Matching with Externalities

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Abstract

We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to allow for externalities, we establish that stable matchings exist when agent choices satisfy substitutability. In addition, we show that the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, remain valid despite the presence of externalities even though the standard fixed-point techniques do not apply. Furthermore, we establish novel comparative statics on externalities.

1 Introduction

Externalities are present in many two-sided markets. For instance, couples in a labor market pool their resources as do partners in legal or consulting partnerships. As a result, the preferences of an agent may depend on the contracts signed by the partner(s). Likewise, a firm's hiring decisions are affected by how candidates compare to competitors' employees. Finally, because of technological requirements of interoperability, an agent's purchase decisions may change because of other agents' decisions.

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In this paper, we incorporate externalities into the stable matching theory of Gale and Shapley (1962) and Hatfield and Milgrom (2005).[1] We refer to the two sides of the market as buyers and sellers. Each buyer-seller pair can sign many bilateral contracts. Furthermore, each agent is endowed with a choice function that selects a subset of contracts from any given set conditional on other agents' contracts.[2] We build a theory of matching with externalities that both extends to this more general setting some of the key insights of the classical theory without externalities, such as the existence of stable matchings and Gale and Shapley's deferred acceptance (or cumulative offer) algorithm and establishes new insights, including comparative statics on externalities.

Our theory is built on a substitutes condition that extends the classical substitutes condition to the setting with externalities. We require that each agent rejects more contracts from a larger set (as in the classical substitutes condition) and also that each agent rejects more contracts conditional on a matching that reflects better market conditions for his side of the market. We formalize the latter idea in two steps. A matching reflects better market conditions for one side of the market than another matching whenever the first matching is chosen by agents on this side of the market from a larger set conditional on a matching while the second matching is chosen by the agents from a smaller set conditional on the same matching. The second matching then reflects worse market conditions. Furthermore, we also say that a matching reflects better market conditions for one side of the market than another matching whenever the first matching is chosen by agents on this side of the market from some set conditional on some matching while the second matching is chosen by these agents from a smaller set conditional on a matching that reflects worse market conditions. When there are no externalities, this substitutes condition reduces to the classical gross substitutes condition of Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

We start by proposing a deferred acceptance algorithm for the setting with externalities which may be important in potential market design applications. In particular, our version of the algorithm can be viewed as a new auction that performs well in the presence of externali-

[1] Let us stress that even though we derive our results in a general many-to-many matching setting with contracts, the results are new in all special instances of our setting, including many-to-one and one-to-one matching problems.

[2] We formulate most of our results in terms of choice functions satisfying the *irrelevance of rejected contracts*. A choice function satisfies the irrelevance of rejected contracts if removing a rejected contract does not change the chosen set conditional on the same matching. When there are no externalities, this condition reduces to the one used in Aygün and Sönmez (2013). This is a basic rationality axiom: it is satisfied tautologically whenever agents' choice can be rationalized through a strict preference ordering.

ties.[3] Since an agent's choice depends on others' matching, we keep track not only of which offers are already made and rejected but also of the reference matchings that agents on each side use to condition their choice. The construction requires care because after the reference matching has changed an agent on the accepting side might want to go back to a contract that is already rejected, or an agent on the proposing side might want to withdraw a contract already made. To ensure that this does not happen, we construct the initial reference matchings in a preliminary phase of the algorithm.[4] Relatedly, we cannot stop the algorithm as soon as there are no rejections and no new offers: we need to continue until the reference matchings converge. Our construction of initial reference matchings ensures that subsequent reference matchings change in a monotonic way with respect to the "better market conditions" preorder, thus ensuring that from some point on the reference matchings belong to the same equivalence class. While these equivalence classes might consist of many matchings, we further show that the algorithm converges to one of them and never cycles among the members of the same equivalence class. In Section 4, we use a simple example to illustrate these points.

Our first two main results shows that our deferred acceptance algorithm always converges to a stable matching when choice functions satisfy substitutability (Theorem 1), and hence that stable matchings exist (Theorem 2). We focus on the classical short-sighted stability concept in which each agent assumes that other agents do not react to his or her choice. Our results, however, are applicable to many other stability concepts including far-sighted ones because we formulate the results in terms of agents' choice behavior and not in terms of their preferences. As we discuss in Remark 1, agents' choice behavior captures both agents' preferences and their conjectures about the reactions of other agents' to choices.[5]

Our third main result is a comparative statics on the strength of externalities and substitutes. Comparing two profiles of choice functions, we say that *substitutes are stronger* when agents reject more. In addition, we say that a reference choice function has *weaker externalities* than another choice function when the reference choice function reflects better market conditions

[3]See, e.g., Abdulkadiroğlu and Sönmez (2003) and Sönmez and Switzer (2013) for market design applications of deferred acceptance, and Kelso and Crawford (1982) and Hatfield and Milgrom (2005) for the relationship between deferred acceptance and ascending auctions.

[4]The cumulative offer phase of the algorithm builds on the approach of Fleiner (2003) and Hatfield and Milgrom (2005). The preliminary phase of the algorithm has no forerunners. It may be omitted if there is an underlying lattice structure on the set of all matchings; in general, however, such a lattice structure does not exist and neither do side-optimal matchings.

[5] While the study of stability in terms of choice behavior is well established (see e.g. Aygün and Sönmez, 2013), we believe that this conceptual point is new. The choice-based approach allows us to also consider agents whose choice behavior cannot be represented in terms of preferences as long as this choice behavior satisfies the rationality postulate discussed in footnote 2.

(when the market conditions are measured by the reference choice function) than the other choice function. This comparison of the strength of externalities satisfies some natural properties: for instance, the choice function exhibiting no externalities has weaker externalities than any other choice function. We prove that agents on one side of the market face better market conditions as their side of the market exhibits stronger substitutes and weaker externalities and they face worse market conditions if the other side of the market exhibits stronger substitutes and weaker externalities (Theorem 5).

In addition to these results, we extend the classical theory of matching to the setting with externalities. In Section 6.2, we study vacancy-chain dynamics. What are the welfare implications of an agent leaving the market? We show that when agents recontract according to an algorithm akin to the deferred acceptance algorithm (Gale and Shapley, 1962), all agents on the same side are better off and all agents on the other side are worse off (Theorem 6). In the setting without externalities and when agents on one side of the market can sign only one contract, the corresponding results have been proven by Kelso and Crawford (1982) and Crawford (1991). Similarly, our results generalize those of Blum, Roth, and Rothblum (1997) and Hatfield and Milgrom (2005), none of whom looked at the setting with externalities.

Furthermore, we analyze the existence of side-optimal stable matchings, that is, matchings that represent the optimal market conditions. A side-optimal stable matching exists under the additional assumption that there exists a side-optimal matching (Theorem 4). This additional assumption is satisfied trivially in finite settings without externalities, where the existence of side-optimal stable matchings was established already by Gale and Shapley (1962).

We also generalize the rural hospitals theorem of Roth (1986), which states that each hospital gets the same number of doctors in each stable matching in many-to-one matching without externalities (in Appendix A).[6] Our generalization allows different contracts to have different weights that may depend on the quantity, price, or quality of the contracts. For this purpose, we introduce a general law of aggregate demand. An agent's choice function satisfies *the law of aggregate demand* if the weight of contracts chosen from a set conditional on a reference matching is greater than the weight of contracts chosen from a subset conditional on a matching that has worse market conditions than the reference matching. We show that when choice functions satisfy the law of aggregate demand in addition to the aforementioned properties, all stable matchings have the same weight for every agent (Theorem 7). When there are no externalities, this law of aggregate demand reduces to the monotonicity condition of Fleiner

[6]Roth's theorem has been previously extended to more general settings without externalities, see e.g. Hatfield and Milgrom (2005).

(2003).

To the best of our knowledge, our development of comparative statics and results such as the rural hospitals theorem with externalities have no forerunners in the literature analyzing externalities in the setting of (Gale and Shapley, 1962). We thus contribute to the matching literature by showing how one can incorporate externalities into standard models of matching, including matching with contracts (e.g., Hatfield and Milgrom (2005)),[7] by offering new insights, and by showing that many of the insights of the classical literature remain valid in the presence of externalities.[8]

On the other hand, our existence result contributes to a rich literature analyzing the existence and nonexistence results in matching with externalities. In an early influential paper, Sasaki and Toda (1996) showed that stable one-to-one matchings need not exist. Their insight led the subsequent literature to take one of two routes: to modify the stability concept, or to impose assumptions on agents' preferences. Sasaki and Toda's seminal paper belongs to the first strand of literature. They focused on a weak stability concept that allows a pair of agents to block a matching only if they benefit from the block under all possible rematches of the remaining agents. They show that such weak stable matchings exist.[9] In contrast, our paper uses the standard stability concept of Gale and Shapley (1962) and the literature on matching without externalities.[10] We guarantee the existence of stable matchings not by modifying the stability concept but by imposing assumptions on preferences in line with the standard approach of restricting attention to substitutable preferences. While we primarily focus on the standard (short-sighted) stability concept, our results are applicable to many other stability

[7]The matching with contracts approach has not only been useful as a theoretical tool but also as a practical tool to design markets. For example, see Sönmez and Switzer (2013); Sönmez (2013). It has also been extended to the many-to-many matching and more general settings without externalities, see e.g. Ostrovsky (2008). In particular, Ostrovsky showed that stable matchings exists even in the presence of well-behaved complementarities among contracts. See also Azevedo and Hatfield (2013); Che, Kim, and Kojima (2015) who establish general existence of stable matchings allowing for complements in large markets without externalities.

[8]In fact, our main comparative statics result is new even in the setting without externalities as is our synthesis of classical and far-sighted stability.

[9]The rich subsequent literature, e.g., Chowdhury (2004); Hafalir (2008); Eriksson, Jansson, and Vetander (2011); Chen (2013); Gudmundsson and Habis (2013); Salgado-Torres (2011a,b)—maintained the focus on the existence question while refining Sasaki and Toda's weak stability concept by varying the degree to which the rematches of other agents penalize the blocking pair. Bodine-Baron, Lee, Chong, Hassibi, and Wierman (2011) analyze a related weak stability concept in a setting with peer effects.

[10]In line with this literature, a set of agents forms a blocking coalition if it benefits them in the absence of any reaction from the remaining agents. Note that the question of how other agents react to the formation of a blocking coalition is important whether externalities are present or not. In particular, even in the absence of externalities, one might entertain an alternative solution concept in which an agent might be unwilling to enter a blocking coalition if she is concerned that doing so will trigger a chain of events that will lead her to losing a partner she blocks with.

concepts including Sasaki and Toda's and other far-sighted concepts (see Remark 1).

The second strand of the literature analyzes the standard stability concept.[11] Prior work in this second strand of the literature identified several assumptions under which stable matchings exist. Particular attention has been devoted to externalities among couples (Dutta and Massó, 1997; Klaus and Klijn, 2005; Kojima, Pathak, and Roth, 2013; Ashlagi, Braverman, and Hassidim, 2014) and to peer effects among students matched to the same college (Dutta and Massó, 1997; Echenique and Yenmez, 2007; Pycia, 2012; Inal, 2015). We are not restricting our attention to either of these two types of externalities.

Our existence contribution is closest to the few papers that look at standard stability in the general matching problem with externalities. Bando (2012; 2014) studies many-to-one matching allowing externalities in the choice behavior of firms (agents who match with potentially many agents on the other side) but not of workers; he further assumes that each firm's choice function depends on the matching of other firms only through the set of workers hired by other firms, and imposes several other elegant assumptions on firms' choice behavior. Under these assumptions, he proves the existence of stable matchings and analyzes the deferred acceptance algorithm.[12] In another related work, Teytelboym (2012) looks at externalities among agents in a component of a network and shows that a stable matching exists provided agents' preferences are aligned in the sense of Pycia (2012). Finally, Fisher and Hafalir (2014) consider a setting in which each agent cares only about the level of externality in the overall economy (such as pollution) and study the existence of stable matchings when there are such aggregate externalities.[13]

Our work is also related to the exploration of efficiency in markets with externalities (see, e.g., Pigou (1932); Chade and Eeckhout (2014); Watson (2014)); while this literature focuses on efficiency, we focus on stability. Similarly related to our work is the literature on contracting in the presence of externalities (see, e.g., Segal, 1999; Segal and Whinston, 2003) and auctions with externalities (e.g., Jehiel, Moldovanu, and Stacchetti 1996; Jehiel and Moldovanu

[11] We follow this second approach. As discussed above, we also go beyond this second approach by offering a synthesis of standard and far-sighted approaches to stability.

[12] Under Bando's assumptions, there is no need to keep track of the reference matchings in the deferred acceptance algorithm (and hence no need for the preliminary phase that constructs the initial reference matchings), and his algorithm terminates as soon as there are no rejections.

[13] Also of note is Uetake and Watanabe (2012) who provide an empirical analysis of firm mergers using a matching model with externalities, and Mumcu and Saglam (2010) who analyze the question when all matchings in the non-empty collection of top matchings are stable. Baccara, Imrohoroglu, Wilson, and Yariv (2012) analyze stable one-sided allocations with externalities. Hatfield and Kominers (2015) study the existence of competitive equilibria in a multilateral matching setting with externalities, and Baldwin and Klemperer (2015) study the existence of competitive equilibria with indivisibilities allowing for complements. Leshno (2015), a work in progress, looks at large matching markets.

2001).[14] While our focus is on stable matchings, the contracts and auctions literature looks at specific noncooperative games and analyzes their equilibria.

2 Examples

In this section, we provide some examples to motivate and illustrate our work.[15] All these examples satisfy the substitutes condition that we need for the existence of stable matchings as we show in the next section after formally defining the substitutes condition.

We first present our motivating examples and then our simpler but more abstract illustrative example.

2.1 Motivating Examples

For simplicity, we consider only one side of the market in our examples. One could model the other side in the same or a different way because we impose no assumptions relating the choice behavior of agents across sides.

2.1.1 Sharing

Our theory applies to situations in which agents share profits, for instance because they work for the same firm or have some insurance arrangements. The following examples illustrate such situations.

Our theory applies to a labor market with couples in which an agent becomes more selective as his or her partner gets a better job.

Example 1. [Couples in a Local Labor Market] [16] Agents on one side of the market represent workers and agents on the other side of the market represent firms. Workers are either single or are members of exogenously married couples. The labor participation decision of a married man depends on the job of his wife: the better the job she has, the more selective he

[14] See also, e.g., Aseff and Chade (2008); Skreta and Figueroa (2008); Brocas (2013).

[15] As it is well known, even the existence of stable matchings is not guaranteed in the presence of externalities. Consider for instance a one-to-one matching setting between two men m_1 and m_2 and one woman. Regardless of the matching, the woman prefers man m_2 over man m_1 and she prefers either of them to being unmatched; and man m_1 prefers being matched to being unmatched. The preference ranking of man m_2 depends on the matching however: man m_2 prefers being matched to being unmatched if and only if the other man is matched. In this simple situation no matching is stable.

[16] We are grateful to Michael Ostrovsky for providing this example.

becomes. In other words, the outside option of not working becomes more attractive when a man's wife earns more.[17]

A richer example of sharing is as follows.

Example 2. **[Profit Sharing]** Agents on one side of the market represent attorneys organized in law firms. Each attorney can work on up to $k \geq 0$ contracts with clients on the other side of the market; an attorney works on all contracts he or she signs and the attorney can also work on selected contracts signed by others in the same firm. Each contract allows an arbitrary number of attorneys to contribute; the profit an attorney makes from a contract does not depend on how many other attorneys contribute to it.[18] Each attorney prioritizes the contracts she works on, and the profit attorney i earns on a contract depends on whether it is the first, second, etc. contract in attorney i's priorities. We assume that each attorney must prioritize the contracts she signs over other contracts that she works on.

Attorneys choose what contracts to sign and what contracts to work on so as to maximize their profits: An attorney's profit is the sum of the profits from all the contracts she works on whether she signed it or not. We denote by $\lambda(x,i,\ell) \geq 0$ the profit that accrues to attorney i from working on contract x that she prioritizes in position $\ell \in \{1,...,k\}$. For simplicity, let us also assume that there are no indifferences.[19] This example satisfies our assumptions provided

$$\lambda(x,i,1) > \lambda(y,i,\ell)$$

for all contracts x and y as long as attorney i is the signatory of contract x and $\ell > 1$.

2.1.2 Relative comparisons

Our theory also applies to situations in which market participants care about the relative standings of their partners. The following two examples illustrate this.

Example 3. [The Marriage Problem with Influence Hierarchy and Relative Comparisons] Men and women form pairs as in Gale and Shapley (1962); however, how attractive being single is to an agent, say a man, may depend on how many other men are married and how

[17] We assume that there are no externalities for firms (whose preferences satisfy the standard substitutes condition), single workers or the married women.

[18] This assumption and some of our other assumptions can be relaxed.

[19] To formalize this assumption let us define a work schedule ϕ given a non-empty set of k or fewer contracts Y to be a one-to-one mapping $Y \to \{1,...,k\}$. We then assume that no attorney i is indifferent between two different work schedules ϕ_1 and ϕ_2, that is $\sum_{x\in \mathrm{Domain}(\phi_1)} \lambda(x,i,\phi_1(x)) \neq \sum_{x\in \mathrm{Domain}(\phi_2)} \lambda(x,i,\phi_2(x))$.

attractive their partners are. We formalize this dependence as follows. Men are ordered in terms of how influential they are from the most influential man named 1, through the second most influential man named 2, etc. For each man j the set of acceptable women depends on the matching of men who are more influential than he is while his ranking of acceptable women does not depend on other agents' matches. The higher man 1's partner in his ranking, the more selective man j becomes. If man 1 has the same partner in two matchings, then the higher man 2's partner in his ranking, the more selective man j becomes, etc. lexicographically.[20]

Example 4. [Relative Rankings in Hiring] Agents on one side of the market represent colleges and agents on the other side represent academics in a particular field. For each college i and each academic j the productivity of j at i is denoted by $\lambda(i,j) \geq 0$. For simplicity, assume that no two academics have the same productivity at a college. Now, suppose that each college hires at most two academics in the field considered, and that it wants to hire at least one academic and would like to hire another one only if his or her productivity is at least as high as the productivity of all academics in at least half of the other colleges. Formally, the choice function $c_i(X_i|\mu)$ of college i is as follows: from choice set X_i, the college chooses the academic $j \in X_i$ with highest productivity $\lambda(i,j)$, and it chooses a second academic $j' \in X_i$ if and only if $\lambda(i,j')$ is greater than or equal to the productivity of all academics in at least half of the other colleges under matching μ. More generally, we can fix $k \in [0,1]$ and assume that college i chooses a second academic $j' \in X_i$ if and only if $\lambda(i,j')$ is greater or equal than the productivity of academics in at least a fraction k of other colleges.[21]

2.1.3 Interoperability

Our theory also applies to situations in which agents choose basic products with no regard to the choices of others but choose add-ons in a way that depends on others' choices of basic products. For instance, consider buyers who choose between Mac, PC, and Linux computers (and operating systems) in a way that does not depend on other buyers' choices and who take the hardware/operating system choices of others into account when buying productivity software.

[20] Notice that we do not impose any assumptions on how preferences of one agent relate to preferences of another. In particular, one man might prefer woman w over woman w' while another man might have the opposite preference. Also, a woman might prefer man 17 over man 1 while another woman might prefer man 1 over man 17.

[21] We can alternatively include this fraction the college whose choice function despite the self-referentiality of doing so. While we focus our discussion on non-self-referential situations, we can in general allow the choice function of an agent to depend on this agent's choice; see the discussion in footnote 26.

Example 5. [Interoperability and Add-on Contracts] Suppose agents on one side (buyers) sign two types of contracts with sellers on the other side: for instance, agents might be signing primary contracts and add-on (or maintenance) contracts. These two classes of contracts are disjoint.[22] In line with the literature on add-on pricing, suppose that agents ignore the add-on contracts when deciding which primary contracts to sign (Gabaix and Laibson, 2006), and suppose that each agent signs at most one primary contract and that there are no externalities among primary contracts.[23]

We assume that no agent's choice of add-on contracts depends on the other agents' choices of add-on contracts, and we allow a buyer's choice among add-on contracts to depend on his and the other agents' choices of primary contracts in an arbitrary way as long as the buyer rejects weakly more (in the inclusion sense) add-on contracts out of X conditional on μ than he would reject out of X' conditional on μ' whenever $X \supseteq X'$ and the agent prefers his primary contracts in μ to those in μ'.

2.2 Illustrative Example

In the next example, we consider a simple market with a few agents on both sides of the market. This example is used for illustrative purposes in the rest of the paper.

Example 6. Suppose that there are two sellers s_1 and s_2 and two buyers b_1 and b_2. Seller s_1 and buyer b_1 can sign contract x_1 and seller s_1 and buyer b_2 can sign contract x_2. Seller s_2 can sign contract x_3 with buyer b_2 only.

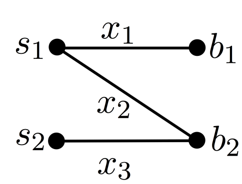

Figure 1: Contractual structure in Example 6.

Buyer b_1 wants to sign contract x_1 regardless of the contracts signed by b_2. Buyer b_2 signs

[22] Similar examples can be written for hardware contracts and software contracts, or contracts on inputs and outputs.

[23] Formally, we assume that each buyer's choice among primary contracts contracts does not depend on other agents' matches nor on the availability of add-on contracts. One reason that the agents ignore add-on contracts when signing primary contracts might be that the agents do not know which add-on contracts are available when signing the primary contracts as in Ellison (2005). We can relax the assumption that each agent signs at most one primary contract and assume instead that each agent's choice among primary contracts satisfies the standard substitutes assumption (see the next section).

contract x_2 whenever it is available but signs contract x_3 only when contract x_2 is not available conditional on buyer b_1 and seller s_1 not signing contract x_1. Choice functions are summarized by the following tables.[24]

	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_2,x_3\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_2\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\{x_3\})$	$\{x_1\}$	$\emptyset$
$c_{b_1}(\cdot\|\emptyset)$	$\{x_1\}$	$\emptyset$

	$\{x_2,x_3\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_2\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$c_{b_2}(\cdot\|\emptyset)$	$\{x_2\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$

Table 1: Buyers' choice functions in Example 6.

3 Model

There is a finite set of agents $\mathscr{I}$ partitioned into buyers, $\mathscr{B}$, and sellers, $\mathscr{S}$, $\mathscr{B}\cup\mathscr{S}=\mathscr{I}$. Agent i's type is denoted as $\theta(i)\in\{b,s\}$. If θ is a type, then $-\theta$ is the other type, that is, $-b\equiv s$ and $-s\equiv b$. Agents interact with each other bilaterally through contracts. Each contract x specifies a buyer $b(x)$, a seller $s(x)$, and terms, which may include prices, salaries and fringe benefits. There exists a finite set of contracts $\mathscr{X}$. For any $X\subseteq\mathscr{X}$, X_i denotes the maximal set of contracts in X involving agent i, that is $X_i\equiv\{x\in X : i\in\{b(x),s(x)\}\}$. Similarly, X_{-i} denotes the maximal set of contracts not involving agent i, that is, $X_{-i}\equiv X\setminus X_i$. We refer to all sets of contracts as matchings and we embed problems such as one-to-one matchings in our model by treating the relevant quota constraints as embedded in agents' choice behavior (discussed below). For instance, we model one-to-one matching markets by assuming that each agent chooses at most one contract from any choice set. Thus, examples of our setting include standard one-to-one and many-to-one matching problems with and without transfers.[25]

Each agent i has a choice function c_i, where $c_i(X_i|\mu_{-i})$ is the set of contracts that i chooses from X_i given that μ_{-i} is the set of contracts signed by the other agents on the same side.[26] We expand the domain of the choice function so that $c_i(X|\mu)=c_i(X_i|\mu_{-i})$. Let $r_i(X|\mu)\equiv$

[24]Columns are indexed by choice sets.

[25]Without affecting any of the results, we could alternatively model one-to-one matching and other matching environments with quota constraints by assuming that only some sets of contracts are matchings. This alternative route is straightforward if agents condition their choice behavior on subsets of contracts rather than on matchings. As is usual in models of matching with contracts, in applications with transfers, we assume that there is a lowest monetary unit.

[26]We could allow choice functions c_i that depend not only on X_i and μ_{-i} but also on μ_i (that is the set of contracts signed by i) with no change in our proofs. See footnote 21 for an example of when such self-referentiality is natural.

$X_i \setminus c_i(X|\mu)$ be the set of contracts rejected by agent i from X_i given matching μ. Similarly define $C^\theta(X|\mu) \equiv \cup_{i\in\theta} c_i(X|\mu)$ and $R^\theta(X|\mu) \equiv \cup_{i\in\theta} r_i(X|\mu)$ to be the set of contracts chosen and rejected from set X by side θ given matching μ, respectively. Note that for any $X, \mu \subseteq \mathscr{X}$ and θ, $C^\theta(X|\mu)$ and $R^\theta(X|\mu)$ form a partition of X since every contract involves one agent from both sides of the market. A **matching problem** is a tuple $(\mathscr{B}, \mathscr{S}, \mathscr{X}, C^b, C^s)$.

Matching μ is **individually rational** for agent i if $c_i(\mu|\mu) = \mu_i$. Less formally, given the remaining contracts, agent i wants to keep all of her contracts. A buyer i and seller j form a **blocking pair** for matching μ if there exists a contract $x \in \mathscr{X}_i \cap \mathscr{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}|\mu) \cap c_j(\mu \cup \{x\}|\mu)$. Matching μ is **stable** if it is individually rational for all agents and there are no blocking pairs. This stability concept is identical to stability studied in settings without externalities (see Roth, 1984).[27]

We illustrate this stability notion using Example 6. Suppose that there are no externalities for sellers and that they choose all available contracts, that is, $C^s(X|\mu) = X$ for any set of contracts X and μ. In this example, $Y = \{x_1, x_2\}$ is a stable matching. First of all, it is individually rational: buyer b_1 always wants to keep contract x_1, buyer b_2 also wants to sign contract x_2, and, likewise, seller s_1 wants to keep both contracts. Furthermore, there are no blocking pairs. The only potential blocking pair is seller s_2 and buyer b_2 with contract x_3. But buyer b_2 does not want to sign contract x_3 given contract x_1, i.e., $x_3 \notin c_{b_2}(Y \cup \{x_3\}|Y)$. Therefore, Y is a stable set.

Remark 1. We take choice functions as primitives of our model.[28] In general, when agents have preferences over matchings (sets of contracts) then these preferences and agents' predictions of how others will react to the changes in a matching allows us to construct the choice functions. In particular, while we focus on standard stability in which agents assume that their choice does not trigger chains of reactions by others, the general choice formulation we study implies that our results are equally applicable to theories of far-sighted stability. In this remark we give two simple examples of how agents' preferences over matchings translate to their choice behavior.

As a preparation, let us note that when there are externalities, preferences range not only

[27] As in the standard settings without externalities, stability defined in terms of individual and pairwise blocking is equivalent to core stability; see Appendix B. Defining stability in terms of agents' choices rather than preferences allows us to be agnostic whether blocking agents expect no further reaction to their blocking, as in canonical stability concepts, or whether blocking agents have more complex expectations about the consequences of them blocking; see Remark 1.

[28] As we explain in this remark, this approach allows us to offer a unified theory of stability that does not depend on blocking agents' hypothesis on how other agents react. This approach has many other benefits (Chambers and Yenmez, 2013) and it has been used in a matching context before (Alkan and Gale, 2003; Fleiner, 2003).

over the sets of contracts that list agents as a buyer or seller but over all contracts. In this case, the alternative approach works as follows. Denote agent i's preference by $\succeq_i$ (and the strict part by $\succ_i$). We assume that $\succeq_i$ is strict if the matching for the rest of the agents is fixed, that is, if $X_{-i} \subseteq \mathscr{X}_{-i}$ is a set of contracts that do not have agent i as a buyer or seller, $X_i, X_i' \subseteq \mathscr{X}_i$ such that $X_i \neq X_i'$, then either $X_i \cup X_{-i} \succ_i X_i' \cup X_{-i}$ or $X_i' \cup X_{-i} \succ_i X_i \cup X_{-i}$. This assumption guarantees that agent i's choice function, which we construct below, is well defined.[29]

1. (*Choice functions without prediction*) We construct the choice of agent i given μ from any set X, $c_i(X|\mu) \subseteq X_i$, as follows:

$$c_i(X|\mu) \cup \mu_{-i} \succeq_i X_i' \cup \mu_{-i} \text{ for every } X_i' \subseteq X_i.$$

This is the choice behavior we assume in the examples of Section 2.[30] We could also analyze these examples with the choice behavior that we discuss next.[31]

2. (*Choice functions with prediction*) For simplicity, we specify the choice behavior for the special case of our model in which each agent signs at most one contract. This is the one-to-one matching problem with contracts.

Let $A(x;\mu)$ be the set of contracts in μ that have to be removed when contract x is added to matching μ. These are the contracts signed by the buyer and seller associated with contract x in μ. More formally,

$$A(\{x\};\mu) \equiv \mu_{b(x)} \cup \mu_{s(x)}.$$

In particular, if x is the empty contract, then $A(\{x\};\mu) = \emptyset$. The choice of agent i from a set X given μ, $c_i(X|\mu) \subseteq X_i$, is then defined as follows:[32]

$$c_i(X|\mu) \cup (\mu_{-i} \setminus A(c_i(X|\mu);\mu)) \succeq_i X_i' \cup \left(\mu_{-i} \setminus A\left(X_i';\mu\right)\right) \text{ for every } X_i' \subseteq X_i, \left|X_i'\right| \leq 1.$$

We can similarly construct choice functions for many-to-one matching markets by appropriately changing the definition of $A(\{x\};\mu)$. In general, any deterministic theory of how agents react to the matching of an agent allows the agent to compare the resulting matchings

[29] In the special case when there are no externalities, each agent's preference depends only on the set of contracts that she signs, i.e., for any $X_i, X_i' \subseteq \mathscr{X}_i$ and $X_{-i} \subseteq \mathscr{X}_{-i}$, we have $X_i \cup X_{-i} \succeq_i X_i' \cup X_{-i} \iff X_i \cup \mathscr{X}_{-i} \succeq_i X_i' \cup \mathscr{X}_{-i}$.

[30] One could easily generalize the above approach as follows. For each μ_{-i}, let there be a strict preference relation $\succeq_i^{\mu_{-i}}$ of agent i. The choice function can be constructed similarly as above: $c_i(X|\mu) \cup \mu_{-i} \succeq_i^{\mu_{-i}} X_i' \cup \mu_{-i}$ for every $X_i' \subseteq X_i$.

[31] For instance, in Example 4 it does not matter for the choice behavior whether the colleges assume that an academic they are hiring is part of the benchmark of other hired academics or not because whether he or she is included in the benchmark does not affect the comparison of this academic's productivity to the benchmark.

[32] Note that this choice behavior is implicit in, for instance, Bando (2012; 2014).

and thus can be easily incorporated in our model.[33]

Our results and analysis remain the same regardless of how choice functions are constructed from agents' preferences. Furthermore, we allow for more general choice behavior including non-rationalizable ones.

3.1 Properties of Choice Functions

To guarantee the existence of stable matchings and mechanisms with desirable properties, we impose more structure on the choice functions. Let us first define the auxiliary concept of consistency.[34]

Definition 1. A preorder $\tilde{\succeq}^{\theta}$ is **consistent** with the choice function C^{θ} if for any $X, X', \mu, \mu' \subseteq \mathscr{X}$,

$$X' \supseteq X \;\&\; \mu' \tilde{\succeq}^{\theta} \mu \implies C^{\theta}\left(X'|\mu'\right) \tilde{\succeq}^{\theta} C^{\theta}\left(X|\mu\right).$$

To define our conditions, we consider consistent preorders. The following lemma establishes the existence and uniqueness of the minimal preorder that is consistent with a side choice function.

Lemma 1. *There exists a minimal preorder that is consistent with the choice function C^{θ}. Furthermore, the minimal preorder is unique.*

Proof. First of all, the preorder on the set $\mathscr{X}$ that includes all possible pairs of matchings is consistent with the choice function C^{θ}. Hence, there exists at least one preorder that is consistent with the choice function C^{θ}. Now, let us construct a minimal preorder consistent with C^{θ}. Suppose that $\{\succeq_1^{\theta}, \succeq_2^{\theta}, \ldots, \succeq_k^{\theta}\}$ is the set of all preorders that are consistent with

[33] In analyzing far-sighted stability based on such deterministic theories, we may need to take care of the possibility that two choices might lead to the same outcome. In such cases, the preferences over final outcomes need to be supplemented with a tie-breaking procedure to determine choice behavior. Such indifference situations never arise in the constructions 1 and 2 above. Theories of far-sighted stability that are not directly based on deterministic assumptions on agents reactions are harder to map into our framework; see, for example, Konishi and Ünver (2007) and Ray and Vohra (2015).

[34] In our context, a **binary relation** $\tilde{\succeq}^{\theta}$ on domain $\mathscr{A}^{\theta}$ is a set of ordered pairs of elements from $\mathscr{A}^{\theta}$. It is **reflexive** if for any $\mu \in \mathscr{A}^{\theta}$, $\mu \tilde{\succeq}^{\theta} \mu$. It is **transitive**, if $\mu_1 \tilde{\succeq}^{\theta} \mu_2$ and $\mu_2 \tilde{\succeq}^{\theta} \mu_3$ imply $\mu_1 \tilde{\succeq}^{\theta} \mu_3$. A reflexive and transitive binary relation is called a **preorder**. In defining our conditions on choice, we set the domain of the preorder to be $\mathscr{A}^{\theta} = 2^{\mathscr{X}}$. Alternatively, we can restrict attention to any smaller domain that contains $\emptyset$ and satisfies $C^{\theta}\left(X|\mu\right) \in \mathscr{A}^{\theta}$ whenever $X \subseteq \mathscr{X}$ and $\mu \in \mathscr{A}^{\theta}$. The minimal such domain is $\mathscr{A}^{\theta} \equiv \bigcup_{t=0,1,\ldots} \mathscr{A}_t^{\theta}$ where $\mathscr{A}_0^{\theta} \equiv \{\emptyset\}$ and $\mathscr{A}_t^{\theta}$ for $t \geq 1$ are defined recursively $\mathscr{A}_t^{\theta} \equiv \{C^{\theta}(X|\mu) : X \subseteq \mathscr{X}, \mu \in \mathscr{A}_{t-1}^{\theta}\} \cup \mathscr{A}_{t-1}^{\theta}$. Since there exists a finite number of contracts, this $\mathscr{A}^{\theta}$ is well defined; it is the set of all matchings that can be reached from the empty set by applying the choice function C^{θ}.

choice function C^{θ}. Define the following binary relation: $\mu' \succeq^{\theta} \mu$ if and only if $\mu' \succeq_j^{\theta} \mu$ for every $j = 1, \ldots, k$. The binary relation $\succeq^{\theta}$ is reflexive and transitive, so it is a preorder. In addition, let $X' \supseteq X$ and $\mu' \succeq^{\theta} \mu$. Then $\mu' \succeq_j^{\theta} \mu$ for every $j = 1, \ldots, k$. By consistency of $\succeq_j^{\theta}$, we get $C^{\theta}(X'|\mu') \succeq_j^{\theta} C^{\theta}(X|\mu)$ for every $j = 1, \ldots, k$. As a result, $C^{\theta}(X'|\mu') \succeq^{\theta} C^{\theta}(X|\mu)$. Therefore, $\succeq^{\theta}$ is also consistent with the choice function C^{θ}. Since the number of preorders is finite, this argument shows that there exists a unique minimal preorder $\succeq^{\theta}$ which is consistent with C^{θ}. □

We define our conditions using this minimal preorder $\succeq^{\theta}$. To simplify exposition, when $\mu' \succeq^{\theta} \mu$ we say that μ' has a *better market condition* than μ for side θ. Sometimes we refer to the preorder as *a side ranking*.

Definition 2. Choice function C^{θ} satisfies **substitutability** if for any $X, X', \mu, \mu' \subseteq \mathscr{X}$,

$$X' \supseteq X \ \& \ \mu' \succeq^{\theta} \mu \implies R^{\theta}\left(X'|\mu'\right) \supseteq R^{\theta}(X|\mu).$$

Less formally, the choice function of side θ satisfies substitutability if it rejects less contracts from a set X given a matching μ than it rejects from a superset of X given a matching μ' that has a better market condition than μ. When $\mu' = \mu$ or when there are no externalities, a choice function satisfies substitutability if the corresponding rejection function is monotone, or equivalently, a contract that is chosen from a larger set is also chosen from a smaller set including that contract. This special case is standard substitutability; it was introduced by Kelso and Crawford (1982) for a matching market with transfers, and generalized to the setting with contracts by Roth (1984).[35] Our definition is more general and incorporates externalities since the choice function of an agent depends on the set of contracts signed by the rest of the agents.

In substitutability, we condition the choice set and rejection set on matchings; in particular, we impose that μ' has a better market condition than μ. This is a novel property. Importantly, when there are no externalities for side θ, the preorder $\succeq^{\theta}$ is defined as the *revealed preference* for agents on side θ.[36] In addition, substitutability reduces to the regular one studied in the literature when there are no externalities as the conditioning on matchings is no longer important. It is also satisfied in the slightly more general setting in which externalities affect agents' preferences but not their choices (for instance, if the agents' utility can be additively separated into utility from one's own contracts and utility from contracts of other agents' on the same

[35] See also Fleiner (2003) and Hatfield and Milgrom (2005). Note that in the presence of externalities, our substitutes assumption imposes a preference restriction even on agents who sign at most one contract.

[36] X is revealed preferred to Y if $C^{\theta}(X \cup Y) = X$.

side of the market). This observation is straightforward; notice that $X \subseteq X'$ implies that the choice out of X' is from a larger set, and hence revealed preferred.

Substitutability can be decomposed into two separate conditions. First is the case when $\mu' = \mu$, which is similar to the standard substitutability: we discuss this in the preceding paragraph. Second, when $X' = X$, we reject more students conditional on a matching that has a better market condition. The conjunction of these two special cases are equivalent to substitutability.

Furthermore, if substitutability is satisfied for a preorder consistent with the choice function, then it is also satisfied for the minimal preorder $\succeq^\theta$. This will be useful in our applications as we do not have to find the minimal preorders consistent with the choice functions but just some preorders consistent with the choice functions. As a result, we can potentially use many preorders $\succeq^\theta$ for each side θ. One example of such a preorder can be defined as follows when agents have preferences over sets of contracts: for any matchings μ and μ', $\mu \succeq^\theta \mu'$ if $\mu \succeq_i \mu'$ for all $i \in \theta$ (in words, side θ prefers matching μ to μ' if all agents in θ prefer μ to μ'). But we are not restricting our attention to such preorders. In particular, the preorder might capture some properties of the underlying fundamentals. For instance, if agents contract over qualities and payments, we might have $\mu \succeq^\theta \mu'$ if the profile of qualities in μ is higher than the profile of qualities in μ' (irrespective of payments, and hence of agents' utilities). In Example 2, where attorneys share profits, we can use the following preorder for the attorneys: $\mu \succeq^\theta \mu'$ if and only if the profit accrued from the contract that has the highest priority is greater in μ compared to that in μ' for every attorney. In Example 4, where colleges care about the relative ranking of their hires, $\mu \succeq^\theta \mu'$ for colleges if and only if the maximum quality of hires in μ is weakly better than that of μ' for each college. In Example 5, where buyers sign primary and add-on contracts, $\mu \succeq^\theta \mu'$ for buyers when the primary contracts in μ are better than the primary contracts in μ' for every buyer.

Next, we introduce a basic rationality axiom for a choice function. Let us stress that this axiom is tautologically satisfied when the choice behavior is rationalizable as in Remark 1 and in our examples.

Definition 3. Choice function C^θ satisfies the **irrelevance of rejected contracts** if for all $X', X, \mu \subseteq \mathscr{X}$, we have

$$C^\theta(X'|\mu) \subseteq X \subseteq X' \implies C^\theta\left(X'|\mu\right) = C^\theta(X|\mu).$$

If choice function C^θ satisfies the irrelevance of rejected contracts, then excluding contracts that are not chosen does not change the chosen set. This is a basic rationality axiom for choice functions. It has been studied in the matching with contracts literature by Aygün and Sönmez (2013) when there are no externalities. They show that, without this condition, substitutability alone does not guarantee the existence of stable matchings; but these two conditions together imply the existence. If choice functions are constructed from preferences as in Remark 1, then the irrelevance of rejected contracts is automatically satisfied.

By construction, C^{θ} satisfies the irrelevance of rejected contracts (or substitutability) if and only if c_i satisfies the irrelevance of rejected contracts (or substitutability) for every agent i on side θ. Therefore, we can impose these two conditions on either agents' choice functions or the choice functions for each side of the market.

3.2 Examples Revisited

Now, we illustrate these properties with our examples. We focus on substitutability because it is straightforward to see that the irrelevance of rejected contracts is satisfied.

Example 1 revisited: Worker choice functions satisfy substitutability for preorder $\succeq^{\theta}$ such that $\mu' \succeq^{\theta} \mu$ when each married woman gets a better job in μ' compared to μ. This preorder is consistent because as there are more contracts available married women are better off since their choice functions do not exhibit externalities. The substitutes condition is satisfied because a married man becomes more selective whenever his wife gets a better job, so he rejects more contracts conditional on μ' compared to μ whenever $\mu' \succeq^{\theta} \mu$.

Example 2 revisited: Attorney choice functions satisfy substitutability if we define the preorder $\succeq^{\theta}$ so that $\mu' \succeq^{\theta} \mu$ if and only if $\max_{x\in\mu'(i)} \lambda(x,i,1) \geq \max_{x\in\mu(i)} \lambda(x,i,1)$ for all agents $i \in \theta$.[37] This preorder is consistent with choice: When more contracts are available, the profitability of the best contract signed by each attorney goes up (irrespective of what contracts other attorneys sign). The substitutability condition holds for each attorney i: When more contracts are available and when the profitability of the best contract signed by other attorneys (and hence the outside option of attorney i) increases, the attorney continues to reject the contracts she previously rejected.

Example 3 revisited: Man choice functions satisfy substitutability if we define the preorder $\succeq^{\theta}$ so that $\mu' \succeq^{\theta} \mu$ if and only if for some man $\mu'(j) \succ_j \mu(j)$ and $\mu'(i) = \mu(i)$ for $i < j$. This preorder is consistent with the choice functions, and the substitutability condition is satisfied as choosing out of larger (in inclusion sense) choice set conditional on a matching

[37] We use the convention that the maximum over the empty set is $-\infty$.

higher in this preorder, each man continues to reject the women he previously rejected.

Example 4 revisited: College choice functions satisfy substitutability if we define the preorder $\succeq^{\theta}$ so that $\mu' \succeq^{\theta} \mu$ if and only if $\max_{j\in\mu'(i)}\lambda(i,j)$ is weakly higher than $\max_{j\in\mu(i)}\lambda(i,j)$ for all colleges i.[38] This preorder is consistent with the choice functions: when more academics are around then the maximum quality of the academics a college hires goes up (whether or not the benchmark quality of academics increases). The substitutability condition is then satisfied: when more academics are around and when the benchmark quality of academics increases, each college continues to reject the academics it previously rejected.

Example 5 revisited: Buyer choice functions satisfy substitutability for the preorder $\succeq^{\theta}$ such that $\mu' \succeq^{\theta} \mu$ when each buyer prefers her primary contracts signed under μ' to those signed under μ. This preorder is consistent: $\succeq^{\theta}$ depends only on primary contracts, and each agent prefers to choose from larger choice sets over choosing from smaller choice sets. It is enough to check substitutability separately for the primary contracts and the add-on contracts: it holds for the primary contracts as the choice over them is not affected by externalities, and it holds for the add-on contracts as we explicitly assumed it.

Example 6 revisited: Using individual buyer choice functions, we can construct a choice function C^b for the buyer side.

	$\{x_1,x_2,x_3\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot\|\{x_1,x_2,x_3\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot\|\{x_1,x_2\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot\|\{x_1,x_3\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot\|\{x_2,x_3\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot\|\{x_1\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\emptyset$	$\emptyset$
$C^b(\cdot\|\{x_2\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot\|\{x_3\})$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$
$C^b(\cdot\|\emptyset)$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_3\}$	$\{x_2\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\emptyset$

Table 2: Buyer-side choice function in Example 6.

We use the following preorder for buyers: $\{x_1,x_2\} \succeq^b \{x_1,x_3\},\{x_1\},\{x_2\} \succeq^b \{x_3\},\emptyset$; $\{x_1,x_3\} \sim^b \{x_1\}$; and $\{x_3\} \sim^b \emptyset$. This preorder is consitent with C^b: for example, $\{x_1,x_2\} \succeq^b \{x_1\}$, so we must have $C^b(\{x_1,x_3\}|\{x_1,x_2\}) \succeq^b C^b(\{x_3\}|\{x_1\})$, which is true since $C^b(\{x_1,x_3\}|\{x_1,x_2\}) = \{x_1\} \succeq^b \emptyset = C^b(\{x_3\}|\{x_1\})$. Likewise, $\{x_1,x_2\} \succeq^b \{x_2\}$ implies $C^b(\{x_1,x_3\}|\{x_1,x_2\}) \succeq^b C^b(\{x_1,x_3\}|\{x_2\})$. Again, this holds because $C^b(\{x_1,x_3\}|\{x_1,x_2\}) = \{x_1\} \succeq^b \{x_1,x_3\} = C^b(\{x_1,x_3\}|\{x_2\})$. Substitutability is satisfied for this consistent preorder. For example, $\{x_1,x_2\} \succeq^b \{x_1\}$, as a result,

[38] When $\mu(i)$ is empty, we set the maximum equal to $-\infty$.

we must have $R^b(\{x_1,x_3\}|\{x_1,x_2\}) \supseteq R^b(\{x_3\}|\{x_1\})$, which is true since $R^b(\{x_1,x_3\}|\{x_1,x_2\}) = \{x_3\} \supseteq \{x_3\} = R^b(\{x_3\}|\{x_1\})$. Likewise, $\{x_1,x_2\} \succeq^b \{x_2\}$ implies $R^b(\{x_1,x_3\}|\{x_1,x_2\}) \supseteq R^b(\{x_1,x_3\}|\{x_2\})$. Again, this holds because $R^b(\{x_1,x_3\}|\{x_1,x_2\}) = \{x_3\} \supseteq \emptyset = R^b(\{x_1,x_3\}|\{x_2\})$. Finally, $\{x_3\} \sim \emptyset$ implies $R^b(X|\{x_3\}) = R^b(X|\emptyset)$ for any set of contracts X, which is true.

4 Deferred Acceptance with Externalities and the Existence of Stable Matchings

As in classical matching theory, a key step in proving the existence of stable matchings is the deferred acceptance algorithm. We describe the version of the algorithm in which sellers make proposals and buyers tentatively accept some of them and reject others. Of course, an analogous algorithm in which buyers propose works as well.

Our generalization of the deferred acceptance algorithm has two phases. First, we construct an auxiliary matching μ^* such that $C^s(\mathscr{X}|\mu^*) \preceq^s \mu^*$. Then, we use μ^* to construct a stable matching in a way resembling the classic deferred acceptance algorithm of Gale and Shapley (1962) and, particularly, its extensions by Adachi (2000); Fleiner (2003); Hatfield and Milgrom (2005): we run the algorithm in rounds, $t = 1, 2, \ldots$. In any round $t \geq 1$, we denote by $A^s(t)$ the set of contracts that have not yet been rejected by buyers, by $A^b(t)$ the set of contracts that have been offered by the sellers to the buyers, and thus the set of contracts held at the beginning of each round is $A^s(t) \cap A^b(t)$. We also track the reference matchings for each side: $\mu^s(t)$ is the seller reference matching and $\mu^b(t)$ is the buyer reference matching.[39]

Deferred Acceptance Phase 1: Construction of an auxiliary matching $\mu^* \in \mathscr{M}^s$ such that $\mu^* \succeq^s C^s(\mathscr{X}|\mu^*)$. Set $\mu_0 \equiv \emptyset$ and define recursively $\mu_k \equiv C^s(\mathscr{X}|\mu_{k-1})$ for every $k \geq 1$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$. We take the minimum m satisfying this property and set $\mu^* = \mu_m$.

[39]The tracking of reference matchings has no counterpart in earlier formulations of the deferred acceptance algorithms of, among many others, Gale and Shapley (1962), Adachi (2000), Roth (1984), Fleiner (2003), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Ostrovsky (2008), Hatfield and Kojima (2010), and Bando (2014). In these papers, there is no need to track reference matchings and the deferred acceptance algorithm terminates when there are no more rejections and no new offers. However, in our setting, the lack of rejections and new offers is not sufficient to stop the algorithm and we need to run it until the reference matchings converge. We run the algorithm in a symmetric way: in each round agents on both sides respond to the offers and rejections from the previous round. This is formally different from the standard approach where agents on the proposing side respond to rejections from the earlier round but the agents on the accepting side respond to offers in the current round. This difference is not substantive: we could run the deferred acceptance algorithm in the latter manner with straightforward adjustments.

We establish below that the matching constructed in phase 1 satisfies the property that $\mu^* \succeq^s C^s(\mathscr{X}|\mu^*)$.

Deferred Acceptance Phase 2 (the cumulative offer process): Construction of a stable matching. Set $A^s(1) \equiv \mathscr{X}$ (no contracts have been rejected by the buyers), $A^b(1) \equiv \emptyset$ (the sellers have made no offers yet), and the reference matchings are $\mu^s(1) = \mu^*$, and $\mu^b(1) = \emptyset$. In each round $t = 1, 2, ...$, we update these sets and matchings as follows:

$$\begin{aligned} A^s(t+1) &\equiv \mathscr{X} \setminus R^b(A^b(t)|\mu^b(t)), \\ A^b(t+1) &\equiv \mathscr{X} \setminus R^s(A^s(t)|\mu^s(t)), \\ \mu^s(t+1) &\equiv C^s(A^s(t)|\mu^s(t)), \\ \mu^b(t+1) &\equiv C^b(A^b(t)|\mu^b(t)). \end{aligned}$$

Thus, the buyers reject some of the contracts offered in $A^b(t)$ given their reference matching $\mu^b(t)$ and the set of not-yet rejected contracts after the round is $A^s(t+1) = \mathscr{X} \setminus R^b(A^b(t)|\mu^b(t))$; the sellers make new offers from the set of contracts that have not been rejected yet, and the set of contracts offered to the buyers after the round is $A^b(t+1) = \mathscr{X} \setminus R^s(A^s(t)|\mu^s(t))$. We also update the reference matchings: at each round, the sellers' reference matching is the matching the sellers would choose out of contracts not yet rejected, and the buyers' reference matching is the matching buyers would choose out of contracts offered so far.

We continue updating these sets until round T such that $A^s(T+1) = A^s(T)$, $A^b(T+1) = A^b(T)$, $\mu^s(T+1) = \mu^s(T)$, and $\mu^b(T+1) = \mu^b(T)$. The outcome of the deferred acceptance is then $A^s(T) \cap A^b(T)$.

The main result of this section establishes that the deferred acceptance algorithm terminates at some round T despite the presence of externalities and, furthermore, it produces a stable matching.

Theorem 1. *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then, the deferred acceptance algorithm terminates, its outcome is stable, and*

$$\mu^s(T) = \mu^b(T) = A^s(T) \cap A^b(T).$$

Let us recognize the following immediate corollary.

Theorem 2. *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then there exists a stable matching.*

In particular, this result implies the existence of stable matchings in all the examples of Section 2.

Before embarking on the proof of Theorem 1, let us notice the similarities and differences with the standard deferred acceptance algorithm, consider an example of how the algorithm runs, and establish two auxiliary properties of the transformation iteratively performed in the second phase of the deferred acceptance algorithm.

4.1 An Illustration of the Deferred Acceptance Algorithm

Similarly to the standard deferred acceptance algorithm, in each round of phase 2, substitutability and the irrelevance of rejected contracts imply that $A^s(t+1) \subseteq A^s(t)$ and $A^b(t+1) \supseteq A^b(t)$, i.e., the sellers make more offers to the buyers while more contracts are rejected by the buyers with each passing round (Lemma 2). As a consequence, the sellers' reference matching worsens and the buyers' reference matching improves. Hence, both of these two sets converges at some round t; however, the algorithm does not necessarily terminate when $A^s(t+1) = A^s(t)$ and $A^b(t+1) = A^b(t)$. Indeed, because of externalities, the set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is not necessarily stable at such a round. Instead, the algorithm converges only when $A^s(t+1) = A^s(t)$, $A^b(t+1) = A^b(t)$, $\mu^s(t+1) = \mu^s(t)$ and $\mu^b(t+1) = \mu^b(t)$. And the set of contracts held at such a round is stable.

The following example, which is a special case of Example 1, illustrates this point and shows the steps of the algorithm. This example also illustrates that our deferred acceptance algorithm can be viewed as an ascending auction in the presence of externalities.

Example 7. Suppose there is one employer f (a firm) and two workers w_1 and w_2. The firm can sign two types of contracts with different wages: a low wage, L, and a high wage, H. The contracts are denoted as follows: $x_{1L} = (f, w_1, L)$, $x_{1H} = (f, w_1, H)$, $x_{2L} = (f, w_2, L)$, and $x_{2H} = (f, w_2, H)$. The firm would like to hire as many workers as it can and pay as low wages as it can. In other words, from any given set of contracts, the firm chooses the contract with the lowest wage associated for each worker.

Notice that in this simple example all contracts involve firm f, and hence its preferences do not depend on the reference matching (i.e., there are no externalities for the firm). Worker w_1's preferences do not depend on the reference matching (that is on what contract w_2 signs) and worker w_1 is willing to work only at the high wage: $x_{1H} \succ_{w_1} \emptyset \succ_{w_1} x_{1L}$. Worker w_2's preferences depend on the contract of worker w_1 (we may think of these two workers as a married couple as in Example 1). More precisely, worker w_2 is willing to work at any wage only if worker w_1 is not employed: if worker w_1 is not employed then worker w_2's preference ranking is $x_{2H} \succ_{w_2} x_{2L} \succ_{w_2} \emptyset$ and if worker w_1 is employed then worker w_2's ranking is $\emptyset \succ_{w_2} x_{2H}, x_{2L}$. The workers' choice functions are constructed from these preferences.

Consider the firm-proposing version of the algorithm. Thus, the firm plays the role of a single seller and the workers play the roles of buyers. The first phase of the algorithm yields $\mu^* = \{x_{1L}, x_{2L}\}$.[40] We then run the second phase as summarized in the following table.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^s(A^s(t)\|\mu^s(t))$	$C^b(A^b(t)\|\mu^b(t))$
$t=1$	$\mathscr{X}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\emptyset$
$t=2$	$\mathscr{X}$	$\{x_{1L}, x_{2L}\}$	$\{x_{1L}, x_{2L}\}$	$\emptyset$	$\{x_{1L}, x_{2L}\}$	$\{x_{2L}\}$
$t=3$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{2L}\}$	$\{x_{1L}, x_{2L}\}$	$\{x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{2L}\}$
$t=4$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$
$t=5$	$\{x_{1H}, x_{2L}, x_{2H}\}$	$\{x_{1L}, x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}\}$
$t=6$	$\{x_{1H}, x_{2H}\}$	$\mathscr{X}$	$\{x_{1H}, x_{2L}\}$	$\{x_{1H}\}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$
$t=7$	$\{x_{1H}\}$	$\mathscr{X}$	$\{x_{1H}, x_{2H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$
$t=8$	$\{x_{1H}\}$	$\mathscr{X}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$
$t=9$	$\{x_{1H}\}$	$\mathscr{X}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$	$\{x_{1H}\}$

Table 3: Steps of the Deferred Acceptance Algorithm.

In the first round, firm f offers low wage contracts to both workers. Workers respond to the offers in the initial set $A^b(1) = \emptyset$.[41] At the end of this round, $A^s(2) = \mathscr{X}$ and $A^b(2) = \{x_{1L}, x_{2L}\}$, and the reference matchings are unchanged. In the second round, firm f faces the same choice problem while workers are now choosing from $A^b(2) = \{x_{1L}, x_{2L}\}$ and thus worker w_1 rejects the offered contract x_{1L}, while worker w_2 accepts x_{2L}.

The algorithm continues to proceed in this way. Notice that between the fouth and fifth rounds the sets of contracts already offered are the same, $A^b(4) = A^b(5)$, as are the sets of contracts not yet rejected, $A^s(4) = A^s(5)$. In the standard deferred acceptance algorithm without externalities, we could stop the algorithm here and set the outcome to the matching $A^s(4) \cap A^b(4) = \{x_{1H}, x_{2L}\}$. In our setting, this matching is not stable as w_2 prefers not to work given that w_1 is working. And, indeed, our deferred acceptance does not converge yet as the new reference matching for the workers is $\mu^b(5) = \{x_{1H}, x_{2L}\}$ which is different from

[40]Since the firm's preferences do not exhibit externalities, this initial matching does not impact how the algorithm runs. However, the initial matching matters for the worker-proposing version of the algorithm which we discuss next.

[41]In the variant of the deferred acceptance algorithm in which workers respond to offers made in the current round (see footnote 39) workers would be reacting to offers in $\{x_{1L}, x_{2L}\}$ with worker w_1 rejecting the offered contract x_{1L}, and worker w_2 accepting x_{2L}. In the symmetric cumulative process version of the algorithm, workers react to round 1 offers only in round 2.

$\mu^b(4) = \{x_{2L}\}$. Given this change of the reference matching, worker w_2 rejects the contract x_{2L}. In round 6, firm f thus raises worker w_2's wage to H, and worker w_2 rejects this high wage offer in round 7. The reference matchings are adjusted in round 8 and by then the algorithm converges: the contract sets and reference matchings are the same in rounds 8 and 9.

The worker-proposing deferred acceptance algorithm works similarly where workers play the role of the sellers and the firm plays the role of the buyer. The workers' initial reference matching μ^* matters and in the first phase of the algorithm we calculate it as follows: We set $\mu_0 = \emptyset$; then $\mu_1 = C^b(\{x_{1L}, x_{1H}, x_{2L}, x_{2H}\}|\emptyset) = \{x_{1H}, x_{2H}\}$, and finally

$$\mu_2 = C^b(\{x_{1L}, x_{1H}, x_{2L}, x_{2H}\}|\{x_{1H}, x_{2H}\}) = \{x_{1H}\}.$$

Since $\mu_3 = \mu_2$, we set $\mu^* = \{x_{1H}\}$. The cumulative-offer phase of deferred acceptance obtains μ^* after the first round.

4.2 A Characterization of Stable Matchings via Fixed Points of a Monotone Function

Let us introduce some notation for the proof of Theorem 1 and the subsequent proofs. Each iteration in the second phase of the deferred acceptance algorithm can be described as the following transformation function

$$f\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathscr{X} \setminus R^b(A^b|\mu^b),\, \mathscr{X} \setminus R^s(A^s|\mu^s),\, C^s(A^s|\mu^s),\, C^b(A^b|\mu^b)\right).$$

Notice that in the deferred acceptance algorithm the reference matchings μ^s, μ^b are always sets of contracts that can be chosen by the two sides of the market. In view of this, we define $\mathscr{M}^\theta = \left\{C^\theta(X|X')\,|X, X' \subseteq \mathscr{X}\right\}$ and define f as a function from $2^{\mathscr{X}} \times 2^{\mathscr{X}} \times \mathscr{M}^s \times \mathscr{M}^b$ into itself.[42]

The deferred acceptance function f has two important properties, monotonicity and stability of its fixed points, that are captured in the following two auxiliary results.

Lemma 2. *Suppose that the choice functions satisfy substitutability. Then, the deferred acceptance transformation function f is monotone increasing with respect to the preorder $\sqsubseteq$ on $2^{\mathscr{X}} \times 2^{\mathscr{X}} \times \mathscr{M}^s \times \mathscr{M}^b$ defined as follows:*

[42] If the domain of the preorder $\preceq^\theta$ is $\mathscr{A}^\theta \subsetneq 2^{\mathscr{X}}$, then we define $\mathscr{M}^\theta = \left\{C^\theta(X|X')\,|X, X' \subseteq \mathscr{X}\right\} \cap \mathscr{A}^\theta$; see footnote 34.

$$(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b) \Longleftrightarrow A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^s \tilde{\mu}^s, \mu^b \succeq^b \tilde{\mu}^b.$$

Proof. Function f is monotonic in $\sqsubseteq$ because for any $A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^s \tilde{\mu}^s, \mu^b \succeq^b \tilde{\mu}^b$, substitutability implies that

$$\begin{aligned} \mathscr{X} \backslash R^b(A^b|\mu^b) &\subseteq \mathscr{X} \backslash R^b(\tilde{A}^b|\tilde{\mu}^b), \\ \mathscr{X} \backslash R^s(A^s|\mu^s) &\supseteq \mathscr{X} \backslash R^s(\tilde{A}^s|\tilde{\mu}^s), \end{aligned}$$

and consistency implies that

$$\begin{aligned} C^s(A^s|\mu^s) &\preceq^s C^s(\tilde{A}^s|\tilde{\mu}^s), \\ C^b(A^b|\mu^b) &\succeq^b C^b(\tilde{A}^b|\tilde{\mu}^b). \end{aligned}$$

Therefore, $(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$ implies that $f(A^s, A^b, \mu^s, \mu^b) \sqsubseteq f(\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$. □

When choice functions are substitutable, a matching is stable if and only if it can be supported as a fixed point of f.

Theorem 3. *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then a matching μ is stable if and only if there exist sets of contracts $A^s, A^b \subseteq \mathscr{X}$ such that (A^s, A^b, μ, μ) is a fixed point of the deferred acceptance transformation function f.*

The proof is provided in Appendix C.

4.3 Proof of Theorem 1

First, let us consider the first phase of deferred acceptance and check that $\mu^* \succeq^s C^s(\mathscr{X}|\mu^*)$. By construction, $\mu_k \in \mathscr{M}^s$ for every $k \geq 1$. By the irrelevance of rejected contracts, we get $C^s(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \succeq^s \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathscr{X} \supseteq \emptyset$ and consistency imply that

$$\mu_1 = C^s(\mathscr{X}|\emptyset) \succeq^s C^s(\emptyset|\emptyset) = \emptyset = \mu_0.$$

For the general case, $\mu_k \succeq^s \mu_{k-1}$ and $\mathscr{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^s(\mathscr{X}|\mu_k) \succeq^s C^s(\mu_k|\mu_{k-1}) = \mu_k.$$

Therefore, $\{\mu_k\}_{k\geq 1}$ is a monotone sequence with respect to the preorder $\succeq^s$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$; we take the minimum m satisfying this property and set $\mu^* = \mu_m$. Then,

$$C^s(\mathscr{X}|\mu_m) = \mu_{m+1} = \mu_n \preceq^s \mu_m$$

where the monotonicity comparison follows as $\preceq^s$ is transitive.

It remains to show that the second phase of deferred acceptance converges and that the resulting matching is stable. It is easy to see that $f(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsubseteq (\mathscr{X}, \emptyset, \mu^*, \emptyset)$, since $C^s(\mathscr{X}|\mu^*) \preceq^s \mu^*$ by construction and $C^b(\emptyset|\emptyset) = \emptyset \succeq^b \emptyset$ by reflexivity of $\succeq^b$. By Lemma 2, f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^k(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsubseteq f^{k-1}(\mathscr{X}, \emptyset, \mu^*, \emptyset)$ for every k. We consider two separate cases. Suppose first that this sequence converges. Therefore, there exists k such that $f^{k-1}(\mathscr{X}, \emptyset, \mu^*, \emptyset) = f^k(\mathscr{X}, \emptyset, \mu^*, \emptyset)$. As a result, $f^{k-1}(\mathscr{X}, \emptyset, \mu^*, \emptyset)$ is a fixed point of f. Let $(A^{*s}, A^{*b}, \mu^{*s}, \mu^{*b}) \equiv f^{k-1}(\mathscr{X}, \emptyset, \mu^*, \emptyset)$. By Lemma 4, $\mu^{*s} = \mu^{*b}$ and μ^{*b} is a stable matching by Theorem 3.

Otherwise, if the sequence does not converge, there exists a subsequence $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^{n+1}(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq \ldots \sqsupseteq f^m(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^{m+1}(\mathscr{X}, \emptyset, \mu^*, \emptyset) = f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset)$ because the number of contracts is finite. By transitivity of the preorder $\sqsupseteq$ and the previous inequality, we get $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) = f^{m+1}(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^m(\mathscr{X}, \emptyset, \mu^*, \emptyset) \sqsupseteq f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset)$. Let $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) = (A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ and $f^m(\mathscr{X}, \emptyset, \mu^*, \emptyset) = (A_2^s, A_2^b, \mu_2^s, \mu_2^b)$. By definition of $\sqsupseteq$, we get that $A_1^s = A_2^s$, $A_1^b = A_2^b$, $\mu_1^s \sim^s \mu_2^s$, and $\mu_1^b \sim^b \mu_2^b$. Now, by construction $C^s(A_2^s|\mu_2^s) = \mu_1^s$ and by substitutability $C^s(A_2^s|\mu_2^s) = C^s(A_1^s|\mu_1^s)$, which imply that $C^s(A_1^s|\mu_1^s) = \mu_1^s$. Similarly, we get that $C^s(A_1^s|\mu_1^b) = \mu_1^b$. Furthermore, by substitutability, $\mathscr{X} \backslash R^b(A_2^b|\mu_2^b) = \mathscr{X} \backslash R^b(A_1^b|\mu_1^b)$ and, by construction, $\mathscr{X} \backslash R^b(A_2^b|\mu_2^b) = A_1^b$, which imply $\mathscr{X} \backslash R^b(A_1^b|\mu_1^b) = A_1^b$. Similarly, we get $\mathscr{X} \backslash R^s(A_1^s|\mu_1^s) = A_1^s$. Therefore, $(A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ is a fixed point of f. This shows that the sequence converges as in the previous paragraph, so there exists a stable matching.

4.4 Comments

Note that the proof above does not rely on Tarski's fixed point theorem, which is routinely used in the matching literature.[43] In fact, Tarski's fixed point theorem cannot be directly applied in our setting because even though f is monotone increasing, the domain of f does not have to be a (complete) lattice. In addition, there do not have to exist matchings that are optimal for buyers or sellers. As a result, the domain of f does not have extremal points, so the standard approach of applying f to the extreme points to get a fixed point fails. Furthermore, the binary relation $\sqsubseteq$ on the domain of f is not a partial order, which means that even if extreme points in the domain existed applying f would not necessarily converge to a fixed point as the preorder $\sqsubseteq$ could cycle.[44]

Theorems 1 and 2 establish that stable matchings exist when choice functions satisfy substitutability and the irrelevance of rejected contracts. Both conditions are necessary in the sense that when only one of them is satisfied there may not be any stable matchings: Example 1 of Aygün and Sönmez (2013) satisfies substitutability for the revealed preference but there exists no stable matching (because the irrelevance of rejected contracts fails). In the next example, the irrelevance of rejected contracts is satisfied but there exists no stable matching.

Example 8. Suppose that there are two buyers b_1, b_2 and one seller, s_1. There is only one contract associated with every seller-buyer pair. Let the contract between b_1 and s_1 be x_1 and the contract between b_2 and s_1 be x_2. Since there is only one seller, there are only externalities for buyers. Agents have the following preferences:

$$\begin{aligned}
\succeq_{b_1} &: \{x_1\} \succ \emptyset, \{x_2\} \succ \{x_1, x_2\}; \\
\succeq_{b_2} &: \{x_1, x_2\} \succ \{x_1\}, \emptyset \succ \{x_2\}; \\
\succeq_{s_1} &: \{x_1, x_2\} \succ \{x_1\} \succ \{x_2\} \succ \emptyset.
\end{aligned}$$

Construct agents' choice functions from their preferences. As a result, the choice functions satisfy the irrelevance of rejected contracts. Yet there exists no stable matching. To see this, first note that $\emptyset$ is not a stable matching because (b_1, s_1) forms a blocking pair with contract x_1. Second, $\{x_1\}$ is not a stable matching because (b_2, s_1) forms a blocking pair with contract x_2.

[43] For example, see Adachi (2000).

[44] Hatfield and Kojima (2010); Sönmez and Switzer (2013) also do not rely on Tarski for a different reason: their choice functions do not satisfy the standard substitutes condition.

Third, $\{x_2\}$ is not a stable matching because it is not individually rational for buyer b_2. Finally, $\{x_1,x_2\}$ is not a stable matching because it is not individually rational for buyer b_1.

5 Side-Optimal Stable Matchings

A key insight in the standard theory of stable matchings without externalities is that not only do stable matchings exist but also side-optimal stable matchings exist. The counterpart of this insight with externalities is given by the following:

Theorem 4. *Suppose that the choice functions satisfy substitutability, the irrelevance of rejected contracts, and, in addition, for side θ there exists a set of contracts $\bar{\mu}^\theta$ such that for any $\mu \in \mathscr{M}^\theta$, $\bar{\mu}^\theta \succeq^\theta \mu$. Then, there exists a stable matching $\hat{\mu}$ such that for any stable matching μ, we have $\hat{\mu} \succeq^\theta \mu$ and $\hat{\mu} \preceq^{-\theta} \mu$; that is, $\hat{\mu}$ is the θ-optimal and $(-\theta)$-pessimal stable matching. Furthermore, matching $\hat{\mu}$ can be obtained by running the second phase of side-θ-proposing deferred acceptance with μ^* set to $\bar{\mu}^\theta$.*

In the standard theory, side optimality is measured with respect to the preference rankings of agents on this side. This standard result is subsumed when $\succeq^\theta$ is derived from agents' preferences (as in the in-text example at the beginning of Section 3.1).

The assumption that there exists a set of contracts $\bar{\mu}^\theta$ such that for any $\mu \in \mathscr{M}^\theta$, $\bar{\mu}^\theta \succeq^\theta \mu$ is not innocuous but it is satisfied in all the motivating examples of Section 2. We comment more on this assumption below.

Proof. Without loss of generality assume that $\theta = s$. For any $(A^s,A^b,\mu^s,\mu^b) \in 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times \mathscr{M}^s \times \mathscr{M}^b$ we have $(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq (A^s,A^b,\mu^s,\mu^b)$. Therefore, $(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq f(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset)$. By Lemma 2, the deferred-acceptance transformation function f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^{k-1}(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq f^k(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset)$ for every k. Since $2^{\mathscr{X}} \times 2^{\mathscr{X}} \times \mathscr{M}^s \times \mathscr{M}^b$ is a finite set, this sequence converges at some point as in the proof of Theorem 1, so there exists k such that $f^{k-1}(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) = f^k(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset)$. Therefore, $f^{k-1}(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset)$ is a fixed point of f. By Lemma 4 there is $\left(\hat{A}^s,\hat{A}^b,\hat{\mu},\hat{\mu}\right)$ that is equal to $f^{k-1}(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset)$. Theorem 3 tells us that $\hat{\mu}$ is a stable matching.

We next show that $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. Let μ be any stable matching. By Theorem 3, there exist A^s and A^bsuch that (A^s,A^b,μ,μ) is a fixed point of f. Since $(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq (A^s,A^b,\mu,\mu)$ and f is monotonic increasing, f can be applied repeatedly while preserving the order. Therefore, $f^k(\mathscr{X},\emptyset,\bar{\mu}^s,\emptyset) \sqsupseteq f^k(A^s,A^b,\mu,\mu)$ for every

k, which implies $\left(\hat{A}^s, \hat{A}^b, \hat{\mu}, \hat{\mu}\right) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $\hat{\mu} \succeq^s \mu$ and $\hat{\mu} \preceq^b \mu$, so $\hat{\mu}$ is the seller-optimal and buyer-pessimal stable matching. $\square$

The assumption that there exists a set of contracts $\bar{\mu}^\theta$ such that for any $\mu \in \mathscr{M}^\theta$, $\bar{\mu}^\theta \succeq^\theta \mu$ plays a crucial role in the proof of Theorem 4. In the absence of externalities, this assumption is automatically satisfied when $\succeq^\theta$ is defined as $\mu \succeq^\theta \mu'$ if and only if for every $i \in \theta$, $c_i(\mu(i) \cup \mu'(i)) = \mu(i)$ (or, if and only, if all agents on side θ prefer μ over μ'). Indeed, we can take $\bar{\mu}$ to be the set of contracts which assigns each agent on side θ his unconstrained best set of contracts.[45] Furthermore, for this preorder $\succeq^\theta$ substitutability and irrelevance of rejected contracts are equivalent to the standard ones without externalities. Thus, Theorem 4 subsumes the standard insight that, in the absence of externalities, there exists a θ-optimal stable matching with respect to $\succeq^\theta$ if preferences satisfy substitutability and the irrelevance of rejected contracts. This matching is also $(-\theta)$-pessimal.

Furthermore, our assumption on $\bar{\mu}$ is equivalent to the following: for any two matchings μ and μ', there exists another matching $\tilde{\mu}$ such that $\tilde{\mu} \succeq^\theta \mu$ and $\tilde{\mu} \succeq^\theta \mu'$. In fact, in light of our analysis of the first phase of deferred acceptance, it is enough to impose this assumption on matchings μ such that $C^\theta(\mathscr{X}|\mu) \preceq^\theta \mu$.

6 Comparative Statics and "Vacancy Chain" Dynamics

In this section, we first present a comparative statics result that goes beyond the classic theory of stable matchings. Then we look at the welfare implications of an agent retiring from the market.

6.1 Comparative Statics on Strength of Externalities and Substitutes

How do stable matchings change when externalities and substitutability are strengthened? To answer this question, we first introduce the notions of having weaker externalities and stronger substitutability.

Definition 4. Choice function $\hat{C}^\theta$ exhibits **stronger substitutability** than choice function C^θ if $R^\theta(X|\mu) \subseteq \hat{R}^\theta(X|\mu)$ for any $\mu, X \subseteq \mathscr{X}$.

[45]Notice that this point remains true regardless of whether all sets of contracts are matchings or only some sets of contracts are matchings because of some feasibility constraints as, for instance, in one-to-one matching. This is so because we allow $\bar{\mu}$ to be any set of contracts.

Strengthening the substitutes means that agents choose fewer contracts or reject more. Equivalently, we can think of shrinking the choice function so that agents choose only a subset of the previously chosen contracts.[46] To get a sense of this assumption, consider for instance Example 4 (in its general, quantile form). In this example, the larger k is the stronger substitutability of the colleges' choice function. In Example 2, the choice functions satisfy stronger substitutability as an attorney's profits from contracts signed by the attorney decrease relative to his profits from working on contracts signed by other attorneys.

Definition 5. Choice function $\hat{C}^\theta$ exhibits **weaker externalities** than choice function C^θ if $\hat{C}^\theta(X|\mu)\,\hat{\succeq}^\theta C^\theta(X|\mu)$ for any $\mu, X \subseteq \mathscr{X}$.

Note that if choice function $\hat{C}^\theta$ exhibits no externalities then it has weaker externalities than any other choice function when $\hat{\succeq}^\theta$ is the revealed preference for side θ. In the context of Example 4, the positive externalities are weaker when the benchmark ratio k is higher. Notice that the choice function when $k=\infty$ and the choice function when $k=0$ exhibit no externalities, and thus have weaker externalities than the intermediate choice functions.

In the result below, we consider two seller choice functions C^s and $\hat{C}^s$. Suppose that preorder $\succeq^s$ is consistent with C^s and preorder $\hat{\succeq}^s$ is consistent with $\hat{C}^s$. Assume that both choice functions satisfy the irrelevance of rejected contracts and substitutability.

Theorem 5. *Suppose that $\hat{C}^s$ exhibits stronger substitutability and weaker externalities than C^s. Then for any (C^b, C^s)-stable matching μ there exists a $(C^b, \hat{C}^s)$- stable matching μ^* such that $\mu \succeq^b \mu^*$ and $\mu^* \hat{\succeq}^s \mu$.*

In the context of Example 4, as colleges raise the hiring benchmark, the quality of academics hired in stable matchings increases. Whenever the side-optimal and side-pessimal stable matchings exist, the market conditions are better for buyers in the buyer-optimal $\hat{\succeq}$-stable matching than in the buyer-optimal $\succeq$-stable matching; and the converse holds for the sellers. When one side of the market faces no externalities, then the preorder $\succeq^\theta$ that ranks μ above μ^* whenever all agents on this side prefer μ over μ^* is consistent with this side's choice behavior. Hence, if, say, buyers face no externalities then they would all prefer μ over μ^*. This gives us:

Corollary 1. *Suppose that $\hat{C}^s$ does not exhibit any externalities and that $\hat{C}^s$ has stronger substitutes than C^s. Then for any (C^b, C^s)-stable matching μ there exists a $(C^b, \hat{C}^s)$- stable matching*

[46] In the terminology of Echenique and Yenmez (2012), choice function C^θ is an expansion of choice function $\hat{C}^\theta$ if for any $\mu, X \subseteq \mathscr{X}$, $C^\theta(X|\mu) \supseteq \hat{C}^\theta(X|\mu)$. This is equivalent to the stronger substitutes comparison above. Note that the result of this subsection specialized to the setting without externalities does not have a counterpart in Echenique and Yenmez (2012).

μ^* *such that all buyers prefer* μ *over* μ^*.

Proof of Theorem 5. For any $A^s, A^b, \mu^s, \mu^b \subseteq \mathscr{X}$, let

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathscr{X} \backslash R^b(A^b|\mu^b),\, \mathscr{X} \backslash \hat{R}^s(A^s|\mu^s),\, \hat{C}^s\left(A^s|\mu^s\right), C^b(A^b|\mu^b)\right).$$

Since μ is a (C^s, C^b)-stable matching, there exist $A^s, A^b \subseteq \mathscr{X}$ such that (A^s, A^b, μ, μ) is a fixed point of f (Theorem 3). By Lemma 4, $C^s(A^s|\mu) = C^b(A^b|\mu) = \mu$. By strong substitutes, $X \setminus \hat{R}^s(A^s|\mu) \subseteq X \setminus R^s(A^s|\mu)$; by weaker externalities, $\hat{C}^s\left(A^s|\mu\right) \hat{\succeq}^s C^s(A^s|\mu)$. Hence, $(A^s, A^b, \mu, \mu) = f(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}(A^s, A^b, \mu, \mu)$. Since $\hat{f}$ is monotone $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}^k(A^s, A^b, \mu, \mu)$ for all $k \geq 1$. Since the number of contracts is finite, there exists k such that $\hat{f}^{k-1}(A^s, A^b, \mu, \mu)$ is a fixed point of $\hat{f}$ as in the proof of Theorem 2. By Lemma 4, $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) = (\hat{A}^s, \hat{A}^b, \mu^*, \mu^*)$, and by Theorem 3, μ^* is a $(\hat{C}^s, C^b)$-stable matching. By construction, $\mu^* \hat{\succeq}^s \mu$ and $\mu \succeq^b \mu^*$. □

6.2 Vacancy Chain Dynamics

Let us consider the classic retirement problem in matching. Suppose that agent $i \in \theta$ retires. Then all of the contracts that agent i has signed are annulled. Some agents may be affected by the removal of these contracts. Therefore, agents may want to add new contracts, or they may want to remove some of the existing contracts. But the addition or removal of a new contract may also affect the remaining agents in the market, which may lead to other changes in the set of contracts. We analyze such changes and show that there is a *vacancy chain dynamics* (Crawford, 1991; Blum, Roth, and Rothblum, 1997) that leads to a stable matching in which agents on side θ are better off and agents on side $-\theta$ are worse off. Similar vacancy chain dynamics have been studied in different matching markets without externalities (e.g., Kelso and Crawford, 1982; Hatfield and Milgrom, 2005). Our construction shows that vacancy chain dynamics extend to the setting with externalities.

Without loss of generality, we fix the choice functions of agents other than some seller i while we compare two possible choice functions of seller i, say c_i and $\hat{c}_i$, where this agent rejects all contracts under $\hat{c}_i$. Let the corresponding rejection functions be r_i and $\hat{r}_i$, respectively. Less formally, the retirement of seller i is interpreted as no offers being accepted by seller i and so all offers being rejected by her. Thus, she prefers the empty set of contracts to any other set regardless of the contracts signed by the rest of the agents. On the other hand, the rejection set for the buyers is the same. For any $X, \mu \subseteq \mathscr{X}$, $\hat{C}^s\left(X|\mu\right) \equiv \hat{c}_i(X|\mu) \cup \bigcup_{j \in s \setminus \{i\}} c_j(X|\mu)$.

We assume that C^s satisfies substitutability and the irrelevance of rejected contracts for preorder $\succeq^s$. In addition, assume that $\hat{C}^s$ satisfies substitutability and the irrelevance of rejected contracts for preorder $\hat{\succeq}^s$. Likewise, C^b satisfies substitutability and the irrelevance of rejected contracts for preorder $\succeq^b$. Notice that in the contexts of our motivating examples, all these assumptions are satisfied.

To study the vacancy-chain dynamics, we need to modify the function f. For any $A^s, A^b, \mu^s, \mu^b \subseteq \mathscr{X}$,

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathscr{X} \setminus R^b(A^b|\mu^b),\ \mathscr{X} \setminus \hat{R}^s(A^s|\mu^s),\ \hat{C}^s(A^s|\mu^s),\ C^b(A^b|\mu^b)\right).$$

Let $(A^s(0), A^b(0), \mu^s(0), \mu^b(0))$ be the initial matching that is stable with seller i present in the market. After removing seller i from the market, agents start recontracting dynamically. This process can be described through the function $\hat{f}$. Let $(A^s(t), A^b(t), \mu^s(t), \mu^b(t)) \equiv \hat{f}(A^s(t-1), A^b(t-1), \mu^s(t-1), \mu^b(t-1))$. We call this the **vacancy chain dynamics**. In our setting, $\hat{f}$ is monotonic since we impose the substitutes and irrelevance of rejected contracts assumptions both on the original choice function profile and on the profile when agent i rejects all offers (or, equivalently, has retired).

Theorem 6. *Suppose that $\hat{C}^s$ exhibits stronger positive externalities than C^s. Let (A^s, A^b) be a $(\hat{C}^s, C^b)$-stable set of contracts with stable matching $\mu \equiv A^s \cap A^b$. Then the vacancy chain dynamics converges to $(A_s^*, A_b^*, \mu^*, \mu^*)$ where μ^* is a $(\hat{C}^s, C^b)$-stable matching such that $\mu^* \hat{\succeq}^s \mu$ and $\mu \succeq^b \mu^*$.*

The assumption that $\hat{C}^s$ exhibits stronger positive externalities than C^s is satisfied in Example 4. Thus, in this example the closure of one of the colleges leads to an increase in the quality of academics hired by the remaining colleges.

Proof. Since (A^s, A^b) is a stable set of contracts, (A^s, A^b, μ, μ) is a fixed point of f. By Lemma 4, $C^s(A^s|\mu) = C^b(A^b|\mu) = \mu$. By definition, $\mathscr{X} \setminus \hat{R}^s(A^s|\mu) \subseteq \mathscr{X} \setminus R^s(A^s|\mu)$, and $\hat{C}^s(A^s|\mu) = \mu_{-i}$. By stronger externalities, we have $\mu_{-i} \hat{\succeq}^s \mu$. Hence, $(A^s, A^b, \mu, \mu) = f(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}(A^s, A^b, \mu, \mu)$. Since $\hat{f}$ is monotone $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) \hat{\sqsubseteq} \hat{f}^k(A^s, A^b, \mu, \mu)$ for all $k \geq 1$. Since the number of contracts is finite, there exists k such that $\hat{f}^{k-1}(A^s, A^b, \mu, \mu)$ is a fixed point of $\hat{f}$ as in the proof of Theorem 2. By Lemma 4, $\hat{f}^{k-1}(A^s, A^b, \mu, \mu) = (\hat{A}^s, \hat{A}^b, \mu^*, \mu^*)$, and by Theorem 3, μ^* is a stable matching in the market without seller i. By construction, $\mu^* \hat{\succeq}^s \mu$ and $\mu \succeq^b \mu^*$. $\square$

7 Conclusion

In this paper, we have studied a two-sided matching problem with externalities where each agent's choice depends on other agents' contracts. For such settings, we have developed the the theory of stable matchings by introducing conditions on agents' choice behavior. More explicitly, we have studied the existence of stable matchings, side-optimal stable matchings, vacancy-chain dynamics, the deferred acceptance algorithm, comparative statics depending on the strength of externalities and substitutes, and the rural hospitals theorem (which is in Appendix A). Unlike the previous matching literature, we have not relied on fixed point theorems; instead, we have used elementary techniques to overcome the difficulties associated with externalities.

Even though we have studied two-sided markets, our techniques are applicable to more general markets such as the supply chain networks of Ostrovsky (2008) where externalities may naturally appear. This is left open for future research.

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Appendix A: Law of Aggregate Demand and the Rural Hospitals Theorem

We provide a generalization of the law of aggregate demand (Hatfield and Milgrom, 2005) and size monotonicity (Alkan and Gale, 2003; Alkan, 2002), which is due to Fleiner (2003) for markets without externalities. For each contract $x \in \mathscr{X}$, there is a corresponding positive weight denoted by $w(x)$. The generalized law of aggregate demand requires that for agent $i \in \theta$ the total weight of contracts chosen from X conditional on μ is weakly smaller than the total weight of contracts chosen from X' conditional on μ' for any $X' \supseteq X$ and $\mu' \succeq^{\theta} \mu$. For a set of contracts $X \subseteq \mathscr{X}$, let $w(X) \equiv \sum_{x \in X} w(x)$. We provide a formal definition as follows.

Definition 6. Choice function c_i satisfies **the law of aggregate demand** if $i \in \theta$ and for any $X \subseteq X'$ and $\mu \preceq^{\theta} \mu'$ then $w(c_i(X|\mu)) \leq w(c_i(X'|\mu'))$.

Previous definitions in the matching literature are restricted to the settings without externalities, and assume that the weight on all contracts are exactly equal.[47] Under this assumption, the generalized law of aggregate demand reduces to for any $X \subseteq X'$ and $\mu \subseteq \mathscr{X}$, $|c_i(X|\mu)| \leq |c_i(X'|\mu)|$. In terms of the demand metaphor of Hatfield and Milgrom (2005), all contracts are traded at price one. In contrast, we allow any prices.

We study how the weight of contracts changes for an agent in different stable matchings. We show that the weight remains the same regardless of the stable matching. This extends the rural hospitals theorem of Roth (1986) in two directions:[48] We allow different contracts to have different weights and also preferences of an agent can depend on contracts signed by others.

[47]The only exception is Fleiner (2003).

[48]Hatfield and Milgrom (2005) show the rural hospitals theorem for the many-to-one matching with contracts setup in which there are no externalities.

Theorem 7. *Suppose that choice functions satisfy substitutability, the irrelevance of rejected contracts, and the law of aggregate demand for a weight function w. In addition, there exists a matching $\bar{\mu}^\theta$ such that for any $\mu \in \mathscr{M}^\theta$, $\bar{\mu}^\theta \succeq^\theta \mu$ for side θ. Then for any two stable matchings μ and μ', $w(\mu_i) = w(\mu'_i)$ for every agent i.*

Proof. Without loss of generality assume that $\theta = s$. Then by Theorem 4, there exists a stable matching μ^*, which is seller-optimal and buyer-pessimal simultaneously. We show that for any stable matching μ, $w(\mu_i) = w(\mu_i^*)$. As it is shown in the proof of Theorem 4 f has two fixed points $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) such that $(A^{*s}, A^{*b}, \mu^*, \mu^*) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $A^{*s} \supseteq A^s$, $A^{*b} \subseteq A^b$, $\mu^* \succeq^s \mu$ and $\mu^* \preceq^b \mu$. Now by the law of aggregate demand for any $i \in S$, $w(c_i(A^{*s}|\mu^*)) \geq w(c_i(A^s|\mu))$, which is equivalent to $w(\mu_i^*) \geq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When this is summed over all sellers, we get $w(\mu^*) \geq w(\mu)$. Similarly, for any $i \in B$, $w(c_i(A^{*b}|\mu^*)) \leq w(c_i(A^b|\mu))$, which is equivalent to $w(\mu_i^*) \leq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When summed over all buyers, this implies $w(\mu^*) \leq w(\mu)$. Therefore, $w(\mu^*) = w(\mu)$, moreover, all of the individual inequalities must hold as equalities implying that for any agent i, $w(\mu_i^*) = w(\mu_i)$. $\square$

Remark 2. When all the weights are positive, substitutability and the law of aggregate demand imply the irrelevance of rejected contracts. This is easy to see: Suppose that $X', X, \mu \subseteq \mathscr{X}$ are such that $c_i(X'_i|\mu) \subseteq X_i \subseteq X'_i$ for agent i. Then substitutability implies that $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$. Since weights are positive we get $w(c_i(X_i|\mu)) \geq w(c_i(X'_i|\mu))$. Now, since $X_i \subseteq X'_i$, the law of aggregate demand implies that $w(c_i(X_i|\mu)) \leq w(c_i(X'_i|\mu))$. Consequently, we need to have $w(c_i(X_i|\mu)) = w(c_i(X'_i|\mu))$. Since all weights are positive and $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$, we get $c_i(X_i|\mu) = c_i(X'_i|\mu)$, the desired conclusion.

In addition, under these assumptions an agent's choice from the same set conditional on two ranked matchings needs to be the same. Let $i \in \theta$ be an agent. Suppose that $X, \mu, \mu' \subseteq \mathscr{X}$ are such that $\mu \preceq^\theta \mu'$. Then, by substitutability, $c_i(X|\mu) \supseteq c_i(X|\mu')$. But the law of aggregate demand implies that $w(c_i(X|\mu)) \leq w(c_i(X|\mu'))$. Since all weights are positive, we get that $c_i(X|\mu) = c_i(X|\mu')$. However, this argument does not mean that we cannot have externalities because the choice conditional on two matchings that are not ranked with respect to $\succeq^\theta$ can still be different.

Appendix B: Core Stability

A set $X \subseteq \mathscr{X}$ **blocks** matching μ if $X \nsubseteq \mu$ and for all $i \in \mathscr{I}$ we have $X_i \subseteq c_i(\mu \cup X|\mu)$. Less formally, conditional on matching μ, every agent who is associated with a contract in X wants to have all contracts in X associated with her. In this case, X is also called a **blocking set** for μ. Without externalities, this stability concept is the same as core stability (Roth and Sotomayor, 1990). Therefore, a matching is **core stable** if it is individually rational matching and it does not have any blocking.

Proposition 1. ***[Equivalence of Stability and Core Stability]*** *A matching is stable if and only if it is core stable.*

See Roth and Sotomayor (1990), Echenique and Oviedo (2004), and Hatfield and Kominers (2014) for earlier developments of this equivalence. To prove the proposition it is enough to prove the following

Lemma 3. *Suppose X blocks matching μ and choice functions satisfy substitutability. Then for every $x \in X \setminus \mu$, $\{x\}$ blocks μ.*

Proof. If X is a blocking set, then $X \subseteq C^s(\mu \cup X|\mu) \cap C^b(\mu \cup X|\mu)$. Take any $x \in X \setminus \mu$. Since choice function c_i satisfies substitutability, we have $r_i(\mu \cup \{x\}|\mu) \subseteq r_i(\mu \cup X|\mu)$ for every agent i. This implies $x \in c_i(\mu \cup \{x\}|\mu)$ for every i, so $x \in C^s(\mu \cup \{x\}|\mu) \cap C^b(\mu \cup \{x\}|\mu)$. Therefore, $\{x\}$ is a blocking set for μ. $\square$

Appendix C: Proof of Theorem 3

Let us first observe that the fixed points of f satisfy the following:

Lemma 4. *Let $\left(A^s, A^b, \mu^s, \mu^b\right)$ be a fixed point of the deferred-acceptance transformation function f. Then $A^s \cup A^b = \mathscr{X}$ and*

$$\mu^s = \mu^b = A^s \cap A^b = C^b(A^b|\mu^b) = C^s(A^s|\mu^s).$$

Proof. $A^s \cup A^b = A^s \cup [\mathscr{X} \setminus R^s(A^s|\mu^s))] \supseteq A^s \cup [\mathscr{X} \setminus A^s] = \mathscr{X}$, so

$$A^s \cup A^b = \mathscr{X}.$$

Similarly, $A^s \cap A^b = A^s \cap [\mathscr{X} \setminus R^s(A^s|\mu^s))] = A^s \setminus R^s(A^s|\mu^s) = C^s(A^s|\mu^s)$, which implies $C^s(A^s|\mu^s) = A^s \cap A^b$. Analogously for b, $C^b\left(A^b|\mu^b\right) = A^s \cap A^b$. Finally, $\mu^\theta = C^\theta\left(A^\theta|\mu^\theta\right)$ implies

$$\mu^s = \mu^b = A^s \cap A^b = C^b(A^b|\mu^b) = C^s(A^s|\mu^s).$$

□

Proof of Theorem 3. Suppose that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. Before we show that μ is a stable matching, we need the following.

Claim 1. Suppose that choice functions satisfy substitutability and the irrelevance of rejected contracts. Then matching μ is stable.

Proof. Suppose for contradiction that μ is not stable. Then there are three possibilities, all of which we proceed to rule out.

1. Matching μ is not individually rational for some seller j, that is $c_j(\mu|\mu) \subsetneq \mu_j$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, $C^s(A^s|\mu) = \mu$ and $A^s \supseteq \mu$. But substitutability and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^s by agent j, that is $x \notin C^s(A^s|\mu)$, a contradiction.

2. Matching μ is not individually rational for some buyer i, that is $c_i(\mu|\mu) \subsetneq \mu_i$.This is analogous to the previous case since f treats buyers and sellers symmetrically.

3. There exists a blocking pair with contract $x \in X \setminus \mu$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 4 $A^s \cup A^b = \mathscr{X}$. Therefore, without loss of generality, assume that $x \in A^b$. Since $\{x\}$ is a blocking set, there exists buyer i such that $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$. Again, since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 4 $C^b(A^b|\mu) = \mu$, which implies that $c_i(A^b|\mu) = \mu_i$. By the irrelevance of rejected contracts, for any set Y such that $A^b \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular, for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}|\mu) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$.

To finish the proof of the theorem, we need to show that if matching μ is stable then there exist sets of contracts A^s, A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. The following is useful in our construction of A^s and A^b.

Claim 2. Suppose that choice functions satisfy substitutability and the irrelevance of rejected contracts. Then the function $M^\theta(\mu) \equiv \max\{X \subseteq \mathscr{X} | C^\theta(X|\mu) = \mu\}$, where the maximum is with respect to set inclusion, is well defined. Moreover, for any contract $z \notin M^\theta(\mu)$, $z \in C^\theta(M^\theta(\mu) \cup z|\mu)$.

Proof. If there are two sets M' and M'' such that $C^{\theta}(M'|\mu) = C^{\theta}(M''|\mu) = \mu$, then (by substitutability)

$$\begin{aligned} C^{\theta}\left(M' \cup M''|\mu\right) &= \left(M' \cup M''\right) \setminus R^{\theta}\left(M' \cup M''|\mu\right) = \left[M' \setminus R^{\theta}\left(M' \cup M''|\mu\right)\right] \cup \left[M'' \setminus R^{\theta}\left(M' \cup M''|\mu\right)\right] \\ &\subseteq \left[M' \setminus R^{\theta}\left(M'|\mu\right)\right] \cup \left[M'' \setminus R^{\theta}\left(M''|\mu\right)\right] = \mu. \end{aligned}$$

If $C^{\theta}(M' \cup M''|\mu)$ was a proper subset of μ, then the irrelevance of rejected contracts would imply that $C^{\theta}(M'|\mu) = C^{\theta}(M''|\mu) = C^{\theta}(M' \cup M''|\mu)$, which is a contradiction. Therefore, $M^{\theta}(\mu)$ is well defined. Let $x \notin M = M^{\theta}(\mu)$. If $x \notin C^{\theta}(M \cup x|\mu)$, then $C^{\theta}(M \cup x|\mu) = C^{\theta}(M|\mu)$ by the irrelevance of rejected contracts. But this implies $C^{\theta}(M \cup x|\mu) = \mu$, which contradicts maximality of M. Hence $x \in C^{\theta}(M \cup x|\mu)$.

Claim 3. Suppose that the matching μ is stable and the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then there exist sets of contracts A^s and A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f.

Proof. By Claim 2, there exists the largest set $M^{\theta}(\mu) \equiv \max\{X \subseteq \mathscr{X} | C^{\theta}(X|\mu) = \mu\}$. Let $A^s \equiv M^s(\mu)$ and $A^b \equiv \mathscr{X} \setminus R^s(A^s|\mu)$. By definition, $A^b = \mathscr{X} \setminus R^s(A^s|\mu)$ and $\mu = C^s(A^s|\mu)$. Thus, we get $A^s \cap A^b = A^s \cap (\mathscr{X} \setminus R^s(A^s|\mu)) = C^s(A^s|\mu) = \mu$. To finish the proof, we need to show $\mu = C^b(A^b|\mu)$ and $A^s = \mathscr{X} \setminus R^b(A^b|\mu)$.

Note that $A^b = \mathscr{X} \setminus R^s(A^s|\mu) = (\mathscr{X} \setminus A^s) \cup C^s(A^s|\mu) = (\mathscr{X} \setminus A^s) \cup \mu$. In particular, $A^b \supseteq \mu$. If $C^b(A^b|\mu) = Y \neq \mu$, there are two cases, both of which contradict stability of μ. First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^b(\mu|\mu) = Y$, implying that μ is not individually rational for some buyers, contradicting stability. Second, if $Y \nsubseteq \mu$, then there exists a $y \in Y \setminus \mu$, and $y \in C^b(\mu \cup \{y\}|\mu)$ by substitutability since $y \in C^b(A^b|\mu)$ and $A^b \supseteq \mu \cup \{y\}$. But we also have that $y \in C^s(A^s \cup \{y\}|\mu)$ by Claim 2. Then $\{y\}$ blocks μ, contradicting stability. Thus, the only case consistent with stability is $C^b(A^b|\mu) = \mu$.

Finally, we show that $A^s = \mathscr{X} \setminus R^b(A^b|\mu) = \mathscr{X} \setminus R^b(\mathscr{X} \setminus R^s(A^s|\mu)|\mu)$. Since $C^b(A^b|\mu) = \mu$, then $\mathscr{X} \setminus R^b(A^b|\mu) = \mathscr{X} \setminus (A^b \setminus \mu) = \mathscr{X} \setminus (((\mathscr{X} \setminus A^s) \cup \mu) \setminus \mu) = \mathscr{X} \setminus (\mathscr{X} \setminus A^s) = A^s$ and we have the result.

Appendix D: Another Illustration of the Deferred Acceptance Algorithm

We use Example 6 to offer another illustration of the deferred acceptance algorithm. Suppose that there are no externalities for sellers and that $C^s(X|\mu) = X$ for any set of contracts X and μ. We need to find a matching μ^* such that $C^s(\mathscr{X}|\mu^*) \preceq^s \mu^*$ where $\mathscr{X} = \{x_1, x_2, x_3\}$. Since $C^s(\mathscr{X}|\mu^*) = \mathscr{X}$, $\mu^* = \mathscr{X}$ works.

The algorithm starts at $A^s(1) = \mathscr{X}$, $A^b(1) = \emptyset$, $\mu^s(1) = \mathscr{X}$, and $\mu^b(1) = \emptyset$. The table below shows the iterations of the algorithm.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^s(A^s(t)\|\mu^s(t))$	$C^b(A^b(t)\|\mu^b(t))$
$t=1$	$\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\emptyset$
$t=2$	$\mathscr{X}$	$\mathscr{X}$	$\mathscr{X}$	$\emptyset$	$\mathscr{X}$	$\{x_1,x_2\}$
$t=3$	$\{x_1,x_2\}$	$\mathscr{X}$	$\mathscr{X}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$
$t=4$	$\{x_1,x_2\}$	$\mathscr{X}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$
$t=5$	$\{x_1,x_2\}$	$\mathscr{X}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$	$\{x_1,x_2\}$

Table 4: Steps of the Deferred Acceptance Algorithm.

For $t = 2$, we first compute $A^s(2) = \mathscr{X} \setminus R^b(A^b(1)|\mu^b(1)) = \mathscr{X}$. For buyers, $A^b(2) = \mathscr{X} \setminus R^s(A^s(1)|\mu^s(1)) = \mathscr{X}$. The chosen contracts for buyers and sellers give us $\mu^s(2) = C^s(A^s(1)|\mu^s(1)) = \mathscr{X}$ and $\mu^b(2) = C^b(A^b(1)|\mu^b(1)) = \emptyset$.

We keep iterating these steps until we arrive at a fixed point, which happens at Step 5. Therefore, the generalized deferred acceptance algorithm produces $A^s(5) \cap A^b(5) = \mu^s(5) = \mu^b(5) = \{x_1, x_2\}$, which is a stable matching.