is unbounded. This is necessary to guarantee that P has an incentive to trade a finite amount.[24]

While the previous analysis gives some confidence that it may be optimal for uninformed P to trade a finite quantity in equilibrium, it is not yet clear whether P's problem has a unique (interior) solution. Namely, from (9) P's expected profit involves an integral over a non-trivial function $Q_2(.)$ that depends on uninformed P's demand and is always positive for $h_2^{UP} > (<)\, 0$ provided that $(\mathbb{E}[\theta|y_1, U] - p_1) > (<)\, 0$.

Lemma 1.2 *If* $(\mathbb{E}[\theta|y_1, U] - p_1) > (<)\, 0$ *then uninformed* P*'s expected profit (9) is strictly log-concave in* $h_2^{UP} > (<)\, 0$.

Proof. Assume without loss of generality that $(\mathbb{E}[\theta|y_1, U] - p_1) > 0$ and $h_2^{UP} > 0$. Taking logs of (9), we obtain $\ln\left(\pi_2^{UP}\right) = \ln\left(h_2^{UP}\right) + \ln\left(\mathbb{E}[Q_2|y_1, U]\right) + \ln\left(\mathbb{E}[\theta|y_1, U] - p_1\right) - \ln\left(Q_1\right)$ and $\partial^2 \ln\left(\pi_2^{UP}\right)/\partial h_2^{UP}\partial h_2^{UP} = -\left(h_2^{UP}\right)^{-2} + \partial^2 \ln\left(\mathbb{E}[Q_2|y_1, U]\right)/\partial h_2^{UP}\partial h_2^{UP}$, which is negative if $\mathbb{E}[Q_2|y_1, U]$ is log-concave. By change of variables $y_2 = h_2^{UP} + s_2$, we can express

$$\mathbb{E}[Q_2|y_1, U] = \int_{-\infty}^{\infty} Q_2(y_2)\, \varphi_s\left(y_2 - h_2^{UP}\right) dy_2. \tag{12}$$

Using Theorem 6 of Prékopa (1973), restated as Theorem A.2 in Appendix A, a sufficient condition for (12) to be log-concave is that the function $Q_2(y_2)\, \varphi_s\left(y_2 - h_2^{UP}\right)$ is log-concave in h_2^{UP} and y_2. We can then derive $\partial^2 \ln \varphi_s\left(y_2 - h_2^{UP}\right)/\partial h_2^{UP}\partial h_2^{UP} = -\sigma_s^{-2}$ and $\partial^2 \ln \varphi_s\left(y_2 - h_2^{UP}\right)/\partial h_2^{UP}\partial y_2 = \partial^2 \ln \varphi_s\left(y_2 - h_2^{UP}\right)/\partial y_2 \partial h_2^{UP} = \sigma_s^{-2}$. As by part 5 of Lemma 1.1 $\partial^2 \ln\left(Q_2(y_2)\right)/\partial y_2 \partial y_2 < 0$, it is immediate that the Hessian[25] is negative definite, and therefore $\mathbb{E}[Q_2|y_1, U]$ and π_2^{UP} are log-concave. The proof for the case $(\mathbb{E}[\theta|y_1, U] - p_1) < 0$ and $h_2^{UP} < 0$ is similar. ■

Since any univariate log-concave function is also quasi-concave with a unique maximum, we can now state our main result[26]:

[24] Suppose instead that learning was bounded (i.e., Q_2 was such that $0 < |Q_2| < 1$ even at the limit when $y_1 \to \pm\infty$) and consider a candidate equilibrium where P trades a finite amount. It is easy to see that this cannot be an equilibrium as P would benefit from deviating to trade an infinite quantity. See also Section 5.

[25] The Hessian is $\begin{vmatrix} -\sigma_s^{-2} & \sigma_s^{-2} \\ \sigma_s^{-2} & \partial^2 \ln\left(Q_2(y_2)\right)/\partial y_2 \partial y_2 - \sigma_s^{-2} \end{vmatrix}$.

[26] An alternative proof of quasi-concavity is to require that the negative of the first derivative of the objective

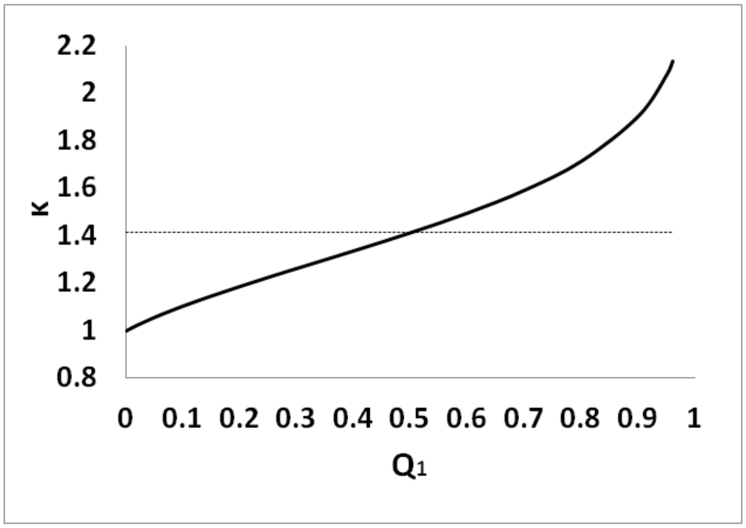


Figure 1: Uninformed P's trading volume at date 2.

Theorem 1 *Uninformed P's unique equilibrium strategy is to demand a finite amount*

$$h_2^{UP} = \bar{h}_2 = \begin{cases} \sigma_s \kappa \text{ if } \mathbb{E}[\theta|y_1, U] - p_1 > 0 \\ -\sigma_s \kappa \text{ if } \mathbb{E}[\theta|y_1, U] - p_1 < 0 \end{cases}, \tag{13}$$

where $\kappa > 1$ for any $Q_1 \in (0,1)$ and κ depends on Q_1 only.

Proof. It is immediate from (9) that $h_2^{UP} < (>)\, 0$ when $\mathbb{E}[\theta|y_1, U] - p_1 > (<)\, 0$ cannot be optimal as it leads to strictly negative profits. By Lemma 1.2, the uninformed P's problem then has a unique maximum at a non-negative h_2^{UP}. Therefore it is sufficient to look at the first order condition only and then impose that in equilibrium beliefs must be consistent with the optimal strategy $h_2^{UP} = \bar{h}_2$. The first order condition, the expression for κ and the proofs of the statements that $\kappa > 0$ and only depends on Q_1 are in Appendix B. ■

Because κ only depends on Q_1, it is most illustrative to present the solution[27] on a graph (see Figure 1). We find that whenever $\mathbb{E}[\theta|y_1, U] \neq p_1 = \mathbb{E}[\theta|y_1]$, it is generally optimal for uninformed P to trade at date 2. The volume traded by P is proportional to the standard deviation of noise trading and is increasing in Q_1. Both effects are intuitive. When the order flow is more noisy (high σ_s), it is harder for the Market to update its beliefs about the state and it is less costly for uninformed P to trade more aggressively. Because the Market's posterior

function is single crossing in h_2^{UP}. The sufficient conditions for single crossing under uncertainty have also been explored by Athey (2002) and Quah and Strulovici (2012). In both cases we can derive that log-concavity of $Q_2(.)$ and $\varphi_s(.)$ are sufficient for strict single crossing.

[27] It is relatively easy to show that if $Q_1 = 0.5$, then $\kappa = \sqrt{2}$, and if $Q_1 \to 0$, then $\kappa = 1$. The other values on Figure 1 are derived using numerical integration.

belief that the state is $R = I$, $Q_2 = \Pr(I|y_1, y_2)$, is increasing in Q_1 (its belief about P's type before date 2 trading), it is clear that a higher Q_1 also makes it less costly for an uninformed P to trade more aggressively as the Market is learning about his type more slowly. Overall P trades a finite quantity as he faces a trade-off between profiting from his superior information and revealing his type too much. This trade-off is fundamentally similar to the one in Kyle (1985), however differently from that setting P's private information is not about the fundamental directly, but about his impact or lack of impact on date 1 price.

It is worth noticing that at the limit, where $Q_1 \to 0$, the quantity traded by P at date 2 converges to exactly one standard deviation, while P's expected date 2 profit converges to zero. If P was known to be uninformed with probability one, i.e., $\eta = 0$, it would imply that $Q_1 = 0$, with probability one. The limit case and the certainty case are not the same. If P's type is known, it is easy to see that any known quantity traded by P (including zero trade, and one standard deviation trade) is an equilibrium, as P would always earn zero profit at any such equilibrium.[28] On the one hand this observation is consistent with Easley and O'Hara's (1991) argument that uninformed traders cannot benefit from trading against risk-neutral agents (the Market) who have at least as much information as they do. On the other hand, it highlights that even a very small probability of P being informed is sufficient for quantitative trading to exist in equilibrium, the traded quantity to be unique, and to generate strictly positive profits.

3.2 Date 1 problem

We define price-contingent strategies as follows:

Definition *P's date 2 strategy is called*[29]
- *trend-following (momentum) for some y_1 if $y_1 > 0$ and $h_2^{UP} > 0$, or $y_1 < 0$ and $h_2^{UP} < 0$*
- *contrarian for some y_1 if $y_1 > 0$ and $h_2^{UP} < 0$, or $y_1 < 0$ and $h_2^{UP} > 0$.*

Note that this definition allows equilibrium strategies to be non-monotonic in the order flow, as it is defined for each realization of $\mathbf{y_1}$. Provided the price is monotonic in the order flow (as it will be in equilibrium), it would be equivalent to define P's strategy through date 1 returns.

[28] When $\Pr(I) = \eta = 0$, then $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{f(y_1)} = 0$. From (9) and (10), it then follows that P's date 2 profit is zero regardless of the quantity it trades. As the Market's on-path beliefs about P's trade need to be consistent with P's strategy, we can construct equilibria with any known quantity traded by P. In all these equilibria, $p_2 = \mathbb{E}[\theta|y_1, y_2] = \mathbb{E}[\theta|y_1] = p_1$, as the Market cannot learn new information from y_2. Our earlier draft contained a formal proof (allowing for any prior, and assuming log-concave noise trading) that with $\eta = 0$, P cannot earn positive profits at either date. We have omitted this proof to save space.

[29] The words "momentum" and "contrarian" only refer to P's strategy. They should not be confused with positive and negative autocorrelation in returns. By the assumption of efficient markets (2), there is zero autocorrelation by construction. See also Section 3.4.

So far, we have shown that whenever the date 1 price differs from uninformed P's expectations of the fundamental P's optimal trading strategy is price-contingent. In this section we verify this conjecture and study the direction of P's trading. Given the symmetry of the distribution, it is natural to expect that in state $R = U$, the informed trader K's optimal demand is some real number $\bar{g}_U > 0$ if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $-\bar{g}_U$ if $\theta = -\bar{\theta}$; and uninformed trader P does not trade. In state $R = I$, the total demand by informed traders K and P is a real number $\bar{g}_I > 0$ if $\theta = \bar{\theta}$, zero if $\theta = 0$ and $-\bar{g}_I$ if $\theta = -\bar{\theta}$. Given these beliefs, we can derive the expressions and main properties of $\mathbb{E}[\theta|y_1, R]$, the price, and Q_1 as described by the following lemma.

Lemma 2.1 *For the equilibrium price and conditional expectations of the fundamental, it holds that*

1. *The price is given by*
$$p_1(y_1) = \bar{\theta}\frac{M_n(y_1) - M_p(y_1)}{M_n(y_1) + M_p(y_1)} \tag{14}$$
where $M_n(y_1) \equiv \frac{1-\gamma}{2}\left(\eta\varphi_s(y_1 - \bar{g}_I) + (1-\eta)\varphi_s(y_1 - \bar{g}_U) + \frac{\gamma}{1-\gamma}\varphi_s(y_1)\right)$
and $M_p(y_1) \equiv \frac{1-\gamma}{2}\left(\eta\varphi_s(y_1 + \bar{g}_I) + (1-\eta)\varphi_s(y_1 + \bar{g}_U) + \frac{\gamma}{1-\gamma}\varphi_s(y_1)\right)$;

2. *The conditional expectation of the fundamental is*
$$\mathbb{E}[\theta|y_1, R] = \bar{\theta}\frac{\varphi_s(y_1 - \bar{g}_R) - \varphi_s(y_1 + \bar{g}_R)}{\varphi_s(y_1 - \bar{g}_R) + \varphi_s(y_1 + \bar{g}_R) + \frac{2\gamma}{1-\gamma}\varphi_s(y_1)}; \tag{15}$$

3. *The updated probability of state $R = I$ is*
$$Q_1(y_1) = \Pr(I|y_1) = \frac{\eta\left(\varphi_s(y_1 - \bar{g}_I)\frac{1-\gamma}{2} + \varphi_s(y_1)\gamma + \varphi_s(y_1 + \bar{g}_I)\frac{1-\gamma}{2}\right)}{M_n(y_1) + M_p(y_1)}; \tag{16}$$

4. *The price is increasing in the order flow, i.e.,* $p_1'(y_1) > 0$;

5. *The price is symmetric around zero, i.e.,* $p_1(y_1) = -p_1(-y_1)$;

6. *It holds that* $\lim_{y_1 \to \infty} p_1(y_1) = \bar{\theta}$ *and* $\lim_{y_1 \to -\infty} p_1(y_1) = -\bar{\theta}$;

7. $\bar{\theta} - p_1(y_1) > 0$ *for all (finite)* y_1;

Proof. See Appendix B. ■

Lemma 2.1 confirms some reasonable and desirable properties of date 1 price, e.g., the price is increasing in the order flow, symmetric around zero and always between $-\bar{\theta}$ and $\bar{\theta}$. If the state is $R = U$, then the expected profit (4) of K can be written as

$$\pi_1^{UK} = h_1^{UK} \int_{-\infty}^{\infty} \left(\theta - p\left(h_1^{UK} + s_1\right)\right) \varphi_s\left(s_1\right) ds_1. \tag{17}$$

If the state is $R = I$ then the expected profit (3) of trader $J \in \{K, P\}$ can be written as

$$\pi_1^{IJ} = h_1^{IJ} \int_{-\infty}^{\infty} \left(\theta - p\left(h_1^{IK} + h_1^{IP} + s_1\right)\right) \varphi_s\left(s_1\right) ds_1. \tag{18}$$

We also need to explore whether the traders' objective function has a unique maximum. One sufficient condition for this would be that $\theta - p\left(y_1\right)$ is log-concave, but this only holds for some parameters. However log-concavity is only a sufficient, but not a necessary condition for a unique maximum. What we need is that the trader's profit is quasi-concave in own demand, i.e., that $-\frac{\partial \pi_1^{RJ}}{\partial h_1^{RJ}}$ is a single crossing function of h_1^{RJ}. In Appendix B we prove that this is always the case for $\theta = 0$ and we identify some conditions where this is also the case for $\theta = \left\{-\bar{\theta}, \bar{\theta}\right\}$. An intuitive sufficient condition is that the slope of the price does not decrease too rapidly, which is the case in the examples considered. Provided that the informed trader's problem has a unique maximum in own demand, we can state:

Proposition 2 *There is a pure strategy equilibrium at date 1, where the following holds:*

1. *Informed traders' demand is given by*

$$h_1^{UK} = \begin{cases} \bar{g}_U = \sigma_s \mu_U \text{ if } \theta = \bar{\theta} \\ 0 \text{ if } \theta = 0 \\ -\bar{g}_U = -\sigma_s \mu_U \text{ if } \theta = -\bar{\theta} \end{cases} \quad \text{and } h_1^{IK} = h_1^{IP} = \begin{cases} \frac{\bar{g}_I}{2} = \sigma_s \frac{\mu_I}{2} \text{ if } \theta = \bar{\theta} \\ 0 \text{ if } \theta = 0 \\ -\frac{\bar{g}_I}{2} = -\sigma_s \frac{\mu_I}{2} \text{ if } \theta = -\bar{\theta} \end{cases},$$

   where μ_U and μ_I only depend on η and γ.

2. *Total demand by informed traders in the event of news ($\theta = \bar{\theta}$ or $\theta = -\bar{\theta}$) is always higher in absolute value in state $R = I$ compared to state $R = U$, i.e., $g_I > g_U$ (equivalently $\mu_I > \mu_U$).*

3. *In state $R = U$, the uninformed trader P does not trade at date 1, i.e., $h_1^{UP} = 0$.*

Proof. See Appendix B. ■

Proposition 2 states some intuitive properties of date 1 equilibrium. First, informed traders face the standard trade-off as in Kyle (1985) and Holden and Subrahmanyam (1992). On the one hand, whenever they have private information that indicates $\theta \neq 0$ they earn positive expected

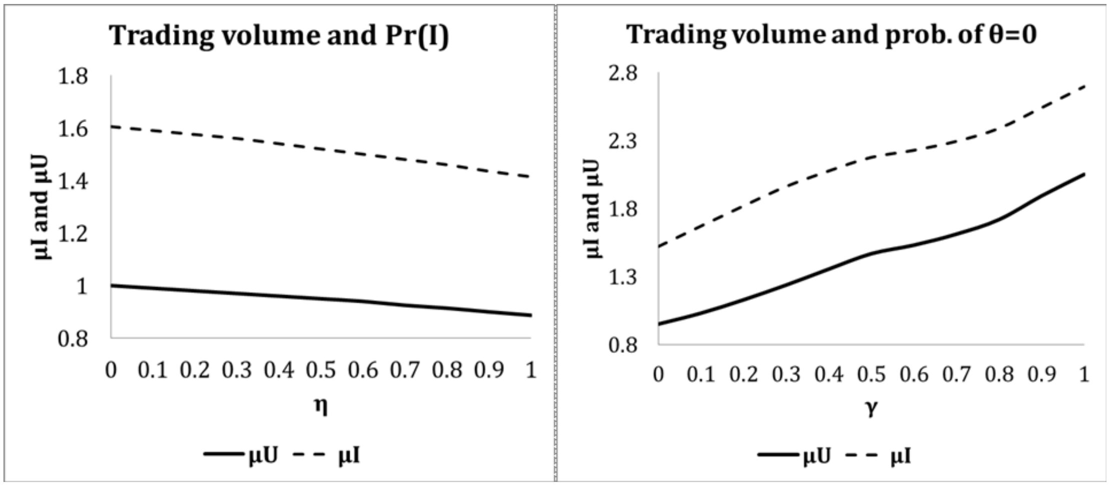


Figure 2: Informed trading volume for different values of η and γ.

profits from trading, so they have an incentive to trade a high volume. On the other hand, they know that due to market impact, trading a high volume reveals information about the fundamental (and also—less importantly for these traders—about the state R) to the Market. Therefore, they trade a finite amount and the price will not adjust immediately to equal the fundamental value.

The trading volume is always proportional to the standard deviation of noise trading. This is because informed traders benefit on average at the expense of noise traders and more noise allows them to hide private information more easily. Because the equilibrium price is proportional to the fundamental (see (14)), the magnitude of the fundamental value does not affect the informed trader's optimal strategy, but clearly profits are higher if $\bar{\theta}$ is higher. By Proposition 2 we know that the optimal strategy only depends on two parameters that are between 0 and 1. Figure 2 illustrates these dependences by plotting on the vertical axis μ_U and μ_I against η (on the left panel, assuming $\gamma = 0$) and against γ (on the right panel, assuming $\eta = 0.5$). These plots are qualitatively similar for different values of η and γ. First, if the prior probability of the state with two informed traders $R = I$ (i.e., $\Pr(I) = \eta$) is higher, then the informed traders are trading less aggressively. This is because $\bar{g}_I > \bar{g}_U$ and the Market expects more informed trading and is updating its beliefs faster. This in turn increases the informed traders' market impact and reduces their willingness to trade aggressively. Second, if the prior probability of "no news" (γ) is higher, the informed traders trade more aggressively whenever they observe $\theta \neq 0$. This is because by Bayes' rule the Market is relatively reluctant to update its beliefs toward the more extreme realizations of the fundamental. This reluctance reduces the market impact of the informed traders and gives them incentives to trade more.

The most important part of Proposition 2 is part 3 which states that the total order flow by informed traders is different in the two states, as we can see when comparing the expressions for

p_1, $\mathbb{E}[\theta|y_1, I]$ and $\mathbb{E}[\theta|y_1, U]$. If the state is $R = U$, then P again obtains superior information exactly because he knows that he did not trade and we can explore the direction of his trade at date 2.

3.3 Direction of price contingent trading

This section identifies the main non-monotonicity results. To build intuition, we start by examining the special case of the two-point prior, i.e., for now we take $\gamma = 0$.

Proposition 3 *When the prior distribution of the fundamental is a symmetric two-point distribution, it holds that*

$$\mathbb{E}[\theta|y_1, U] < (>)\, \mathbb{E}[\theta|y_1, I], \text{ for any } y_1 > (<)\, 0.$$

Whenever $0 < \eta < 1$, *the optimal strategy of* P *in state* $R = U$ *at date 2 is contrarian.*

Proof. See Appendix B. ■

With a discrete two-point distribution we find that if the true state is $R = U$, i.e., P is uninformed, then P's optimal strategy at date 2 is always contrarian. Note that when assuming a two-point prior we are focusing on an environment where any "news" about the fundamental is either "good" or "bad" and the Market always expects informed traders to trade. Any positive order flow is more likely to be associated with $\theta = \bar{\theta}$ compared to $\theta = -\bar{\theta}$. Furthermore, by part 2 of Proposition 2, we know that two informed traders would always trade a larger quantity in absolute value than one informed trader and therefore whenever the order flow is positive it holds that $\Pr\left(\bar{\theta}|y_1, I\right) > \Pr\left(\bar{\theta}|y_1, U\right)$ and $\Pr\left(-\bar{\theta}|y_1, I\right) < \Pr\left(-\bar{\theta}|y_1, U\right)$.[30] As the Market prices the asset considering that both states are possible, it tends to overprice the asset whenever the order flow is positive and the true state is $R = U$. Effectively in such state the Market tends to underestimate the probability that it was a positive noise trading shock rather than the demand of the informed traders to have generated a positive total order flow.

The above conclusion is specific to a two-point distribution with no mass in the center. With a three-point prior, we can establish some more general properties about the direction of price-contingent trading.

Proposition 4 *When the prior distribution of the fundamental is a symmetric three-point distribution and* $R = U$, *then for any* $0 < \eta < 1$ *the following conditions hold for different order flows*

[30] This is straightforward to verify using part 3 of lemma 3 and the properties of log-concave functions in Appendix A.

1. *For order flows* $|y_1| \geq \left|\frac{\bar{g}_I+\bar{g}_U}{2}\right|$, *P always pursues a contrarian strategy at date 2.*

2. *For order flows* y_1 *in the neighborhood of zero (i.e., for* $y_1 \to 0$*), P pursues a trend-following strategy at date 2 iff the following condition holds.*

$$\frac{1+\exp\left(\frac{\mu_I^2}{2}\right)\frac{\gamma}{1-\gamma}}{1+\exp\left(\frac{\mu_U^2}{2}\right)\frac{\gamma}{1-\gamma}} > \frac{\mu_I}{\mu_U} \tag{19}$$

3. *Provided that (19) holds, there exists a threshold order flow in the interval of* $\left(0, \frac{\bar{g}_I+\bar{g}_U}{2}\right)$ *below which P's optimal strategy is trend-following and above which it is contrarian.*

Proof. See Appendix B. ■

Proposition 4 shows that with a three-point prior both trend-following and contrarian strategies are possible at date 2. We also gain further insights on how the characteristics of the prior distribution drive the direction of price-contingent trading.

Part 1 of Proposition 4 shows that, when the date 1 order flow is large in absolute value, then at date 2 uninformed P always pursues a contrarian strategy. The reason for this is similar to our argument about the two-point prior. Intuitively, high order flows in state $R = U$ are relatively more likely to be driven by high noise trading shocks compared to what the Market expects. For example, if the true state is $R = U$, then any order flow that exceeds $\bar{g}_U$ must mean that there was a positive noise trading shock, while the Market will still consider order flows between $\bar{g}_U$ and $\bar{g}_I$ to be potentially reflecting small or even negative noise trading shocks. And this generates incentives for P to pursue a contrarian strategy.

Part 2 of Proposition 4 shows that if the probability of baseline news is large enough, then at least for small order flows uninformed P's optimal strategy at date 2 is trend-following. Namely, there is a threshold level for γ, above which this inequality (19) holds[31] and back-of-the-envelope calculations indicate that this threshold is quite low.[32] This observation highlights the fact that for trend-following trading there should be at least some mass in the center of the distribution. The intuition for why at small order flows it is optimal for P to pursue a trend-following strategy again relates to part 2 of Proposition 2. Consider for example a small positive order flow. If the true state is $R = U$, then the Market is now reluctant to believe that it is driven by informed traders who observed $\bar{\theta}$ (as it considers the possibility that two informed

[31] Note that the right hand side of (19) is always bigger than 1 as $\mu_I > \mu_U$ by point 2 in Proposition 3. The right hand side is 1 if $\gamma = 0$, strictly increasing in γ and converges to $\exp\left(\mu_I^2\right)/\exp\left(\mu_U^2\right)$ when $\gamma \to 1$. We can also verify that $\exp\left(\mu_I^2\right)/\exp\left(\mu_U^2\right) > \mu_I/\mu_U$ at the limit. This is because $\exp\left(\mu_R^2\right)/\mu_R$ is strictly increasing in μ_R for any $\mu_R > 0.5$. Hence (19) will hold at $\gamma \to 1$ if $\mu_I > \mu_U > 0.5$. It can also be shown that μ_U is at its lowest when $\eta = 1$ and $\gamma = 0$, and from Figure (2) that in such case μ_U is noticeably higher than 0.5.

[32] For example, if $\eta = 0.5$ then the threshold is around $\gamma \approx 0.21$.

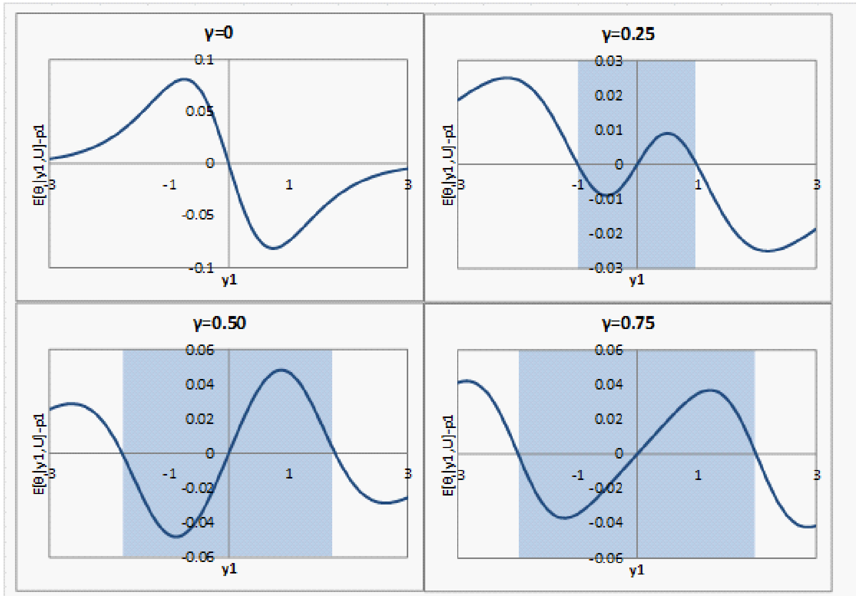


Figure 3: The difference $\mathbb{E}\left[\theta|y_1, U\right] - p_1$ for different values of γ.

traders who would trade a much larger quantity, $\bar{g}_I$ in total, while the actual informed trading could have been at most $\bar{g}_U$) and sets the price relatively close to zero. Because uninformed P knows at date 2 that his trading did not contribute to date 1 order flow, he benefits from trend-following trading on average. What is crucially different in the situation where not just extreme events, but also moderate events regarding the fundamental are possible is that the Market tends to underestimate the probability that there were very good (or bad) news regarding the fundamental when it observes a small order flow, as it would expect two informed traders to always trade much more aggressively than one. This tendency to underestimate the possibility of big fundamental news is what gives incentives for P to pursue a trend-following strategy.

Figure (3) illustrates the equilibrium difference, $\mathbb{E}\left[\theta|y_1, U\right] - p_1$, (vertical axis) for different values of γ, assuming that $\eta = 0.5$. On the horizontal axis, there is always the date 1 order flow, y_1 and the shaded area point out the values of y_1 where P's optimal strategy is trend-following. We can see that when γ is high enough, then there is a set of order flows around zero where $|\mathbb{E}\left[\theta|y_1, U\right]| > |p_1|$ and uninformed P's optimal strategy at date 2 is trend-following. As the informed trading volume is proportional to the standard deviation of noise trading, the values along the horizontal axis reflect the order flows normalized by the standard deviation of noise trading. We can see that already at $\gamma = 0.25$ order flows up to the magnitude of one standard deviation of noise trading will lead to trend-following trading. The magnitude of such order flows doubles if $\gamma = 0.75$. At very high order flows in absolute value, it always holds that $|\mathbb{E}\left[\theta|y_1, U\right]| < |p_1|$ and uninformed P's optimal strategy at date 2 is contrarian.

The three-point distribution also allows to derive richer empirical implications. We find that price-contingent traders are likely to react differently when they observe order flows of different magnitude. It is plausible to expect that quantitative traders who typically trade in the direction of past price changes will adjust their behavior and become contrarian at extreme order flows that are most likely driven by noise trading shocks.

3.4 Predictability of order flow and the effect of price-contingent trading on market efficiency.

Here we point out some natural consequences of equilibrium price-contingent trading under either the semi-strong or weak form of market efficiency.

Proposition 5 *While there is no predictability in returns, the order flow is predictable.*
Proof. The lack of predictability in returns is immediate and is due to imposing the efficient market condition (2). By construction $p_2 = \mathbb{E}[\theta|y_1, y_2]$ and $p_1 = \mathbb{E}[\theta|y_1]$, and by application of the law of iterated expectations, it is clear that $\mathbb{E}[p_2 - p_1|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1, y_2]|y_1] - p_1 = \mathbb{E}[\theta|y_1] - p_1 = 0$. At the same time by Theorem 1 we know that if the state is $R = U$ then P trades at date 2 a known amount $\bar{h}_2$. Therefore, $\mathbb{E}[y_2|y_1] = \Pr(I|y_1)\mathbb{E}[y_2|y_1, I] + \Pr(U|y_1)\mathbb{E}[y_2|y_1, U] = Q_1\mathbb{E}[s_2|y_1, I] + (1 - Q_1)\mathbb{E}\left[\bar{h}_2 + s_2|y_1, U\right] = (1 - Q_1)\mathbb{E}\left[\bar{h}_2|y_1, U\right] \neq 0$. ■

In Kyle (1985) and Holden and Subrahmanyam (1992) and subsequent models that build on their framework, imposing the market efficiency condition implies *both* the lack of predictability of returns *and* the lack of predictability of the order flow. This is because traders' types are known in these models. Matters differ considerably in our more general setting, because the Market cannot be perfectly sure of whether there is a price-contingent trader P or not, but the Market still knows that if there is one, he will trade in a predictable direction, described in Propositions 3 and 4. For example, if the optimal strategy is trend-following, the Market expects a positive order flow with some probability; if the actual order flow is instead zero, the price will fall as a result.

It should also be noted that the type of price-contingent trading we analyze as emerging in a fully rational setting without other frictions on average facilitates price discovery by moving prices closer to the fundamental. In state $R = U$, the best estimate of the fundamental conditional on all the information apart from the fundamental itself is $\mathbb{E}[\theta|y_1, U]$, and not $\mathbb{E}[\theta|y_1]$, so that uninformed P's price-contingent trading on average pushes date 2 price p_2 closer to $\mathbb{E}[\theta|y_1, U]$.

Importantly, in our model there is also no sense in which contrarian trading is more stabilizing than momentum trading. For example, it is true that in our setting a rare situation can arise whereby prices change purely because of a noise trading shock and P's optimal trend-following strategy moves prices further away from the fundamental, but similarly there can be a rare situation whereby following some draws of noise trading P's optimal contrarian trading delays information about the fundamental from being reflected into prices. As a result, while both contrarian and momentum trading are on average stabilizing, both can end up pushing prices away from fundamentals.

4 Discussion of Special Cases and Extensions

In this Section we discuss some special cases and alternative assumptions and extensions. In Section 4.1 we allow both traders P and K to trade at date 2, too; in Section 4.2 we examine alternative distributions of the fundamental; and in Section 4.3 we discuss other assumptions such as number of traders, trading periods and distribution of noise trading.

4.1 Allowing both traders K and P to trade at date 2 when they are informed

We have derived our main results in Sections 3 and 4 under the assumption that K and P cannot trade at date 2 if they are informed. This way, we have derived our results in the clearest manner, abstracting from the well-studied incentives of strategic informed traders to split their orders. On the one hand, the assumption that informed traders can trade less frequently could be justified by arguing that simple price-contingent trades are easy to program in a quantitative and systematic manner and can therefore be implemented much faster than any fundamental information-based trades. On the other hand, it can also be argued that informed traders benefit from algorithms as well, as an appropriate algorithm is capable of quickly breaking up a large informed order into several smaller orders. Whichever view is more realistic, we show in this Section that allowing all traders to trade at both dates does not qualitatively alter our main findings.

Namely, we alter the assumptions (3) and (4), and assume instead that if $R = I$, then trader $J = \{K, P\}$ is forward-looking and solves

$$\max_{h_2^{IJ}} \pi_2^{IJ} = \mathbb{E}\left[h_2^{IJ}\left(\theta - p_2\right)|y_1, \theta, I\right] \text{ at date 2,} \tag{20}$$

$$\max_{h_1^{IJ}} \pi_1^{IJ} = \mathbb{E}\left[h_1^{IJ}\left(\theta - p_1\right) + h_2^{IJ}\left(\theta - p_2\right)|\theta, I\right] \text{ at date 1.}$$

If $R = U$, then informed trader K is also forward-looking and solves

$$\max_{h_2^{UK}} \pi_2^{UK} = \mathbb{E}\left[h_2^{UK}(\theta - p_2)|y_1, \theta, U\right] \text{ at date 2,} \tag{21}$$
$$\max_{h_1^{IJ}} \pi_1^{UK} = \mathbb{E}\left[h_1^{UK}(\theta - p_1) + h_2^{UK}(\theta - p_2)|\theta, U\right] \text{ at date 1.}$$

All other assumptions remain unchanged. First, we prove the following result.

Lemma 5 *Provided that $\eta > 0$, uninformed P trading zero at date 2 can not be an equilibrium strategy.*

Proof. See Appendix B. ■

This result is in the spirit of Theorem 1 - uninformed P has incentives to trade at date 2, as he has superior information compared to the Market. This result does not hinge on any particular distribution of the prior.

Next, we examine the direction of trading, and assume a three-point prior, (1). This problem is noticeably more complex than in the basic setting, and does not have a straightforward analytical solution - one complication being that all agents' trades at date 2 are non-trivial functions of date 1 order flow. We can nevertheless show that the demand of each trader must be proportional to the standard deviation of noise trading, as in the basic setting.

We then derive the expressions that characterize the equilibrium, and focus on the numerical results.[33] Similarly to the basic setting, uninformed P cannot gain from trading at date 1. At date 2, it is immediate from (5) that the trader P gains from buying (selling) the asset, i.e., $h_2^{UP} > (<)\, 0$, if and only if $\mathbb{E}[\theta|y_1, U] - \mathbb{E}[p_2|y_1, U] > (<)\, 0$ in equilibrium. Figure 4 plots these differences for different values of γ (the order flow is normalized by dividing it by σ_s, $\bar{\theta} = 1$ and $\eta = 0.5$) . Comparing Figure (3) and Figure (4), we can see that the results are remarkably similar. It remains true that P benefits from contrarian trading at high values of the order flow as well as in the case in which there is no mass in the centre of the prior distribution of the fundamental (low γ). If there is enough mass in the centre of distribution, then P benefits from trend-following trading at low absolute values of order flow. What changes is the magnitude - we can see that the difference between P's expectations of the fundamental and P's expectations of the price is smaller than in the basic setting.

In Appendix C we also report the equilibrium demand of traders at date 2 for different realizations of date 1 order flow and γ. The results are intuitive: 1) informed trading volume is decreasing in date 1 order flow (either because $\theta = \bar{\theta}$ and there is an increasingly lower possibility to buy the asset at a favorable price, or $\theta = \{-\bar{\theta}, 0\}$ and there are increasingly

[33] Technical notes with expressions that characterize the equilibrium and the MATLAB code that solves the problem are available as a technical supplement.

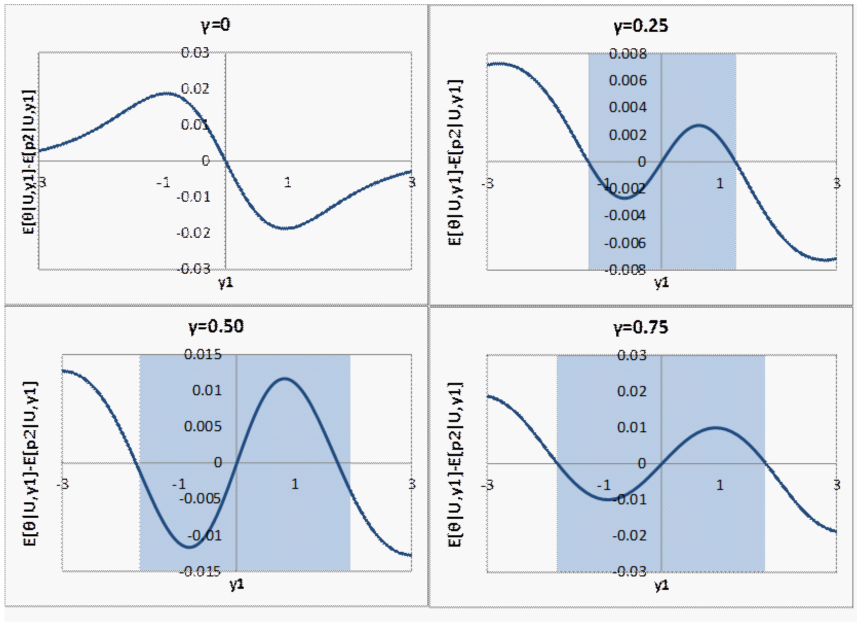


Figure 4: The difference $\mathbb{E}\left[\theta|y_1,U\right]-\mathbb{E}\left[p_2|y_1,U\right]$ for different values of γ

more possibilities to benefit by taking a position against date 1 noise trading shock); 2) as in the basic setting two informed traders demand a higher total quantity than one; 3) trading volume at $\theta=\left\{-\bar{\theta},\bar{\theta}\right\}$ is higher in absolute value if γ is higher. There is also a Kyle effect - informed traders strategically trade less in absolute value at date 1 than in the basic setting where splitting the orders is assumed away. Table 1 illustrates this by comparing informed trader's date 1 total demand in the basic setting of Section 3 and 4 with the setting in this Section.

Table 1: Total demand by informed traders at date 1[34]

	Basic setting		Forward looking	
state	R=U	R=I	R=U	R=I
$\gamma=0$	0.9496	1.5195	0.8698	1.4413
$\gamma=0.25$	1.1808	1.8931	0.9979	1.7199
$\gamma=0.5$	1.4683	2.1758	1.1566	1.9614
$\gamma=0.75$	1.7647	2.427	1.3658	2.2093

Due to the presence of informed traders, the volume traded by P at date 2 is smaller than in the basic setting. Nevertheless, allowing for the possibility of informed trading does not eliminate the possibility of profitable price-contingent trading.

When does K lose due to the presence of uninformed P? While the volume traded by uninformed P is small compared to the volume traded by informed K, it is intuitive that K's returns on information must be lower whenever P and K trade in the same direction, i.e., provided that date 1 order flow is positive, this holds for $\theta = \bar{\theta}$ whenever P's optimal strategy is trend-following and for $\theta = \left\{-\bar{\theta}, 0\right\}$ whenever P's optimal strategy is contrarian. In the case of other outcomes, K benefits not just from trading against noise traders, but also from taking an opposite position to uninformed P. If the state is $R = U$, then on average uninformed P gains from price-contingent trading, and therefore K typically earns lower profits than he would if he were the only strategic trader active at date 2. Nevertheless, competing with P reduces K's profits even more in the state $R = I$, where P is informed.

4.2 Alternative assumptions about the prior distribution of the fundamental

One may also wonder whether the main conclusions of our baseline setting are robust to considering other prior distributions. Let us consider any symmetric and continuous prior with density $f_\theta(\theta)$ and support $[-\bar{\theta}, \bar{\theta}]$, where $\bar{\theta} > 0$ is finite, and $f_\theta(\theta) > 0$ for any $\left(-\bar{\theta}, \bar{\theta}\right)$. As symmetry implies that $f_\theta(\theta) = f_\theta(-\theta)$, it follows that the prior mean $\mathbb{E}[\theta] = 0$, provided that it exists. We assume it to be the case and maintain all the other assumption of Section 3, such as θ being independent of s_t and R, and informed traders only trade at date 1. From the latter, it is clear that the date 2 problem remains unchanged, and as long as $\mathbb{E}[\theta|y_1, U] \neq p_1 \neq \mathbb{E}[\theta|y_1, I]$ and $Q_1 \neq 0$, P's date 2 trading strategy, if uninformed, is given by Theorem 1.

We proceed in two steps. First, by using insights from monotone comparative statics literature (see Milgrom and Shannon, 1994, and Edlin and Shannon, 1998), we can show that the Market's and P's expectations will be generally different, which then implies that P benefits from quantitative trading at date 2. Second, by exploiting the properties of the likelihood ratio

$$L(\theta) \equiv \frac{f(\theta|y_1, I)}{f(\theta|y_1, U)} = \frac{f(y_1|U)}{f(y_1|I)} \frac{\varphi_s(y_1 - g_I(\theta))}{\varphi_s(y_1 - g_U(\theta))}, \tag{22}$$

and highlighting some additional, but intuitive conditions, we can show that also our non-monotonicity results hold more generally.

In line with the standard approach in comparative statics, suppose that the informed trader's objective function is strictly quasi-concave and continuously differentiable. Let us further assume that $p_1'(y_1) > 0$.[35] We can then establish:

[35] Price being monotonically increasing in order flow is both intuitive and desirable in the context of trading in financial markets. Furthermore, this conjecture is consistent with Proposition 7. Namely, the claim that $g_R'(\theta) > 0$, follows only from supermodularity of trader's objective function. This in turn implies that $\mathbb{E}[\theta|y_1, R]$ is increasing in y_1, because the likelihood ratio $\frac{f(\tilde{y}_1|\theta,I)}{f(y_1|\theta,I)} = \frac{\varphi_s(\tilde{y}_1 - g_R(\theta))}{\varphi_s(y_1 - g_R(\theta))}$ is increasing in θ for any $\tilde{y}_1 > y_1$, and R. While $p_1(y_1)$ also depends on $Q_1(y_1)$, any effect trough this term is typically small.

Proposition 7 *If date 1 price is increasing in the order flow and the informed traders' problem is quasi-concave (with interior maximum), then:*

1. *The total demand of informed traders, $g_R(\theta)$ in state $R \in \{I, U\}$* is *strictly increasing in θ.*

2. *It holds that $g_I(\theta) > (<)\, g_U(\theta)$ for any $\theta > (<)\, 0$, and $g_I(0) = g_U(0) = 0$.*

Proof. See Appendix B. ■

The most important implication of Proposition 7 is that, holding fixed any realization of the fundamental, two informed traders trade a higher quantity in equilibrium than one informed trader. This implies that conditional on date 1 order flow y_1, the state R and the fundamental θ are not independent. In particular, $\Pr(U|\theta, y_1)$ is generally a function of θ.[36] This allows to conclude that in general there will be a difference between uninformed P's and the Market's expectations. Namely, it holds that

$$\mathbb{E}[\theta|y_1, U] - p_1 = \frac{\mathbb{E}[\theta \cdot \Pr(U|\theta, y_1)\,|y_1]}{\Pr(U|y_1)} - \mathbb{E}[\theta|y_1] = \frac{Cov(\theta, \Pr(U|\theta, y_1))}{\Pr(U|y_1)},$$

where we used the market efficiency condition (2) and Bayes' rule $f(\theta|y_1, U) = \frac{f(\theta|y_1)\Pr(U|\theta, y_1)}{\Pr(U|y_1)}$. As θ and $\Pr(U|\theta, y_1)$ are not independent, it implies that their covariance will generally not be zero. Furthermore, $\eta > 0$, and the ratio $\frac{f(y_1|U)}{f(y_1|I)}$ being finite for finite y_1, further implies $Q_1 > 0$.[37] Therefore, our main result that price-contingent trading is profitable in a setting where there is uncertainty about traders' types extends well beyond the baseline setting.

For further concreteness and inferences about the direction of P's trade, let us make the following conjectures about equilibrium strategies:

- Conjecture 1. $g_I'(\theta) > g_U'(\theta)$;

- Conjecture 2. $D(\theta) \equiv g_I(\theta) + g_U'(\theta)\frac{g_I(\theta) - g_U(\theta)}{g_I'(\theta) - g_U'(\theta)}$ is monotonically increasing in θ.

Conjecture 1 states that at any given value of θ, two informed traders' joint demand is more sensitive to θ, than one informed traders' demand. Or in other words, competing informed traders trade more aggressively jointly, which is intuitive considering the Cournot-like competition between these traders in our setting. For conjecture 2, notice that the slope of $D(\theta)$ is

[36] Using Bayes' rule and independence of θ and R, we find that $\Pr(U|\theta, y_1) = \frac{f_s(y_1 - g_U(\theta))(1-\eta)}{f_s(y_1 - g_U(\theta))(1-\eta) + f_s(y_1 - g_I(\theta))\eta}$

[37] Namely, from Bayes' rule and independence of θ and R, it follows that $Q_1 = \Pr(I|\theta, y_1) = \frac{\eta}{\eta + (1-\eta)\frac{f(y_1|U)}{f(y_1|I)}}$, where $\frac{f(y_1|U)}{f(y_1|I)} = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} f(\theta)\varphi_s(y_1 - g_U(\theta))s\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f(\theta)\varphi_s(y_1 - g_U(\theta))s\theta}$, which is clearly finite for finite y_1.

primarily driven by $g_I'(\theta) > 0$ and $g_U'(\theta) > 0$ (see Part 1 in Proposition 7). It further rules out major differences in second derivatives of informed traders' strategies, by imposing that $g_I''(\theta)$ cannot be too high compared to $g_U''(\theta)$.

For illustration, it is worth noticing that if informed trader's equilibrium strategies are reasonably well approximated by first (or second) order Taylor approximation around zero, these conjectures indeed hold. Namely, the first order Taylor approximation around zero gives $g_U(\theta) \approx \frac{\theta}{2\tilde{\lambda}}$ and $g_I(\theta) \approx \frac{2\theta}{3\tilde{\lambda}}$, where $\tilde{\lambda} = \int_{-\infty}^{\infty} p_1'(s_1) f_s(s_1) ds_1$ is the expected price impact around $\theta = 0$, and similar to "Kyle lambda". Because a symmetric fundamental implies symmetric strategies and equilibrium price, the second order Taylor approximation gives the same result. In such a case $g_I'(\theta) = \frac{2}{3\tilde{\lambda}} > \frac{1}{2\tilde{\lambda}} = g_U'(\theta)$, and $D(\theta) = \frac{7\theta}{6\lambda}$ is indeed increasing in θ. While it is unclear whether these conjectures hold with any $f_\theta(\theta)$, their validity can always be easily confirmed after numerically solving the problem.

This conjectures are sufficient for deriving clear predictions about the likelihood ratio $L(\theta)$, (see (22)). Namely, the log-likelihood is

$$\ln(L(\theta)) = \ln\left(\frac{f(\theta|y_1, I)}{f(\theta|y_1, U)}\right) = \ln\left(\frac{f(y_1|U)}{f(y_1|I)}\right) + \ln(\varphi_s(y_1 - g_I(\theta))) - \ln(\varphi_s(y_1 - g_U(\theta))).$$

Using the expression for the normal density, we obtain that

$$\frac{\partial \ln(L(\theta))}{\partial \theta} = \frac{(g_I'(\theta) - g_U'(\theta))}{\sigma_s^2}(y_1 - D(\theta)).$$

For clarity, let us focus on the case with $y_1 > 0$.[38] Under the conjectures above, it follows that if $y_1 \geq D(\bar{\theta})$, the likelihood ratio $L(\theta)$ is monotonically increasing. And if $0 < y_1 < D(\bar{\theta})$, the likelihood ratio $L(\theta)$ is unimodal with maximum at $\theta^* = D(y_1^{-1})$. As $sgn(\mathbb{E}[\theta|y_1, U] - p_1) = -sgn(\mathbb{E}[\theta|y_1, I] - \mathbb{E}[\theta|y_1, U])$, we focus on the latter difference. Integration by parts gives

$$\mathbb{E}[\theta|y_1, I] - \mathbb{E}[\theta|y_1, U] = \int_{-\bar{\theta}}^{\bar{\theta}} (F(\theta|y_1, U) - F(\theta|y_1, I)) d\theta, \tag{23}$$

which highlights that the difference in expectations is driven by properties of cumulative distribution functions.

The implications of a monotone likelihood ratio are well known from Milgrom (1981). This allows us to already conclude that if the prior distribution is bounded, then at least at high realizations of y_1, satisfying $y_1 \geq D(\bar{\theta})$, $F(\theta|y_1, I)$ first order stochastically dominates $F[\theta|y_1, U]$. It then follows that $\mathbb{E}[\theta|y_1, I] > \mathbb{E}[\theta|y_1, U]$ and at high enough order flows, uninformed P's optimal date 2 strategy is contrarian, because at high enough order flows $L(\theta)$ is monotonically

[38] Due to the symmetry of the problem, the case with $y_1 < 0$ is a mirror image of this case.

increasing in θ.[39]

The implications of a unimodal likelihood ratio are less commonly explored[40], but is highly relevant in cases where, as in our setting, we need to compare distributions with different dispersions. We derive the following general result:

Proposition 8 *If Condition 1 and 2 hold, and $0 < y_1 < D\left(\bar{\theta}\right)$, i.e., when $L(\theta)$ is unimodal, we obtain the following:*

1. *There exist order flows satisfying $\bar{y}_{co} \leq y_1 < D\left(\bar{\theta}\right)$, where $F(\theta|y_1, I) < F(\theta|y_1, U)$ for any $\theta \in (-\bar{\theta}, \bar{\theta})$, i.e., $F(\theta|y_1, I)$ first order stochastically dominates $F(\theta|y_1, I)$. The threshold $\bar{y}_{co}$ is a solution of $L\left(\bar{\theta}\right) = 1$.*

2. *There exist order flows satisfying $0 < y_1 \leq \bar{y}_{mo}$, where $F(\theta|y_1, I) > F(\theta|y_1, U)$ for any $\theta \in (-\bar{\theta}, \bar{\theta})$, i.e., $F(\theta|y_1, U)$ first order stochastically dominates $F(\theta|y_1, I)$. The threshold $\bar{y}_{co}$ is a solution of $L\left(-\bar{\theta}\right) = 1$.*

3. *If $\bar{y}_{mo} < y_1 < \bar{y}_{co}$, then $F(\theta|y_1, U) - F(\theta|y_1, I)$ is a single crossing function for any $\theta \in (-\bar{\theta}, \bar{\theta})$.*

Proof. See Appendix B. ■

Proposition 8 enables to further assess the sign of (23) as properties of cumulative distribution functions have straightforward implications on the difference in expected values in state I and U. In general, neither distribution first order stochastically dominates the other one over all possible realizations of the order flow. This is exactly what drives the non-monotonicity results that our model predicts.

Consistently with our baseline setting, Proposition 8 highlights that there are values of y_1 at which trend-following strategies are optimal, and these values are concentrated around zero. Namely, Part 2 of the proposition highlights that close to $y_1 = 0$, the distribution $F(\theta|y_1, U)$ first order stochastically dominates $F(\theta|y_1, I)$, which by (23) further implies that $E[\theta|y_1, I] < E[\theta|y_1, U]$. With a continuous prior and $f_\theta(\theta) > 0$ for $\left(-\bar{\theta}, \bar{\theta}\right)$, it is always possible that informed traders observe a realization of θ close to the prior mean $E[\theta] = 0$ and choose to trade a small amount, as in our baseline setting.

Part 1 and our overall analysis of the likelihood ratio properties further highlights that as long as the prior distribution is bounded, there exists a large enough date 1 order flow at which

[39] A similar result holds also when the distribution of noise trading is bounded but has sufficiently wider support compared to support of θ.

[40] Hopkins and Kornienko (2007) and Ramos, Ollero, and Sordo (2000) emphasize the importance of unimodal likelihood ratio, and derive related results. However, as these papers are mainly interested in second order stochastic dominance, their results are derived under the assumption that the sign of differences in expected values does not change. Instead, our problem requires assessing the sign of these differences.

P's optimal date 2 strategy must be contrarian, because the distribution $F(\theta|y_1, I)$ first order stochastically dominates $F(\theta|y_1, U)$. We would argue that assuming bounded support is more realistic, after all it is somewhat difficult to imagine an underlying asset with unbounded support. In any event, a similar appoach can also be applied to priors with unbounded support.[41]

Part 3 of Proposition 8 shows that there is more generally also an area where $F(\theta|y_1, U) - F(\theta|y_1, I)$ is single crossing in θ, which by (23) implies that the difference in expected values is a sum of a negative and a positive term, which can go either way. This implies that the order flows at which trend-following and contrarian strategies prevail are wider than those implied by part 1 and 2. How wide the areas where different strategies prevail are depends on the prior distribution $f_\theta(\theta)$. Given the intuition developed in Section 3, we would expect "hump-shaped" distributions to generate more trend-following strategies than "U-shaped" ones.

More technically, Parts 1 and 3 give examples of first order stochastic dominance that do not rely on the monotone likelihood ratio property. Indeed, the monotone likelihood ratio property is sufficient, but not necessary condition for first order stochastic dominance.

4.3 Number of traders, normal noise trading and other assumptions

We have assumed that in any state there is always an informed trader K. The reason was to guarantee that asset prices always reflect fundamental information at least from K, which P may learn from prices. The presence of trader K allows for a rich set of effects and generates a rationale for trend-following trading under some conditions. However, as we discussed in the introduction, uncertainty about P's type alone is sufficient for rational price-contingent trading to emerge and P's optimal strategy is always contrarian (see Supplementary Appendix).

It would be trivial to add more type K and type P traders. All the effects would be the same as long as the number of sophisticated traders of type K and P is finite. The reason why K and P trade finite amounts and earn returns on their information is because they have market impact and they are aware of it. If the number of type P traders were infinite, then they would be indistinguishable from the Market; if the number of type K traders were infinite, then in the limit prices would tend towards strong-form market efficiency, but also towards an information acquisition paradox in the spirit of Grossman and Stiglitz (1980). Second, it would also be possible to add more trading rounds in which uninformed P can trade. This would complicate

[41] In our earlier draft we derived an example with a Normal prior. While normality is a very common assumption in the literature (e.g., Kyle 1985, and Holden and Subrahmanyam 1992), this assumption is not innocuous in our setting. Not only does the Normal distribution have unbounded support, it also has thin tails. As a result, it can be shown that a normal prior implies that P's date 2 optimal strategiy is always trend-following. (These results are avalable upon request). We would further expect that even if the prior is unbounded, but has "fat tails", it is plausible that P's optimal strategy is non-monotonic for similar underlying mechanisms.

the model as P would likely have a Kyle's (1985) type of incentive to split his orders and reveal information more slowly. However, it is intuitive that the Market will then still be imperfectly and slowly learning about the true state until the price eventually converges to $\mathbb{E}[\theta|y_1, R]$.

As discussed in Section 3, we view the noise traders in our model as capturing a large number of traders who trade for idiosyncratic reasons outside the main focus of our model. Therefore, the main argument for assuming normally distributed noise trading stems from the central limit theorem. However, technically, many realistic properties of our model rely on the less restrictive assumption of log-concave noise trading. Indeed, a log-concave distribution guarantees that the Market updates at date 2 in the "correct direction" - that is, in state $R = U$, if trader P submits a positive quantity in equilibrium, then higher order flows at date 2 always signal a higher posterior probability that the state is indeed $R = U$. It also guarantees that the expected value $\mathbb{E}[\theta|y_1, R]$ is increasing in date 1 order flow, which in turn often implies that also the price is increasing in order flow. Both of these properties hold because log-concavity implies the monotone likelihood ratio property (MLRP). These properties are realistic in the context of financial markets and guarantee that sophisticated large traders in our model face a meaningful trade-off in the spirit of Kyle (1985). Namely, an informed trader (either directly due to superior information about the fundamental or indirectly due to superior knowledge of his own past actions) benefits from trading a higher volume due to positive expected returns, but trading a higher volume is costly due to market impact as it reveals more about his private information—whether about the fundamental or about his own type. In our proofs we frequently relied only on log-concavity rather than on the explicit form of the normal density.

5 Empirical Implications

5.1 Understanding Quantitative Trading

Our theory explains why quantitative trading is more profitable for large financial institutions than for to retail investors. While hedge funds and quantitative traders are shrouded in secrecy and systematic data is thus hard to come by, it is becoming increasingly evident that quantitative trading with algorithms generates large profits on a regular basis.[42] These regular profits are hard to reconcile with a view of quantitative trading as mere implementation of standard portfolio selection models, and do suggest the need to examine the micro-foundation of quantitative trading strategies.

Our paper does offer one such micro-foundation. In our equilibrium, when trader P ends

[42] Quantitative hedge funds such as Citadel, CQS, Renaissance Technologies, and others, which implement multiple trading strategies with a strong emphasis on directional trading, feature regularly among the top performing hedge funds, e.g., see http://media.bloomberg.com/bb/avfile/rMz9ZuocMhKo.

up uninformed about the fundamental, he trades a systematic non-zero quantity based on past prices, whose direction–trend-following or contrarian–depends on the magnitude of the order flow. Because this trade depends only on observable price movements and on P being uninformed, it can be implemented by an automated algorithm. By contrast, when P ends up informed (e.g., he becomes aware of rumors of a takeover bid), he trades on that information, thereby disregarding or overriding the algorithm (e.g., see the illustration based on rumors about the Merrill Lynch and Bank of America merger in Narang (2013, p.15-16)). Thus, in our equilibrium quantitative trading needs human supervision: it is the very possibility of submitting an informed order at some point that makes quantitative trading systematically profitable.

As a result, our model rationalizes price-contingent strategies by Commodity Trading Advisors (CTAs) in futures markets and by hedge funds such as AQR and others in equity markets. CTAs are popular recent investment vehicles that execute profitable trend-following strategies in futures markets at daily, weekly, and monthly frequencies (e.g., see Clenow (2013), and Baltas and Kosowski (2014)). It has also been observed that various hedge funds execute profitable contrarian strategies in equities at weekly (Lehmann (1990)) and monthly (Jegadeesh (1990)) frequency. Our model can account for both types of strategies, because the direction of quantitative trading is determined by two probabilities that can be assessed in the context of a given asset. The first one is the probability that there are privately informed traders. At high frequencies, where "news events" are very unlikely, the Market may be unsure whether there are any informed traders. In such case, we would expect contrarian quantitative trading as in the example in our Supplementary Appendix. The second one, provided that informed traders are present, is the probability at which private information is expected to confirm that the fundamental is close to its prior mean (i.e., how much mass is in the center of the distribution of the prior). If private information is likely to confirm the prior, we would expect to observe trend-following strategies, which can explain why CTAs profit from such strategies. If private information is more likely to indicate either "good" or "bad" news (e.g., a firm succeeding or failing in its' profitable takeover bid), then we would expect contrarian trading, which could explain some profitable strategies in equity markets. Our setting can further explain the profitability of more complex strategies that are non-linear and non-monotonic in past order flow or returns.[43]

More broadly, we can think of two ways to take our model to the data, depending on the type

[43] It seems that, at least anecdotally, some hedge funds do trade systematically in a price-contingent manner according to the predictions of our model. In an unsolicited personal communication, the Founder and Chief Investment Officer of a London-based quantitative fund wrote to us: "My own trading models explicitly use the same nonlinear trading functions that you show in the paper, i.e., sometimes trend following, other times contrarian." See also Martin and Bana (2012) and Martin and Zou (2012).

of data that is available to the econometrician, whether publicly available data or proprietary trading data.

Post-mortem testing. In this case, the econometrician observes ex post proprietary data about algorithms and actual trades that were implemented in the past, which were unavailable to the market maker and the other traders at the time of the trades. In this case the econometrician could, based on the algorithm and the actual trades, back out if the trades followed the algorithm or not (in which case they were likely informed ones); and if the trades followed the algorithm, whether their size and direction accord with the model's predictions. Because this data would refer to past trades and was unavailable to the market-maker at the time of the trades (hence, it is a '*post-mortem*' test), there would be no issue that the market maker could trade against the quantitative trader and undo its profits.

Real-time testing. If the econometrician and the market maker observe the same data, perhaps in real time, then the only way to test the model is to look at average observable quantities. In this sense, our main prediction is that trading volume should be serially correlated, and that the direction of the serial correlation should depend in a non-monotonic manner on the same forces that determine whether algorithmic trading is trend-following or contrarian. In fact, while in this case neither the econometrician nor the market maker observe which trader is informed and which is not, they both know that, if there is a prevalence of trend-following trading then trading volume will be positively auto-correlated, and if there is a prevalence of contrarian trading then trading volume will be negatively auto-correlated.

In the latter case of real-time testing, there are two main alternative hypotheses about the drivers of order flow predictability. The first alternative hypothesis is the behavioral view that order flow predictability should be driven by return predictability. Return predictability is ruled out in our model, but drives order flow predictability in the models of Barberis et al. (1998), Daniel et al. (1998), and Hong and Stein (1999). Therefore, observing order flow predictability without return predictability would be evidence consistent with our model and against the behavioral view.

Even observing order flow predictability without return predictability would not be uniquely consistent with our model. In fact, the other alternative hypothesis is that order flow predictability comes from serially correlated noise trading, as in, e.g., Cespa and Vives (2015). To dig deeper and try to distinguish whether the source of order-flow predictability is rational quantitative trading as in our model, or serially correlated noise trading, one can then look at whether order flow predictability comes with a prevalence of large trades that have market impact, as in our model, or with a prevalence of small orders with no market impact, as in Cespa and Vives (2015) and other models with serially correlated noise trading.

5.2 Market Quality and Crashes

In terms of the impact of the introduction of quantitative trading on various aspects of market quality such as volatility and liquidity, we find that quantitative trading is on average stabilizing, in the sense that price contingent trading typically moves prices closer to the fundamental, consistent with the empirical evidence of Hendershott et al. (2011) and the practitioners' accounts in Kissell (2014), Durenard (2013), and Narang (2013). However, there is a concern that in particular circumstances quantitative trading can propagate adverse negative shocks and generate instability, as in the Quant Meltdown of August 2007, and the Flash Crash of May 6, 2010. For example, the report on the "events of May 6," (CFTC and SEC (2010)) stated that a large 'mistaken' sell order triggered algorithms to start selling; soon after the volume of sell orders increased, and algorithms started to buy. Eventually, many algorithms incurred large losses and just stopped trading, so that the mismatch of supply and demand became so large that the entire system went to a halt for a few minutes.

Remarkably, while not specifically designed to describe these events, our model does capture some of their key features. First, quants did not trigger either episode—the trigger was a noise trading shock such as the 'mistake' by a large investor in 2010; and a series of large trades on the news of problems with subprime mortgages in 2007. Second, and consistent with our model, the initial response of quants in both cases was trend-following trading, as long as total order flow was 'small enough'. Third, and again consistent with our model, when total order flow became larger, quants started pursuing contrarian strategies. On the other hand, by its very design our model does not capture the failures of market efficiency that occurred when many quantitative strategies just stopped trading and prices could no longer equate supply and demand. Most important, though, the events of August 2007 and May 2010 underscore a key feature of our model: quantitative trading through algorithms is profitable on average, as it is better able to chase information than the rest of the market, but it can occasionally end up chasing noise trading shocks, thereby incurring losses.[44]

6 Concluding Remarks

We have presented a theory of quantitative trading as an automated price-contingent strategy under human supervision. We establish that price-contingent trading is the optimal strategy of large rational agents in a setting in which there is uncertainty about whether large traders are informed about the fundamental. We provide conditions under which price-contingent trading is trend-following (momentum) or contrarian in equilibrium. A robust implication of our results

[44]See Mendel and Shleifer (2012) for a related account of the Quant Meltdown of 2007.

is that the order flow is predictable from current prices even if the market is semi-strong efficient and future returns are thus unpredictable.

Our model explains why hedge funds and other large financial institutions who engage in quantitative trading strategies are systematically profitable; and it explains why the secrecy of their strategies, trading portfolios, and exposures is key to their success. By having market impact and by being relatively less known than other agents, hedge funds can learn any information that is reflected into prices better than any other investor who does not perfectly know their trading strategies and portfolios. As a result, hedge funds can successfully implement a broader range of strategies, such as trend-following and contrarian trading, than individual and retail investors without market impact that would lose money from those same strategies.

Our model suggests that quantitative trading does not need to reflect market inefficiency or manipulation. In fact, despite the assumption of strong-form market efficiency, the contrarian, momentum and non-monotonic trading strategies that we derive bear striking similarity to many data driven strategies used by quantitative funds.

Of course, in the real world quantitative strategies can be a lot more sophisticated than our equilibrium strategies, and can use as input an array of quantifiable public information in addition to prices and order flows. One robust insight of our model is that quantifiable information can arise from superior knowledge of market participants' trading styles rather than economic fundamentals as traditionally thought. Extending our model to capture the additional nuances of real-world quantitative strategies would seem to be an interesting area for future research.

A Background theorems and lemmas

Lemma A.1 *If* $f_s(.)$ is strictly log-concave, then it holds that

$$\frac{f_s(x_2-c_2)}{f_s(x_2-c_1)} > \frac{f_s(x_1-c_2)}{f_s(x_1-c_1)} \text{ for any } x_2 > x_1 \text{ and } c_2 > c_1. \tag{24}$$

Proof. By definition of log-concavity it must hold that
$\alpha \ln(f_s(x_1-c_2)) + (1-\alpha)\ln(f_s(x_2-c_1)) < \ln(f_s(\alpha(x_1-c_2)+(1-\alpha)(x_2-c_1)))$ and
$(1-\alpha)\ln(f_s(x_1-c_2)) + \alpha\ln(f_s(x_2-c_1)) < \ln(f_s((1-\alpha)(x_1-c_2)+\alpha(x_2-c_1)))$ for any $0<\alpha<1$. Let $\alpha = \frac{x_2-x_1}{x_2-x_1+c_2-c_1}$. Then $\ln(f_s(\alpha(x_1-c_2)+(1-\alpha)(x_2-c_1))) = \ln(f_s(x_1-c_1))$ and
$\ln(f_s((1-\alpha)(x_1-c_2)+\alpha(x_2-c_1))) = \ln(f_s(x_2-c_2))$. Adding up the inequalities, we obtain that $\ln(f_s(x_1-c_2))+\ln(f_s(x_2-c_1)) < \ln(f_s(x_1-c_1))+\ln(f_s(x_2-c_2))$. Exponentiating both sides and rearranging, we obtain (24). ■

Note that, in probability theory, this implies that if we interpret c as a signal about some random variable such that $x = c + s$, where the density $f_s(s)$ is strictly log-concave, then the conditional distribution of $f(x|c) = f_s(x-c)$ satisfies the strict monotone likelihood ratio property (MLRP).

Corollary A.1.1 *If* $f_s(.)$ is strictly log-concave and symmetric ($f_s(s) = f_s(-s)$), then for any $x > 0$, it holds that

$$f_s(x-c) > (<) f_s(x+c) \text{ for any } c > (<) 0$$

Proof. For the case $c > 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_2 > c_1 = 0$. By (24) $\frac{f_s(x-c)}{f_s(x)} > \frac{f_s(-x-c)}{f_s(-x)} = \frac{f_s(x+c)}{f_s(x)} \implies f_s(x-c) > f_s(x+c)$. For the case $c < 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_1 < c_2 = 0$ to obtain that $\frac{f_s(x)}{f_s(x-c)} > \frac{f_s(-x)}{f_s(-x-c)} = \frac{f_s(x)}{f_s(x+c)} \implies f_s(x+c) > f_s(x-c)$. ■

Theorem A.2 (Prékopa (1973) Theorem 6) Let $f(\mathbf{x},\mathbf{y})$ be a function of $n+m$ variables where $\mathbf{x}$ is an n-component and $\mathbf{y}$ is an m-component vector. Suppose that f is logarithmic concave in $\mathbb{R}^{n+m}$ and let A be a convex subset of $\mathbb{R}^m$. Then the function of the variable $\mathbf{x}$:

$$\int_A f(\mathbf{x},\mathbf{y})d\mathbf{y}$$

is logarithmic concave in the entire space $\mathbb{R}^n$.

B Proofs

Proof of the remaining parts of Theorem 1

Assume that $\mathbb{E}[\theta|y_1,U]-p_1>0$. It is clear from (9) that the optimal demand h_2^{UP} cannot be negative. Because by Lemma 1.2 P's problem at date 2 is log-concave, it is sufficient to explore the first order condition. Using (9), (10), (11), (12) and noticing that $\frac{\partial\varphi_s\left(y_2-h_2^{UP}\right)}{\partial h_2^{UP}}=\frac{y_2-h_2^{UP}}{\sigma_s^2}\varphi_s\left(y_2-h_2^{UP}\right)$, we obtain that

$$\frac{\partial\pi_2^{UP}}{\partial h_2^{UP}}=\int_{-\infty}^{\infty}\frac{Q_1\varphi_s(y_2)}{Q_1\varphi_s(y_2)+(1-Q_1)\varphi_s\left(y_2-\bar{h}_2\right)}\left(1-\frac{\left(h_2^{UP}\right)^2}{\sigma_s^2}+\frac{h_2^{UP}y_2}{\sigma_s^2}\right)\varphi_s\left(y_2-h_2^{UP}\right)dy_2 \quad (25)$$

Define $\kappa\equiv\frac{\bar{h}_2}{\sigma_s}$ and $z\equiv\frac{y_2}{\sigma_s}$, where $dy_2=\sigma_s dz$, which implies that $\varphi_s\left(y_2-\bar{h}_2\right)=\frac{1}{\sigma_s}\phi(z-\kappa)$ and $\varphi_s(y_2)=\frac{1}{\sigma_s}\phi(z)$, where $\phi(.)$ is the p.d.f. of a standard normal. The optimal demand must solve $\frac{\partial\pi_2^{UP}}{\partial h_2^{UP}}=0$ and it must hold in equilibrium that optimal demand $\left(h_2^{UP}\right)^*=\bar{h}_2=\kappa\sigma_s$. Using all this, in (25), we obtain that κ is the positive solution of

$$\int_{-\infty}^{\infty}\frac{Q_1\phi(z)}{Q_1\phi(z)+(1-Q_1)\phi(z-\kappa)}\left(1-\kappa^2+\kappa z\right)\phi(z-\kappa)\,dz=0, \quad (26)$$

which we know to be unique by Lemma 1.2. Because σ_s does not enter in (26), it also proves that P's demand is proportional to σ_s and only depends on Q_1. The proof for the case $\mathbb{E}[\theta|y_1,U]-p_1<0$ is similar and in such a case we need the unique negative solution of (26). It is easy to verify that if $\kappa>0$ solves (26), then also $-\kappa>0$ solves (26).

Next let us prove that $\kappa>1$ by contradiction. Suppose instead that $0<\kappa<1$ solves (26). From (26), it must then be the case that $\kappa\int_{-\infty}^{\infty}z\frac{Q_1\phi(z)}{Q_1\phi(z)+(1-Q_1)\phi(z-\kappa)}\phi(z-\kappa)\,dz<0$. Using that $\phi(.)$ is an even function, we can rewrite this as

$$\kappa\int_0^{\infty}zQ_1\phi(z)\left(\frac{1}{Q_1\frac{\phi(z)}{\phi(z-\kappa)}+(1-Q_1)}-\frac{1}{Q_1\frac{\phi(z)}{\phi(z+\kappa)}+(1-Q_1)}\right)dz<0$$

Because $\phi(.)$ is log-concave, it holds that $\phi(z-\kappa)>\phi(z+\kappa)$ for all $z,\kappa>0$ by Corollary A.1.1 from Appendix A. This implies that $\frac{1}{Q_1\frac{\phi(z)}{\phi(z-\kappa)}+(1-Q_1)}>\frac{1}{Q_1\frac{\phi(z)}{\phi(z+\kappa)}+(1-Q_1)}$. So all terms inside the integral are non-negative for all $z\geq 0$ (with strict inequality for $z>0$), which leads to a contradiction and therefore $0<\kappa<1$ does not hold.

Proof of Lemma 2.1

For parts 1-3 note that (7) implies that, $p_1(y_1)=Q_1\mathbb{E}[\theta|y_1,I]+(1-Q_1)\mathbb{E}[\theta|y_1,U]$. By the law of total expectations $\mathbb{E}[\theta|y_1,R]=\bar{\theta}\Pr\left(\theta=\bar{\theta}|y_1,R\right)-\bar{\theta}\Pr\left(\theta=-\bar{\theta}|y_1,R\right)$ and by Bayes' rule $\Pr(\theta|y_1,R)=\frac{1-\gamma}{2}f(y_1|\theta,R)/f(y_1|R)$, where $f\left(y_1|\theta=\bar{\theta},R\right)=\varphi_s(y_1-\bar{g}_R)$; $f(y_1|\theta=0,R)=\varphi_s(y_1)$; $f\left(y_1|\theta=-\bar{\theta},R\right)=\varphi_s(y_1+\bar{g}_R)$ and $f(y_1|R)=f\left(y_1|\theta=\bar{\theta},R\right)\frac{1-\gamma}{2}+f\left(y_1|\theta=-\bar{\theta},R\right)\frac{1-\gamma}{2}+\gamma f(y_1|\theta=0,R)$. By Bayes' rule $Q_1(y_1)=\frac{\eta f(y_1|I)}{\eta f(y_1|I)+(1-\eta)f(y_1|U)}$. Combining all this proves

parts 1-3. For part 4, note that $\partial\varphi_s(y_1-c)/\partial y_1 = -\frac{y_1-c}{\sigma_s^2}\varphi_s(y_1-c)$ for any constant c, and therefore $M_n'(y_1) = -\frac{y_1}{\sigma_s^2}M_n(y_1) + M_{ng}(y_1)$ and $M_p'(y_1) = -\frac{y_1}{\sigma_s^2}M_p(y_1) - M_{pg}(y_1)$, where $M_{ng}(y_1) \equiv \frac{1-\gamma}{2}\left(\eta\frac{\bar{g}_I}{\sigma_s^2}\varphi_s(y_1-\bar{g}_I) + (1-\eta)\frac{\bar{g}_U}{\sigma_s^2}\varphi_s(y_1-\bar{g}_U)\right) > 0$ and $M_{pg}(y_1) \equiv$
$\frac{1-\gamma}{2}\left(\eta\frac{\bar{g}_I}{\sigma_s^2}\varphi_s(y_1+\bar{g}_I) + (1-\eta)\frac{\bar{g}_U}{\sigma_s^2}\varphi_s(y_1+\bar{g}_U)\right) > 0$. Using the above and differentiating, $p_1'(y_1) = 2\bar{\theta}\frac{M_{ng}(y_1)M_p(y_1)+M_{pg}(y_1)M_n(y_1)}{(M_n(y_1)+M_p(y_1))^2} > 0$. Parts 5-7 are straightforward when using (14), the expression of the normal density, and the fact that $\varphi_s(.)$ is an even function that is always non-negative and positive at finite values.

Quasiconcavity of informed traders' date 1 problem.

Let us focus on the state $R = U$. If $\theta = 0$ then from (17), we obtain the first order condition $-\frac{\partial\pi_1^{UK}}{\partial h_1^{UK}} = \int_{-\infty}^{\infty}\left(p\left(h_1^{UK}+s_1\right) + h_1^{UK}p'\left(h_1^{UK}+s_1\right)\right)\varphi_s(s_1)\,ds_1 =$
$= \int_{-\infty}^{\infty}\left(p(y_1) + h_1^{UK}p'(y_1)\right)\varphi_s\left(y_1-h_1^{UK}\right)dy_1 = 0$. It is clear that $h_1^{UK} = 0$ satisfies the first order condition as $p(y_1)\varphi_s(y_1)$ is an odd function of y_1 and therefore $-\frac{\partial\pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=0} = 0$. Furthermore, the negative of the first derivative can be expressed as
$-\frac{\partial\pi_1^{UK}}{\partial h_1^{UK}} = \int_0^{\infty}p(y_1)\left(\varphi_s\left(y_1-h_1^{UK}\right) - \varphi_s\left(y_1+h_1^{UK}\right)\right)dy_1 + h_1^{UK}\int_{-\infty}^{\infty}p'(y_1)\varphi_s\left(y_1-h_1^{UK}\right)dy_1$. Note that $\varphi_s(.) \geq (>)\,0$, for all (some) y_1 and h_1^{UK}, $p(y_1) \geq (>)\,0$ for all (some) $y_1 \geq 0$ and $p'(y_1) \geq 0$ for all (some) y_1. Furthermore, from Corollary A.1.1 in Appendix A, we know that $\varphi_s\left(y_1-h_1^{UK}\right) > \varphi_s\left(y_1+h_1^{UK}\right)$ if and only if $h_1^{UK} > 0$ and $y_1 > 0$. Therefore, $-\frac{\partial\pi_1^{UK}}{\partial h_1^{UK}}$ is strictly single-crossing, which proves that the objective function (17) is quasiconcave and achieves the maximum at $h_1^{UK} = 0$.

If $\theta = \bar{\theta}$, then it is clear from (17) that $h_1^{UK} < 0$ cannot be the best response as it leads to negative expected profits, and there would be a profitable deviation to $h_1^{UK} = 0$. The negative of the first derivative is now $-\pi'\left(h_1^{UK}\right) = -\frac{\partial\pi_1^{UK}}{\partial h_1^{UK}} = \int_{-\infty}^{\infty}\left(p\left(h_1^{UK}+s_1\right) + h_1^{UK}p'\left(h_1^{UK}+s_1\right) - \bar{\theta}\right)\varphi_s(s_1)\,ds_1$
$= \int_{-\infty}^{\infty}\left(p(y_1) + h_1^{UK}p'(y_1) - \bar{\theta}\right)\varphi_s\left(y_1-h_1^{UK}\right)dy_1$. The solution on $-\pi'\left(h_1^{UK}\right) = 0$ is a unique maximum if $-\pi'\left(h_1^{UK}\right)$ is a strictly single crossing function—that is $-\pi'(h) \geq 0$ implies that $-\pi'\left(\tilde{h}\right) > 0$ for any $0 < h < \tilde{h}$. Using the expression for $-\pi'\left(h_1^{UK}\right)$ we require that
$\left(\tilde{h}-h\right)\int_{-\infty}^{\infty}p'(y_1)\varphi_s\left(y_1-\tilde{h}\right)ds_1 + \int_{-\infty}^{\infty}\left(hp'(y_1) + p(y_1) - \theta\right)\varphi_s\left(y_1-\tilde{h}\right)dy_1 > 0$. The first term is clearly positive. The second term can be written as
$\int_{-\infty}^{\infty}\left(hp'(y_1) + p(y_1) - \theta\right)\frac{\varphi_s\left(y_1-\tilde{h}\right)}{\varphi_s(y_1-h)}\varphi_s(y_1-h)\,dy_1$, where $\frac{\varphi_s\left(y_1-\tilde{h}\right)}{\varphi_s(y_1-h)}$ is increasing in y_1 due to log-concavity (see Lemma A.1).

Notice that if $hp'(y_1) + p(y_1) - \theta$ is single crossing in y_1, we can prove that this integral is non-negative similarly to Lemma 5 and Extension to Lemma 5 in Athey (2002). Namely, suppose that $hp'(y_1) + p(y_1) - \theta$ is single crossing in y_1 then there exists $y_1 = \bar{y}$ such that $hp'(y_1) + p(y_1) - \theta < (>)\,0$ for any $y_1 < (>)\,\bar{y}$. Furthermore, it is clear that $\frac{\varphi_s\left(y_1-\tilde{h}\right)}{\varphi_s(y_1-h)} <$

$(>) \frac{\varphi_s(\bar{y}-\tilde{h})}{\varphi_s(\bar{y}-h)}$ for any $y_1 < (>) \bar{y}$. Then we obtain that

$$\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s\left(y_1 - \tilde{h}\right)}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h)\, dy_1 =$$
$$\int_{-\infty}^{\bar{y}} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s\left(y_1 - \tilde{h}\right)}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h)\, dy_1 +$$
$$\int_{\bar{y}}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s\left(y_1 - \tilde{h}\right)}{\varphi_s(y_1 - h)} \varphi_s(y_1 - h)\, dy_1 >$$
$$\frac{\varphi_s\left(\bar{y} - \tilde{h}\right)}{\varphi_s(\bar{y} - h)} \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h)\, dy_1 = \frac{\varphi_s\left(\bar{y} - \tilde{h}\right)}{\varphi_s(\bar{y} - h)} \cdot (-\pi'(h)) \geq 0,$$

where the first inequality follows from the monotonicity of $\frac{\varphi_s(y_1-\tilde{h})}{\varphi_s(y_1-h)}$. Overall in such case $-\pi'(h) \geq 0$ indeed implies that $-\pi'\left(\tilde{h}\right) \geq 0$. Note that a sufficient (but not necessary) condition for $hp'(y_1) + p(y_1) - \theta$ to be single crossing in y_1 is that $\frac{\theta - p(y_1)}{p'(y_1)}$ is decreasing in y_1, i.e., $\theta - p(y_1)$ is log-concave.[45]

We can also identify a somewhat more general sufficient condition for the term $\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1-\tilde{h})}{\varphi_s(y_1-h)} \varphi_s(y_1 - h)\, dy_1$ to be non-negative using Chebyshev's integral inequality. Namely, using Theorem A.5 from Appendix A, it holds that the sufficient condition for $\int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1-\tilde{h})}{\varphi_s(y_1-h)} \varphi_s(y_1 - h)\, dy_1 \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h)\, dy_1 = -\pi'(h)$ is that for every t

$$\frac{\int_{-\infty}^{t} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h)\, dy_1}{\int_{-\infty}^{t} \varphi_s(y_1 - h)\, dy_1} \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h)\, dy_1.$$

This condition can also be written as

$$\mathbb{E}\left[hp'(h + s_1) + p(h + s_1) | s_1 \leq t - h\right] \geq \mathbb{E}\left[hp'(h + s_1) + p(h + s_1) | s_1 > t - h\right].$$

As $\mathbb{E}\left[p(h + s_1) | s_1 \leq t - h\right] \geq \mathbb{E}\left[p(h + s_1) | s_1 > t - h\right]$ due to the fact that the price is increasing in the order flow, this condition essentially requires that the slope of $p(y_1)$ at high order flows is not too small compared to the slope at small order flows and is less restrictive than requiring $hp'(y_1) + p(y_1) - \theta$ to be single crossing.

While numerically both sufficient conditions clearly hold for a wide set of parameters, to the best of our knowledge there are no more mathematical results that we can apply to our setting

[45] Using the results from Lemma 2.1, we can prove that this is indeed the case for η close to 0 or 1 and $\gamma = 0$.

to derive further analytical results. Overall, the necessary (and the least restrictive) condition for quasiconcavity is that if $-\pi'(h) > 0$ then

$\left(\tilde{h}-h\right)\int_{-\infty}^{\infty} p'(y_1)\,\varphi_s\left(y_1-\tilde{h}\right) ds_1 + \int_{-\infty}^{\infty}\left(hp'(y_1)+p(y_1)\right)\left(\frac{\varphi_s(y_1-\tilde{h})}{\varphi_s(y_1-h)}-1\right)\varphi_s(y_1-h)\,dy_1 > 0$,

which appears to always hold, at least numerically.

As the problem is symmetric, similar arguments apply for $\theta = -\bar{\theta}$ as well as for the quasiconcavity in own demand in the state $R = I$.

Proof of Proposition 2

We already know from the previous part that when $\theta = 0$, the unique solution is $h_1^{UK} = h_1^{IK} = h_1^{IP} = 0$. So let us focus on the case $\theta = \bar{\theta}$. Provided that the trader's problem has a unique maximum in own demand, we focus on the first order conditions.

$$\begin{aligned}
-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} &= \int_{-\infty}^{\infty}\left(p(y_1)+h_1^{UK}p'(y_1)-\bar{\theta}\right)\varphi_s\left(y_1-h_1^{UK}\right)dy_1 = 0 \\
-\frac{\partial \pi_1^{IJ}}{\partial h_1^{IJ}} &= \int_{-\infty}^{\infty}\left(p(y_1)+h_1^{IJ}p'(y_1)-\bar{\theta}\right)\varphi_s\left(y_1-h_1^{IK}-h_1^{IP}\right)dy_1 = 0
\end{aligned}$$

which by integration by parts can be also expressed as

$$\begin{aligned}
-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} &= \int_{-\infty}^{\infty}\left(\bar{\theta}-p(y_1)\right)\left(\frac{h_1^{UK}\left(h_1^{UK}-y_1\right)}{\sigma^2}-1\right)\varphi_s\left(y_1-h_1^{UK}\right)dy_1 = 0 \qquad (27) \\
-\frac{\partial \pi_1^{IJ}}{\partial h_1^{IJ}} &= \int_{-\infty}^{\infty}\left(\bar{\theta}-p(y_1)\right)\left(\frac{h_1^{IJ}\left(h_1^{UK}+h_1^{IK}-y_1\right)}{\sigma^2}-1\right)\varphi_s\left(y_1-h_1^{IK}-h_1^{IP}\right)dy_1 = 0
\end{aligned}$$

It is straightforward to prove that K and P and must trade the same quantity in equilibrium in state $R = I$ and that the solution is symmetric for $\theta = \bar{\theta}$ and $\theta = -\bar{\theta}$. In equilibrium the Market's beliefs must be consistent with optimal strategies, i.e., it must hold that $h_1^{IK} = h_1^{IP} = \frac{\bar{g}_I}{2}$ and $h_1^{UK} = \bar{g}_U$. Define $\mu_R \equiv \frac{\bar{g}_R}{\sigma_s}$ for $R \in \{I, U\}$ and $z \equiv \frac{y_2}{\sigma_s}$. Using the expression for normal density we can express $\varphi_s(y_2-\bar{g}_R) = \frac{1}{\sigma_s}\phi(z-\mu_R)$, $\varphi_s(y_2) = \frac{1}{\sigma_s}\phi(z)$ and $\varphi_s(y_2+\bar{g}_R) = \frac{1}{\sigma_s}\phi(z+\mu_R)$, where ϕ is the p.d.f. of a standard normal. Using (14), we then find that

$$\breve{p}(z) \equiv p_1(z\sigma_s) = \bar{\theta}\frac{\eta\phi(z-\mu_I)+(1-\eta)\phi(z-\mu_U)-\eta\phi(z+\mu_I)-(1-\eta)\phi(z+\mu_U)}{\eta\phi(z-\mu_I)+(1-\eta)\phi(z-\mu_U)+\eta\phi(z+\mu_I)+(1-\eta)\phi(z+\mu_U)+\frac{2\gamma}{1-\gamma}\phi(z)}$$

that clearly does not depend on σ_s and it holds that $p_1(z\sigma_s) = -p_1(-z\sigma_s)$. Using these in (27) and equating $\frac{\partial \pi_1^{IJ}}{\partial h_1^{IJ}} = 0$ for $J \in \{K, P\}$; $\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}} = 0$, we find that μ_I and μ_U are the positive solutions of

$$\frac{\partial \pi_1^{IJ}}{\partial h_1^{IJ}}|_{h_1^{IJ}=\bar{g}_I}= -\left(1-\frac{\mu_I^2}{2}\right)\int_{-\infty}^{\infty}\left(\bar{\theta}-\breve{p}\left(z\right)\right)\phi\left(z-\mu_I\right)dz+\frac{\mu_I}{2}\int_{-\infty}^{\infty}\left(\bar{\theta}-\breve{p}\left(z\right)\right)z\phi\left(z-\mu_I\right)dz=0 \tag{28}$$

$$\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_U}= -\left(1-\mu_U^2\right)\int_{-\infty}^{\infty}\left(\bar{\theta}-\breve{p}\left(z\right)\right)\phi\left(z-\mu_U\right)dz+\mu_U\int_{-\infty}^{\infty}\left(\bar{\theta}-\breve{p}\left(z\right)\right)z\phi\left(z-\mu_U\right)dz=0$$

For part 2 notice that from (28), we can express the first order condition of trader $J \in \{K,P\}$ in state $R=I$ as

$$\frac{\partial \pi_1^{IJ}}{\partial h_1^{IJ}}|_{h_1^{IJ}=\bar{g}_I} = -\frac{1}{2}\int_{-\infty}^{\infty}\left(\bar{\theta}-\breve{p}\left(z\right)\right)\phi\left(z-\mu_I\right)dz-\frac{1}{2}\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_I}=0$$

By part 6 of Lemma 2.1 $\left(\bar{\theta}-p_1\left(y_1\right)\right)>0$ for all finite y_1. Therefore, also $\left(\bar{\theta}-\breve{p}\left(z\right)\right)>0$ for all finite z and $\left(\bar{\theta}-\breve{p}\left(z\right)\right)\phi\left(z-\mu_I\right)\geq 0$ with strict inequality for some z. This implies that it must hold that

$$-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_I}>0.$$

Because $-\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}$ is a single-crossing function and $\frac{\partial \pi_1^{UK}}{\partial h_1^{UK}}|_{h_1^{UK}=\bar{g}_U}=0$, it then follows that $\bar{g}_I>\bar{g}_U$.

For the uninformed trader's strategy, we need to verify that it is optimal for him to trade zero. We now verify that the first order condition of his problem indeed holds at zero. Define $\Delta\left(y_1\right)\equiv\mathbb{E}\left[\theta|y_1,U\right]-\mathbb{E}\left[\theta|y_1,I\right]$ and $Q_{1U}\left(y_1\right)\equiv\mathbb{E}\left[Q_2|y_1,U\right]$. By (15) in Lemma 2.1 (and also by Lemma A.1 in Appendix A), it holds that $\Delta\left(y_1\right)=-\Delta\left(-y_1\right)$. Also, it is clear from (16) that it holds that $Q_1\left(y_1\right)=Q_1\left(-y_1\right)$. Using this in (10) and (12) we confirm that (12) $Q_{1U}\left(y_1\right)=Q_{1U}\left(-y_1\right)$.

Recalling uninformed P's optimal trading strategy at date 2 from (13) in Theorem 1 and using (7) and (9), we can then find P's expected profit at date 2 conditional on y_1 as

$$\pi_2^{UP}=\begin{cases}\sigma_s\kappa Q_{1U}\left(y_1\right)\Delta\left(y_1\right) & \text{if } \Delta\left(y_1\right)>0\\ -\sigma_s\kappa Q_{1U}\left(y_1\right)\Delta\left(y_1\right) & \text{if } \Delta\left(y_1\right)<0\end{cases}$$

Suppose that at date 1, uninformed P trades h_1^{UP}, then he also knows that the distribution of the total order flow is $f_s\left(y_1-h_1^{UP}-\bar{g}_U\right)$ if $\theta=\bar{\theta}$; $f_s\left(y_1-h_1^{UP}\right)$ if $\theta=0$ and $f_s\left(y_1-h_1^{UP}+\bar{g}_U\right)$ if $\theta=-\bar{\theta}$. Using all this, $\mathbb{E}\left[\theta|U\right]=0$, and we can use the law of iterated expectations to express the expected profit of uninformed P before date 1 trading as

$$\pi_1^{UP} = -h_1^{UP} \int_{-\infty}^{\infty} p_1(y_1)\left(m\left(y_1 - h_1^{UP}\right) - m\left(y_1 + h_1^{UP}\right)\right) dy_1 +$$
$$\int_{\Delta(y_1)>0} \sigma_s \kappa Q_{1U}(y_1)\,\Delta(y_1)\left(m\left(y_1 - h_1^{UP}\right) + m\left(y_1 + h_1^{UP}\right)\right) dy_1,$$

where $m(x) \equiv \frac{1-\gamma}{2}\varphi_s(x - \bar{g}_U) + \gamma\varphi_s(x) + \frac{1-\gamma}{2}\varphi_s(x + \bar{g}_U)$. Because of symmetry $\varphi_s(.,)$ it holds that $m(x) = m(-x)$ and $m'(x) = -m'(-x)$.

The first derivative of the profit is

$$\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}} = -\int_0^{\infty} p_1(y_1)\left(m\left(y_1 - h_1^{UP}\right) - m\left(y_1 + h_1^{UP}\right)\right) dy_1$$
$$+ h_1^{UP}\int_0^{\infty} p_1(y_1)\left(m'\left(y_1 - h_1^{UP}\right) + m'\left(y_1 + h_1^{UP}\right)\right) dy_1$$
$$- \int_{\Delta(y_1)<0} \sigma_s \kappa Q_{1U}(y_1)\,\Delta(y_1)\left(m'\left(y_1 - h_1^{UP}\right) - m'\left(y_1 + h_1^{UP}\right)\right) dy_1$$

Replacing in $h_1^{UP} = 0$, we can now verify that $\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}|_{h_1^{UP}=0} = 0$. For the intuition that $h_1^{UP} = 0$ is also a global maximum, notice that $-\int_0^{\infty} p_1(y_1)\left(m\left(y_1 - h_1^{UP}\right) - m\left(y_1 + h_1^{UP}\right)\right) dy_1 = -\int_{-\infty}^{\infty}\left(p_1\left(s_1 + h_1^{UP}\right) - p_1\left(s_1 - h_1^{UP}\right)\right) m(s_1)\, ds_1$. Due to increasing prices, the first term is negative iff $h_1^{UP} > 0$. Also, using integration by parts, the second term is
$h_1^{UP}\int_0^{\infty} p_1(y_1)\left(m'\left(y_1 - h_1^{UP}\right) + m'\left(y_1 + h_1^{UP}\right)\right) dy_1 = -h_1^{UP}\int_0^{\infty} p_1'(y_1)\left(m\left(y_1 - h_1^{UP}\right) + m\left(y_1 + h_1^{UP}\right)\right) dy_1$
and negative iff $h_1^{UP} > 0$. Both of these effects alone would guarantee that $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ is strictly single crossing at 0 as any trading by P at date 1 would lead to short term losses in expectations. The sign of the last term is ambiguous and reflects the fact that by deviating to a non-zero demand at date 1, P could affect the probability he expects the market to assign on him being informed at date 2 and the area where P would pursue different price-contingent strategies. However, it can be verified that this term is relatively small compared to the first two terms and $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ remains single crossing at 0. This is true at least as long as η is not too close to one.

Proof of Proposition 3

Assuming $\gamma = 0$, we obtain from (15) that $sgn\left(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]\right) = sgn\left(\frac{\varphi_s(y_1 - g_U)}{\varphi_s(y_1 + g_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(y_1 + \bar{g}_I)}\right) = sgn\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right)$. Because $\varphi_s(.)$ is log-concave and $\bar{g}_I > \bar{g}_U$ by part 2 in Proposition 2, it holds by the property of log-concave distributions in Lemma A.1 in Appendix A that $sgn\left(\frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(-y_1 - \bar{g}_U)} - \frac{\varphi_s(y_1 - \bar{g}_I)}{\varphi_s(-y_1 - \bar{g}_I)}\right) = -1$ if $y_1 > -y_1 \Leftrightarrow y_1 > 0$ and

$sgn\left(\frac{\varphi_s(y_1-\bar{g}_U)}{\varphi_s(-y_1-\bar{g}_U)} - \frac{\varphi_s(y_1-\bar{g}_I)}{\varphi_s(-y_1-\bar{g}_I)}\right) = 1$ if $y_1 < -y_1 \Leftrightarrow y_1 < 0$. By (7) $sgn\left(\mathbb{E}\left[\theta|y_1, U\right] - \mathbb{E}\left[\theta|y_1, I\right]\right) = sgn\left(\mathbb{E}\left[\theta|y_1, U\right] - p_1\right)$ for any $0 < Q_1 < 1$, which is true for any $0 < \eta < 1$. Uninformed P's optimal strategy at date 2, equation (13) in Theorem 1, and the definition of contrarian strategy in Section 4.2 complete the proof.

Proof of Proposition 4

To prove part 1 we use (15) to find that $sgn\left(\mathbb{E}\left[\theta|y_1, U\right] - \mathbb{E}\left[\theta|y_1, I\right]\right) = sgn\left(\frac{\varphi_s(y_1-\bar{g}_U)}{\varphi_s(y_1+\bar{g}_U)} - \frac{\varphi_s(y_1-\bar{g}_I)}{\varphi_s(y_1+\bar{g}_I)} + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1+\bar{g}_U)\varphi_s(y_1+\bar{g}_I)} B\left(y_1\right)\right)$, where $B\left(y_1\right) \equiv \varphi_s\left(y_1 - \bar{g}_U\right) - \varphi_s\left(y_1 + \bar{g}_U\right) - \varphi_s\left(y_1 - \bar{g}_I\right) + \varphi_s\left(y_1 + \bar{g}_I\right)$. Consider $y_1 > 0$ and let us focus on the sign of $B\left(y_1\right)$. Because $\varphi_s\left(.\right)$ has a maximum at zero and is decreasing for any positive values, it also holds for any $y_1 > 0$ and $\bar{g}_I > \bar{g}_U$ that $-\varphi_s\left(y_1 + \bar{g}_U\right) + \varphi_s\left(y_1 + \bar{g}_I\right) < 0$. We can then prove that the necessary and sufficient condition for $\varphi_s\left(y_1 - \bar{g}_U\right) - \varphi_s\left(y_1 - \bar{g}_I\right) \leq 0$ is that $y_1 \geq \frac{\bar{g}_I+\bar{g}_U}{2}$. Namely, defining $b \equiv y_1 - \frac{\bar{g}_I+\bar{g}_U}{2}$, it holds that $\varphi_s\left(y_1 - \bar{g}_U\right) - \varphi_s\left(y_1 - \bar{g}_I\right) = \varphi_s\left(b + \frac{\bar{g}_I-\bar{g}_U}{2}\right) - \varphi_s\left(b - \frac{\bar{g}_I-\bar{g}_U}{2}\right)$, which is indeed non-positive if and only if $b \geq 0$ (see Corollary A.1.1 in Appendix A and recall that $\bar{g}_I > \bar{g}_U$). Therefore, for any $y_1 \geq \frac{\bar{g}_I+\bar{g}_U}{2}$ it holds that $B\left(y_1\right) < 0$. From the proof of Proposition 3 (and Lemma A.1 in Appendix A), we already know that $\frac{\varphi_s(y_1-\bar{g}_U)}{\varphi_s(y_1+\bar{g}_U)} < \frac{\varphi_s(y_1-\bar{g}_I)}{\varphi_s(y_1+\bar{g}_I)}$ for any $y_1 > 0$. Therefore, $sgn\left(\mathbb{E}\left[\theta|y_1, U\right] - \mathbb{E}\left[\theta|y_1, I\right]\right) = sgn\left(\mathbb{E}\left[\theta|y_1, U\right] - p_1\right) = -1$ for any $y_1 \geq \frac{\bar{g}_I+\bar{g}_U}{2}$ and $0 < \eta < 1$. The definition of contrarian strategy in Section 4.2 completes this part of the proof. The proof for $y_1 \leq -\frac{\bar{g}_I+\bar{g}_U}{2}$ is similar due to symmetry.

To prove part 2, notice that the function determining the sign of $\mathbb{E}\left[\theta|y_1, U\right] - \mathbb{E}\left[\theta|y_1, I\right]$ can be expressed as

$$S\left(y_1\right) \equiv \left(\frac{\varphi_s(y_1-\bar{g}_U)}{\varphi_s(y_1+\bar{g}_U)} - 1\right)\left(1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1+\bar{g}_I)}\right) - \left(\frac{\varphi_s(y_1-\bar{g}_I)}{\varphi_s(y_1+\bar{g}_I)} - 1\right)\left(1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1+\bar{g}_U)}\right),$$

which using the expression for the normal density becomes

$$S\left(y_1\right) = \left(\exp\left(\frac{2\bar{g}_U y_1}{\sigma_s^2}\right) - 1\right)\left(1 + \frac{\gamma}{1-\gamma}\exp\left(\frac{2\bar{g}_I y_1 + \bar{g}_I^2}{2\sigma_s^2}\right)\right) - \left(\exp\left(\frac{2\bar{g}_I y_1}{\sigma_s^2}\right) - 1\right)\left(1 + \frac{\gamma}{1-\gamma}\exp\left(\frac{2\bar{g}_U y_1 + \bar{g}_U^2}{2\sigma_s^2}\right)\right)$$

It is clear that $S\left(0\right) = 0$. Let us consider ε arbitrarily close to zero. By Taylor approximation, we find that $S\left(\varepsilon\right) = S'\left(0\right)\varepsilon$, where $S'\left(0\right) = \frac{2\bar{g}_U}{\sigma_s^2}\left(1 + \frac{\gamma}{1-\gamma}\exp\left(\frac{\bar{g}_I^2}{2\sigma_s^2}\right)\right) - \frac{2\bar{g}_I}{\sigma_s^2}\left(1 + \frac{\gamma}{1-\gamma}\exp\left(\frac{\bar{g}_U^2}{2\sigma_s^2}\right)\right)$. Using then $\bar{g}_R = \mu_R \sigma_s$, it is clear that $S'\left(0\right) > 0$ iff (19) holds, which by $sgn\left(\mathbb{E}\left[\theta|\varepsilon, U\right] - \mathbb{E}\left[\theta|\varepsilon, I\right]\right) = sgn\left(S\left(\varepsilon\right)\right) = 1\left(-1\right)$ if $\varepsilon > (<) 0$ and definition from Section 4.2 implies a trend-following strategy.

If the condition (19) hold, then $S\left(\varepsilon\right) > 0$ for some small $\varepsilon > 0$, while $S\left(\frac{\bar{g}_I+\bar{g}_U}{2}\right) < 0$. As $S\left(y_1\right)$ is a continuous function of the order flow y_1, there must exist an order flow in the interval of $(\varepsilon, \frac{\bar{g}_I+\bar{g}_U}{2})$, where $S\left(y_1\right)$ changes its sign.

Proof of Proposition 6

Assume that the prior distribution of the fundamental is $f_\theta(\theta)$, with support $\left[-\bar{\theta},\bar{\theta}\right]$.[46] We prove the lemma by contradiction. Suppose that P does not trade at date 2 in state $R=U$. It is first useful to prove the following Claim

Claim 1 *The total volume traded by informed traders at date 2 differs across states.*

Proof. Let us differentiate (20) and (21) with respect to, h_2^{UK}, h_2^{IK} and h_2^{IP}. We can then define the total informed order flow at date 2, in state $R=U$ as $g_{2U}(\theta,y_1)=c$ and in state $R=U$ as $g_{2I}(\theta,y_1)=h_2^{IK}+h_2^{IP}$, we obtain that the following first order conditions must hold in equilibrium

$$\int_{-\infty}^{\infty}(\theta-p_2(y_2))\,\varphi_s(y_2-g_{2I}(\theta,y_1))\left(2+\frac{g_{2I}(\theta,y_1)(y_2-g_{2I}(\theta,y_1))}{\sigma_s^2}\right)dy_2=0$$
$$\int_{-\infty}^{\infty}(\theta-p_2(y_2))\,\varphi_s(y_2-g_{2U}(\theta,y_1))\left(1+\frac{g_{2U}(\theta,y_1)(y_2-g_{2U}(\theta,y_1))}{\sigma_s^2}\right)dy_2=0$$

and $g_{2I}(\theta,y_1)$, and $g_{2U}(\theta,y_1)$ are generally non-zero. Suppose that informed trading volume is the same across states: $g_{2I}(\theta,y_1)=g_{2U}(\theta,y_1)=g_2(\theta,y_1)$. From above we obtain then that it must hold that $\int_{-\infty}^{\infty}(\theta-p_2(y_2))\,\varphi_s(y_1-g(\theta,y_1))\,dy_2=0$ for all θ. But this leads to a contradiction, as it would imply that informed traders obtain zero profits when trading optimal quantity. Hence, it must be the case that $g_{2I}(\theta,y_1)\neq g_{2U}(\theta,y_1)$ ■

Let us then consider uninformed P's date 1 problem. For uninformed P not to have incentives to deviate from $h_2^{UP}=0$, it must hold that the first derivative of his date 2 problem (5) $\frac{\partial\pi_2^{UP}}{\partial h_2^{UP}}|_{h_2^{UP}=0}=\mathbb{E}\left[(\theta-p_2)\,|y_1,U\right]+\frac{\partial\mathbb{E}[(\theta-p_2)|y_1,U]}{\partial h_2^{UP}}|_{h_2^{UP}=0}=0$. This implies that

$$\begin{aligned}&\mathbb{E}\left[(\theta-p_2)\,|y_1,U\right]=\mathbb{E}\left[\mathbb{E}\left[(\theta-p_2)\,|\theta,y_1,U\right]|y_1,U\right]=\\&=\frac{1-\eta}{1-Q_1}\int_{-\bar{\theta}}^{\bar{\theta}}\int_{-\infty}^{\infty}(\theta-p_2)\,\varphi_s(y_1-g_{2U}(\theta,y_1))\,\varphi_s(y_1-g_{1U}(\theta))\,f_\theta(\theta)\,dy_2d\theta=0,\end{aligned}$$

where $g_{1U}(\theta)$ is date 1 total order flow by informed trader K. Denoting the date 1 order flow by informed traders in state $R=I$, with $g_{1I}(\theta)$,

$$\begin{aligned}&\mathbb{E}\left[(\theta-p_2)\,|y_1,I\right]=\mathbb{E}\left[\mathbb{E}\left[(\theta-p_2)\,|\theta,y_1,I\right]|y_1,I\right]=\\&=\frac{\eta}{Q_1}\int_{-\bar{\theta}}^{\bar{\theta}}\int_{-\infty}^{\infty}(\theta-p_2)\,\varphi_s(y_1-g_{2I}(\theta,y_1))\,\varphi_s(y_1-g_{1I}(\theta))\,f_\theta(\theta)\,dy_2d\theta\neq0,\end{aligned}$$

[46] We consider continuous distribution, but it is straightforward to extend the proof also to discrete distributions.

as it is sufficient that $\eta > 0$ and $g_{2I}(\theta, y_1) \neq g_{2U}(\theta, y_1)$. It must therefore also hold that

$$\mathbb{E}[(\theta - p_2)|y_1] = Q_1\mathbb{E}[(\theta - p_2)|y_1, I] + (1 - Q_1)\mathbb{E}[(\theta - p_2)|y_1, I] =$$
$$= Q_1\mathbb{E}[(\theta - p_2)|y_1, I] \neq 0$$

However, this leads to a contradiction because it violates the market efficiency condition that must hold in any equilibrium, i.e., by market efficiency conditions and law of iterated expectations, $\mathbb{E}[p_2|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1, y_2]|y_1] = \mathbb{E}[\theta|y_1] \iff \mathbb{E}[(\theta - p_2)|y_1] = 0$. Therefore $h_2^{UP} = 0$ cannot be an equilibrium strategy (for all realizations of y1).

Proof of Proposition 7

Part 1 follows from results are derived using insights from the monotone comparative statics literature. Consider state $R = U$ and denote trader K's expected price when demanding h_1^{UK} as $p_E\left(h_1^{UK}\right) \equiv \int_{-\infty}^{\infty} p_1\left(h_1^{UK} + s_1\right) f_s(s_1)\, ds_1$. We can express (4) as $g_U(\theta) = \arg\max_{h_1^{UK}} h_1^{UK}\left(\theta - p_E\left(h_1^{UK}\right)\right)$. From Milgrom and Shannon (1994) it is known that $g_U(\theta)$ is weakly increasing in θ if the trader's problem has increasing differences (which also implies the payoff is supermodular) in h_1^{UK} and θ. This is indeed the case, because for any $\tilde{\theta} > \theta$ and $\tilde{h}_1^{UK} > h_1^{UK}$, it holds that $\tilde{h}_1^{UK}\left(\tilde{\theta} - p_E(h_1^{UK})\right) - \tilde{h}_1^{UK}\left(\theta - p_E(\tilde{h}_1^{UK})\right) > h_1^{UK}\left(\tilde{\theta} - p_E(h_1^{UK})\right) - h_1^{UK}\left(\theta - p_E(h_1^{UK})\right) \iff (\tilde{h}_1^{UK} - h_1^{UK})(\tilde{\theta} - \theta) > 0$. From Edlin and Shannon (1998), it is also known that $g_U(\theta)$ is strictly increasing if the first derivative of the payoff (profit) is strictly increasing in θ, which is also true in our model, as $\partial h_1^{UK}\left(\theta - p_E\left(h_1^{UK}\right)\right)/\partial h_1^{UK} = \theta - p_E\left(h_1^{UK}\right) - h_1^{UK} p_E'\left(h_1^{UK}\right)$ is clearly increasing in θ. The proof is similar for the state $R = I$, where the same monotone comparative statics establish that K's and P's individual demand is increasing in θ, and so is the sum of their demands.

For part 2, notice that given the above assumptions, it is enough to only look at the first order conditions to find the unique equilibrium demands by all informed traders. Also, it is easy to verify that given θ, both K and P demand the same quantity in state $R = I$. We find that the equilibrium total informed demand $g_R(\theta)$ in state R solves

$$\theta = \int_{-\infty}^{\infty} \left(p_1(s_1 + g_U(\theta)) + g_U(\theta)\, p_1'(s_1 + g_U(\theta))\right) f_s(s_1)\, ds_1$$
$$\theta = \int_{-\infty}^{\infty} \left(p_1(s_1 + g_I(\theta)) + \frac{g_I(\theta)}{2} p_1'(s_1 + g_I(\theta))\right) f_s(s_1)\, ds_1$$

It is straightforward to verify that $g_R(\theta) = -g_R(-\theta)$. As by part 1 $g_R(\theta)$ is invertible, it must also hold that

$$g_U^{-1}(y_\theta) = \int_{-\infty}^{\infty} \left(p_1(s_1+y_\theta) + y_\theta p_1'(s_1+y_\theta)\right) f_s(s_1)\, ds_1, \text{ for } y_\theta \in \left[-g_U\left(\bar{\theta}\right), g_U\left(\bar{\theta}\right)\right] \tag{29}$$

$$g_I^{-1}(y_\theta) = \int_{-\infty}^{\infty} \left(p_1(s_1+y_\theta) + \frac{y_\theta}{2} p_1'(s_1+y_\theta)\right) f_s(s_1)\, ds_1, \text{ for } y_\theta \in \left[-g_I\left(\bar{\theta}\right), g_I\left(\bar{\theta}\right)\right]$$

For y_θ satisfying $-\min\left[g_U\left(\bar{\theta}\right), g_I\left(\bar{\theta}\right)\right] \le y_\theta \le \min\left[g_U\left(\bar{\theta}\right), g_I\left(\bar{\theta}\right)\right]$, it then follows that $g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta) = \frac{y_\theta}{2}\int_{-\bar{s}}^{\bar{s}} p_1'(s_1+y_\theta) f_s(s_1)\, ds_1 > (<)\, 0$ for any $y_\theta > (<)\, 0$. Taking $y_\theta = g_U(\theta) > 0$, we find that $g_U^{-1}(y_\theta) > g_I^{-1}(y_\theta) \iff \theta > g_I^{-1}(g_U(\theta)) \iff g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. The case $y_\theta < 0$ is immediate by symmetry. From this, it also follows that $\min\left[g_U\left(\bar{\theta}\right), g_I\left(\bar{\theta}\right)\right] = g_U\left(\bar{\theta}\right)$, and we have proved part 2 for any $-g_I^{-1}\left(g_U\left(\bar{\theta}\right)\right) \le \theta \le g_I^{-1}\left(g_U\left(\bar{\theta}\right)\right)$. Because $g_I(\theta)$ is increasing in θ, and y_θ outside this area cannot be generated in state $R = U$, the statement in Part 2 holds for any realization of θ.

Proof of Proposition 8

As in the main text, we focus on the case where $y_1 > 0$. Let us define the probability ratio as

$$P_F(\theta) \equiv \frac{F[\theta|I, y_1]}{F[\theta|U, y_1]}, \tag{30}$$

where $F[\theta|R, y_1]$ is c.d.f of θ conditional in R and y_1. Because $\theta|I, y_1$ and $\theta|U, y_1$ have the same support, it holds that $F\left[\bar{\theta}|R, y_1\right] = 1$, for $R = \{I, U\}$. This implies that $P_F\left(\bar{\theta}\right) = 1$, and that

$$\int_{-\bar{\theta}}^{\bar{\theta}} \left(f(\theta|I, y_1) - f(\theta|U, y_1)\right) d\theta = 0 \iff \int_{-\bar{\theta}}^{\bar{\theta}} (L(\theta) - 1) f(\theta|U, y_1)\, d\theta = 0.$$

For the latter to hold, it must be the case that the sign of $L(\theta) - 1$ is not the same over the full support of θ. This and unimodality of $L(\theta)$, further implies that $L(\theta^*) > 1$. By definition of c.d.f., it holds that $P_F\left(-\bar{\theta}\right) = L\left(-\bar{\theta}\right)$.

It is also useful to differentiate (30) to obtain

$$P_F'(\theta) = \frac{f[\theta|U, y_1]}{F[\theta|U, y_1]} \left(L(\theta) - P_F(\theta)\right). \tag{31}$$

Let us first consider the set of lower realization of the fundamental, i.e., we consider any $\theta \in (-\bar{\theta}, \theta^*]$, where $\theta^* = D\left(y_1^{-1}\right) > 0$ is the point where $L(\theta)$ achieves its maximum value. Denote a particular realization of θ that belong to this set with $\tilde{\theta}_L$. We can express $F\left(\tilde{\theta}_L|y_1, I\right) = \int_{-\bar{\theta}}^{\tilde{\theta}_L} f(\theta|I, y_1)\, d\theta = \int_{-\bar{\theta}}^{\tilde{\theta}_L} L(\theta) f(\theta|U, y_1)\, d\theta$. As $L(.)$ is continuous, and $f(\theta|U, y_1)$ is integrable, the first mean value theorem for definite integrals implies that there exists c_θ in $(-\bar{\theta}, \tilde{\theta}_L)$ such that

$$F\left(\tilde{\theta}_L|y_1,I\right)=L\left(c_\theta\right)\int_{-\bar{\theta}}^{\tilde{\theta}_L}f\left(\theta|U,y_1\right)d\theta=L\left(c_\theta\right)F\left(\tilde{\theta}_L|y_1,I\right)$$

From here, $P_F\left(\tilde{\theta}_L\right)=L\left(c_\theta\right)<L\left(\tilde{\theta}_L\right)$, because $L\left(.\right)$ is increasing the the area we consider. By (31), we then find that $P_F'\left(\theta\right)>0$ for any $\theta\in\left(-\bar{\theta},\theta^*\right]$.

Let us then consider the set of higher realization of the fundamental, i.e., $\theta\in\left(\theta^*,\bar{\theta}\right]$, and denote a particular realization of θ that belongs to this set with $\tilde{\theta}_H$. We need to analyze the case $L\left(\bar{\theta}\right)-1\geq 0$ and $L\left(\bar{\theta}\right)-1<0$, separately.

Case 1: Suppose that $L\left(\bar{\theta}\right)-1\geq 0$. Then unimodality of $L\left(.\right)$ implies that $L\left(\theta\right)-1$ is single crossing in interval $(-\bar{\theta},\bar{\theta}]$ and that the crossing point must be at the left of θ^*. It then follows that $L\left(\theta\right)-1>0$ for any $\theta^*<\tilde{\theta}_H\leq\bar{\theta}$ and we obtain that

$$F\left[\tilde{\theta}_H|I,y_1\right]-F\left[\tilde{\theta}_H|U,y_1\right]=\int_{-\bar{\theta}}^{\tilde{\theta}_H}\left(f\left(\theta|I,y_1\right)-f\left(\theta|U,y_1\right)\right)d\theta$$

$$=-\int_{\tilde{\theta}_H}^{\bar{\theta}}\left(f\left(\theta|I,y_1\right)-f\left(\theta|U,y_1\right)\right)d\theta=-\int_{\tilde{\theta}_H}^{\bar{\theta}}\left(L\left(\theta\right)-1\right)f\left(\theta|U,y_1\right)d\theta<0,$$

which implies that $P_F(\tilde{\theta}_H)<1$. By $L\left(\tilde{\theta}_H\right)<1$ and (31) it further implies that $P_F'\left(\tilde{\theta}_H\right)>0$. Combining this with the earlier result that $P_F'\left(\tilde{\theta}_L\right)>0$, we can conclude that $P_F'\left(\theta\right)>0$ for any θ. As $P_F\left(\bar{\theta}\right)=1$, it implies that $F\left[\theta|I,y_1\right]<F\left[\theta|U,y_1\right]$ for any $-\bar{\theta}<\theta<\bar{\theta}$, and we can conclude that $F\left[\theta|I,y_1\right]$ first order stochastically dominates $F\left[\theta|U,y_1\right]$. Using (22) and the functional expression of $\varphi_s\left(.\right)$ we further find that $L\left(\bar{\theta}\right)-1\geq 0$ holds whenever $y_1\geq\bar{y}_{co}$, where $\bar{y}_{co}$ solves $L\left(\bar{\theta}\right)-1=0$, i.e., it solves

$$\bar{y}_{co}=\frac{g_U\left(\bar{\theta}\right)+g_I\left(\bar{\theta}\right)}{2}+\frac{1}{2}\ln\left(\frac{f\left(\bar{y}_{co}|I\right)}{f\left(\bar{y}_{co}|U\right)}\right)\frac{1}{g_I\left(\bar{\theta}\right)-g_U\left(\bar{\theta}\right)}.$$

Notice that $0<\bar{y}_{co}<D\left(\bar{\theta}\right)$.

Case 2: Suppose that $L(\bar{\theta})-1<0$. As $P_F'(\tilde{\theta}_L)>0$ still holds, equation (31) implies that $L(\tilde{\theta}_L)-P_F(\tilde{\theta}_L)>0$, for any $\tilde{\theta}_L=\theta\in(-\bar{\theta},\theta^*]$. This implies that there exists θ, arbitrarily close to $-\bar{\theta}$, such that $L(\theta)>P_F(\theta)$. Combining this with the fact that $L\left(\bar{\theta}\right)<1=L\left(\bar{\theta}\right)$, we can conclude that $L\left(\theta\right)$ and $P_F\left(\theta\right)$ must cross at least once in $(-\bar{\theta},\bar{\theta})$. This further implies that $P_F\left(\theta\right)$ can no longer be monotonically increasing when $L(\bar{\theta})-1<0$. We can further prove that $L\left(\theta\right)$ and $P_F\left(\theta\right)$ cross exactly once in $(-\bar{\theta},\bar{\theta})$. Note that $P_F'(\tilde{\theta}_L)>0$ and $L(\tilde{\theta}_L)>P_F(\tilde{\theta}_L)$ imply that the function $P_F(\theta)$ must be strictly below $L(\theta)$, for any $(-\bar{\theta},\theta^*]$. Because $L(\tilde{\theta}_H)$ is monotonically decreasing, $L(\theta^*)>P_F(\theta^*)$, and $L(\bar{\theta})<1=P_F(\bar{\theta})$, it can only be the case that when $\tilde{\theta}_H$ increases then the sign function $sgn\left(L\left(\tilde{\theta}_H\right)-P_F\left(\tilde{\theta}_H\right)\right)$ follows a sequences such as 1) $+1,-1$; 2) $+1,-1,+1,-1$; 3) $+1,-1,+1,-1,+1,-1$ etc. We can exclude all such sequences apart from 1) by contradiction. Suppose that there exists a subset $\tilde{\theta}'_H \in \left\{\tilde{\theta}_H\right\}$, where $sgn(L(\tilde{\theta}'_H) - P_F(\tilde{\theta}'_H)) = +1$, while there also exist realizations $\tilde{\theta}''_H < \tilde{\theta}'_H$, where $\tilde{\theta}''_H \in \left\{\tilde{\theta}_H\right\}$ and $sgn(L(\tilde{\theta}''_H) - P_F(\tilde{\theta}''_H)) = -1$. For such such realization of $\tilde{\theta}'_H$ to exist, it must be the case that $P'_F(\tilde{\theta}'_H) < 0$. However, this leads to a contradiction given (31).

From here we can conclude that the probability ratio, $P_F(\theta)$, is unimodal. As $P_F(\theta)$ is unimodal, and $P\left(\bar{\theta}\right) = 1$, there can only be two possibilities: either be the case that $P(\theta) > 1$ holds for all $\theta \in (-\bar{\theta}, \bar{\theta})$, in which case $F[\theta|U, y_1] < F[\theta|I, y_1]$ or $P(\theta) - 1$ is single crossing and therefore also $F[\theta|U, y_1] - F[\theta|I, y_1]$ is single crossing in θ.

For $P(\theta) > 1$ to hold for all $\theta \in (-\bar{\theta}, \bar{\theta})$, it must hold that $P\left(-\bar{\theta}\right) = L\left(-\bar{\theta}\right) \geq 1$. From (22), we find that

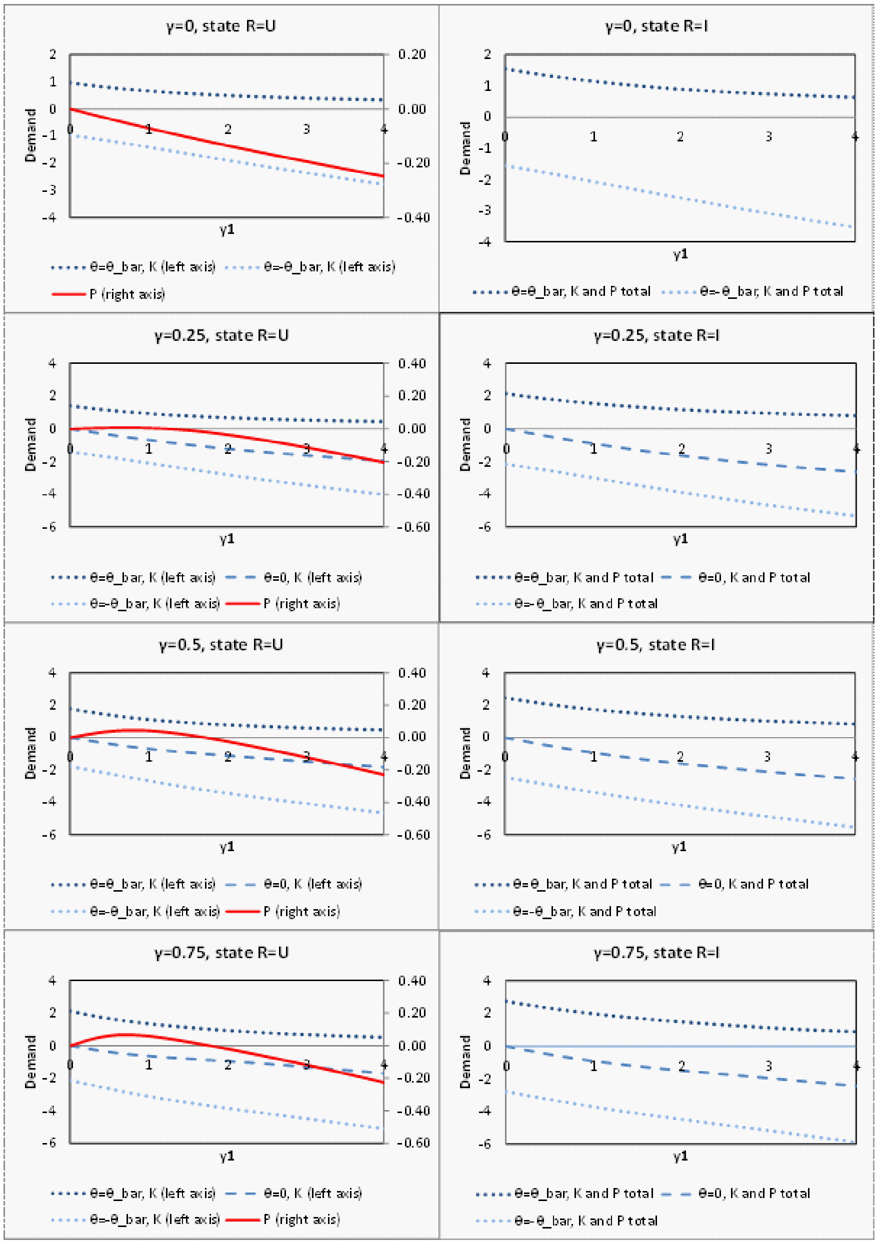
$$L\left(-\bar{\theta}\right) = \frac{f(y_1|U)}{f(y_1|I)} \frac{\varphi_s\left(y_1 + g_I\left(\bar{\theta}\right)\right)}{\varphi_s\left(y_1 + g_U\left(\bar{\theta}\right)\right)} = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} f_\theta(\theta) \frac{\varphi_s(y_1 - g_U(\theta))}{\varphi_s\left(y_1 + g_U\left(\bar{\theta}\right)\right)} d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f_\theta(\theta) \frac{\varphi_s(y_1 - g_U(\theta))}{\varphi_s\left(y_1 + g_U\left(\bar{\theta}\right)\right)} d\theta}. \tag{32}$$

Notice that $\frac{\varphi_s\left(y_1 + g_I\left(\bar{\theta}\right)\right)}{\varphi_s\left(y_1 + g_U\left(\bar{\theta}\right)\right)} < 1$ for any $y_1 > 0$, while $\frac{f(y_1|U)}{f(y_1|I)} > 1$ when y_1 is low in absolute value. For the latter, notice that $\frac{f(y_1|U)}{f(y_1|I)} = \frac{\eta(1-Q_1(y_1))}{Q_1(y_1)(1-\eta)}$ and $\frac{f(y_1|U)}{f(y_1|I)} > 1 \iff Q_1(y_1) < \eta$. Because two informed traders jointly trade less than one, the Market must update its' beliefs such that it expects P to be less likely informed when it observes a low order flow. In particular, using the expressions for φ_s, we find that

$$\lim_{y_1 \to 0} L\left(-\bar{\theta}\right) = \frac{\int_{-\bar{\theta}}^{\bar{\theta}} f_\theta(\theta) \exp\left(\frac{g_U^2\left(\bar{\theta}\right) - g_U^2(\theta)}{2\sigma_s^2}\right) d\theta}{\int_{-\bar{\theta}}^{\bar{\theta}} f_\theta(\theta) \exp\left(\frac{g_I^2\left(\bar{\theta}\right) - g_I^2(\theta)}{2\sigma_s^2}\right) d\theta} = \frac{2\int_0^{\bar{\theta}} f_\theta(\theta) \exp\left(\frac{g_U^2\left(\bar{\theta}\right) - g_U^2(\theta)}{2\sigma_s^2}\right) d\theta}{2\int_0^{\bar{\theta}} f_\theta(\theta) \exp\left(\frac{g_I^2\left(\bar{\theta}\right) - g_I^2(\theta)}{2\sigma_s^2}\right) d\theta} > 1,$$

because $\frac{\partial\left(g_I^2(\theta) - g_I^2(\theta)\right)}{\partial\theta} = g'_I(\theta) g_I(\theta) - g'_U(\theta) g_U(\theta) > 0$ for any $\theta \geq 0$ by Proposition 7 and Conjecture 1 in Section 4.2, which in term implies that $\exp\left(\frac{g_U^2\left(\bar{\theta}\right) - g_U^2(\theta)}{2\sigma_s^2}\right) - \exp\left(\frac{g_I^2\left(\bar{\theta}\right) - g_I^2(\theta)}{2\sigma_s^2}\right) > 0 \Leftrightarrow \exp\left(\frac{g_U^2\left(\bar{\theta}\right) - g_U^2(\theta) - g_I^2\left(\bar{\theta}\right) + g_I^2(\theta)}{2\sigma_s^2}\right) > 1$ for any $0 \leq \theta \leq \bar{\theta}$. This means that at least very close to zero, it holds that $P(\theta) > 1$. Furthermore, this remains to be true for $0 < y_1 < y_1^{co}$, where y_1^{co} is the solution of $L\left(-\bar{\theta}\right) = 1$, where $L\left(-\bar{\theta}\right)$ is given by (32).

C Date 2 demand by strategic traders in the model with forward looking informed traders



Date 2 demand as a function of date 1 order flow. The order flow is normalized by dividing it with the standard deviation of noise trading. We assume $\bar{\theta} = 1$ and $\eta = 0.5$.

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D Supplementary Appendix

D.1 Trader P as the only trader who can be informed

Assume there is no K, the rest of the model is the same as in Section 3 Date 2 problem has the same solution as the basic model in Section 2. Theorem 1 (see (13)) then implies that P will optimally pursue and profit from contrarian strategy if $\mathbb{E}[\theta|y_1, U] - p_1$ has an opposite sign compared to y_1. It is clear that without the presence of any informed traders at date 1, no fundamental information can be contained in date 1 order flow in state $R = U$, i.e., with any symmetric prior $E[\theta|y_1, U] = E[\theta] = 0$. It then follows that $sgn(\mathbb{E}[\theta|y_1, U] - p_1) = -sgn(p_1)$. As long as $sgn(p_1) = sgn(y_1)$ always holds, P's optimal date 2 strategy is contrarian. Furthermore, from (7), we also obtain that in this case $p_1 = Q_1\mathbb{E}[\theta|y_1, I]$ and therfore $sgn(p_1) = sgn(\mathbb{E}[\theta|y_1, I])$. As long as higher order flow is more likely to reflect high θ, i.e., if for any $\tilde{y}_1 > y_1$ the cumulative distribution $F[\theta|\tilde{y}_1, R]$ first order stochastically dominates $F[\theta|y_1, R]$, then $\mathbb{E}[\theta|y_1, I]$ is increasing in y_1. With any symmetric prior, this guarantees that $sgn(p_1) = sgn(y_1)$. It is further well known from Milgrom (1981) that first order stochastic dominance is implied by monotone likelihood ratio property (MLRP) for any prior, that is $\frac{f(\tilde{y}_1|\theta,I)}{f(y_1|\theta,I)}$ is increasing in θ. Because the order flow itself is given by $y_1 = g_I(\theta) + s_1$ where $g_I(\theta)$ is a trade by P when informed, it holds that $\frac{f(\tilde{y}_1|\theta,I)}{f(y_1|\theta,I)} = \frac{\varphi_s(\tilde{y}_1 - g_I(\theta))}{\varphi_s(y_1 - g_I(\theta))}$, which is indeed increasing in θ as long as P's optimal strategy when informed, $g_I(\theta)$, is increasing in θ, because $\varphi_s(.)$ is logconcave (see (24) in Appendix A). Strategies being increasing in θ holds for other priors, see Section 4.2. And it also holds with the the baseline setting with three-point distribution.

In our baseline case, we can conjecture that P trades $g_I(\bar{\theta}) = \bar{g}_I$, $g_I(-\bar{\theta}) = -\bar{g}_I$ and $\bar{g}_I(0) = 0$ if $R = 0$, and trades zero if $R = U$. Under this conjecture, we find using Bayes' rule that

$$\begin{aligned}
\mathbb{E}[\theta|y_1, I] &= \bar{\theta}\Pr\left(\theta = \bar{\theta}|y_1, I\right) - \bar{\theta}\Pr\left(\theta = -\bar{\theta}|y_1, I\right) \\
&= \bar{\theta}\left(\frac{f\left(y_1|\theta = \bar{\theta}, I\right)\Pr\left(\theta = \bar{\theta}|I\right)}{f(y_1|I)} - \frac{f\left(y_1|\theta = -\bar{\theta}, I\right)\Pr\left(\theta = -\bar{\theta}|I\right)}{f(y_1|I)}\right) \\
&= \bar{\theta}\frac{\varphi_s(y_1 - \bar{g}_I) - \varphi_s(y_1 + \bar{g}_I)}{f(y_1|I)}\frac{1-\gamma}{2}
\end{aligned}$$

and

$$Q_1 = \Pr(I|y_1) = \frac{f(y_1|I)\Pr(I)}{f(y_1)} = \frac{f(y_1|I)\,\eta}{f(y_1|I)\,\eta + f(y_1|U)(1-\eta)}.$$

Using that $f(y_1|I) = \frac{1-\gamma}{2}\varphi_s(y_1 - \bar{g}_I) + \gamma\varphi_s(y_1) + \frac{1-\gamma}{2}\varphi_s(y_1 - \bar{g}_I)$, and $f(y_1|U) = \varphi_s(y_1)$, we obtain that

$$p_1(y_1) = \eta\bar{\theta}\frac{\varphi_s(y_1 - g_U) - \varphi_s(y_1 + g_U)}{\eta\frac{1-\gamma}{2}\varphi_s(y_1 - g_U) + \eta\frac{1-\gamma}{2}\varphi_s(y_1 + g_U) + (\eta\gamma + (1-\eta))\varphi_s(y_1)}\frac{1-\gamma}{2}. \tag{33}$$

We can verify that $p_1'(y_1) > 0$, $p_1(y_1) = -p_1(-y_1)$, $p_1'(y_1) = p_1'(-y_1)$. Furthermore, $-\bar{\theta} < p_1(y_1) < \bar{\theta}$ for any finite order flow.

Suppose that $R = I$. From (3), and from $y_1 = h_1^{IJ} + s_1$, we then obtain the first order condition

$$\theta - \mathbb{E}\left[p_1\left(h_1^{IP} + s_1\right)|\theta, I\right] - h_1^{IP}\mathbb{E}\left[p_1'\left(h_1^{IP} + s_1\right)|\theta, I\right] = 0.$$

We can verify the conjectures. Namely, when $\theta = \bar{\theta}$, then $P's$ optimal demand is $h_1^{IP} = \bar{g}_I$, where $\bar{g}_I$ solves

$$\bar{\theta} - \int_{-\infty}^{\infty} p_1(\bar{g}_I + s_1)\varphi_s(s_1)\,ds_1 = \bar{g}_I\int_{-\infty}^{\infty} p_1'(\bar{g}_I + s_1)\varphi_s(s_1)\,ds_1$$

Because $\bar{\theta} > p_1(\bar{g}_I + s_1)$ for an finite values of s_1, and $p_1'(\bar{g}_I + s_1) > 0$, it follows that $\bar{g}_I > 0$. Because $p_1(.)$ is symmetric, it also follows that when $\theta = \bar{\theta}$, then $P's$ optimal demand is $h_1^{IP} = -\bar{g}_I$, and that $h_1^{IP} = 0$ when $\theta = 0$.

The proof that P would indeed choose to trade zero if $R = U$, is similar to the one in the main model (see Proof of Proposition 2 in Appendix A).

We can also see from (33) that if $\eta = 0$, then $p_1(y_1) = 0$ for any order flows, and P cannot profit from superior information at date 2.