Beyond Random Assignment: Credible Inference of Causal Effects in Dynamic Economies*

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Abstract

Random assignment is insufficient for measured treatment responses to recover causal effects (comparative statics) in dynamic economies. We characterize analytically bias probabilities and magnitudes. If the policy variable is binary there is attenuation bias. With more than two policy states, treatment responses can undershoot, overshoot, or have incorrect signs. Under permanent random assignment, treatment responses overshoot (have incorrect signs) for realized changes opposite in sign to (small relative to) expected changes. We derive necessary and sufficient conditions, beyond random assignment, for correct inference of causal effects: martingale policy variable. Infinitesimal transition rates are only sufficient absent fixed costs. Stochastic monotonicity is sufficient for correct sign inference. If these conditions are not met, we show how treatment responses can nevertheless be corrected and mapped to causal effects or extrapolated to forecast responses to future policy changes within or across policy generating processes.

Keywords: Random Assignment, Causal effect, Treatment Response, Structural, Dynamic, Investment, Identification, Bias

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1. Introduction

The goal of most empirical work in economics is to estimate signs and magnitudes of causal effects (comparative statics). Correct identification of causal effect signs is critical given that a theory may be viewed as falsified if it incorrectly predicts a sign. For example, in claiming to document a positive causal relationship between minimum wages and employment, Card and Krueger (1995) write "If accepted, our findings call into question the standard model of the labor market that has dominated economists' thinking for the past half century." Correct estimation of causal effect magnitudes is important since elasticities are key inputs in welfare analysis, thus influencing policy decisions. For example, based on low estimated tax-elasticities, Slemrod (1992) and Aaron (1992) call for less attention to excess burdens. Angrist and Pischke (2009) point out another role for causal effects in stating, "A causal relationship is useful for making predictions about the consequences of changing circumstances or policies; it tells us what would happen in alternative (or 'counterfactual') worlds."

Angrist and Pischke (2010) herald the search for sources of random assignment as a "credibility revolution" in overcoming the causal effect identification problem. They argue that "The most credible research designs are those that exploit random assignment." Their textbook, *Mostly Harmless Econometrics*, states, "The goal of most empirical research is to overcome selection bias, and therefore to have something to say about the causal effect of a variable."

Inspired by the credibility revolution, a varied empirical literature exploits (quasi) random assignment in search of causal effects in dynamic economies. Greenstone (2002) and Greenstone and Chay (2005) exploit random air quality regulation to estimate causal effects on firm activity and house prices, respectively. Deschenes and Greenstone (2007) use annual weather fluctuations to infer causal effects of long-term climate change on firm profitability. An extensive public finance literature, e.g. Cummins, Hassett and Hubbard (1994), treats tax code changes as natural experiments. Romer and Romer (2010) identify exogenous tax changes to estimate their macroeconomic effects. Card and Krueger (1994) analyze minimum wage effects using difference-in-differences. Banerjee, Duflo, Glennerster and Kinnan (2015) and Crepon, Devoto, Duflo and Pariente (2015) estimate causal impacts of microcredit programs on investment using random assignment. Gan (2007) and Chaney, Sraer, Thesmar (2012) exploit real estate price fluctuations to estimate effects of collateral on investment. Werker, Ahmed, and Cohen (2009) use oil price fluctuations to identify causal effects of foreign aid.

It might appear from the preceding discussion that the task of identifying causal effect parameters in dynamic economies is already a "mission-accomplished." Indeed, in surveying the accomplishments of econometricians over the preceding millennium, Heckman (2000) writes:

> The major contributions of twentieth century econometrics to knowledge were the definition of causal parameters within well-defined economic models... [and] the analysis of what is required to recover causal parameters from the data (the identification problem)...

The preceding quotation notwithstanding, the set of identifying assumptions required for recovery of causal effect parameters in dynamic economies is not known. Rather, the same identifying assumption invoked in static settings, random assignment, is informally invoked as the identifying assumption in dynamic settings. For example, the textbook of Angrist and Pischke (2009) does not devote any special attention to inference in dynamic economies, perhaps in the belief that dynamics create no special issues. And such would be the inevitable consequence of a belief that selection is *the* problem to be overcome in empirical work on causal effects.

The overarching objective of this paper is to complete the econometric mission-statement of Heckman (2000) by deriving the required identifying assumptions for the recovery of causal effect parameters in dynamic economies. To accomplish this, we examine the relationship between *causal effects* (comparative statics/differentials) and measured *treatment responses* (empirically measured responses to policy changes). We make four contributions. First, we show random assignment is not generally sufficient for recovery of causal effects in dynamic economies. Second, we characterize analytically bias magnitudes and probabilities. Third, we derive sets of identifying assumptions, beyond random assignment, that must be satisfied in order to correctly identify causal effect signs and/or magnitudes. Finally, we provide an error correction algorithm facilitating correct identification of causal effects when the safe-harbor assumptions are not satisfied. This algorithm also allows one to recover policy invariant structural parameters in dynamic economies, and permits extrapolation robust to the critique of Lucas (1976).

We consider an econometrician seeking to empirically estimate the causal effect parameters implied by a canonical dynamic theory, the q-theory of capital accumulation under quadratic adjustment costs, e.g. Abel and Eberly (1997). Following Abel and Eberly (1994) and Chetty (2012), we also allow for general convex adjustment costs, imperfect reversibility, and fixed costs of investment. The empiricist measures treatment responses with the goal of inferring causal effects, as well as policy-invariant structural parameters. Critical for our exercise is that here we know the true causal effect signs and magnitudes.

We show that random assignment is insufficient for valid inference in dynamic settings. Undoubtedly, some empiricists have a heuristic sense that transience clouds the interpretation of econometric evidence. However, the direction, magnitude and probability of biases have not been analyzed formally, and thus, are not well understood.

Conventional wisdom holds that policy transience results in attenuation bias in the context of real investment (see Slemrod (1992), Aaron (1992), Cummins, Hassett and Hubbard (1994), and Atanasov and Black (2015)). As we show, however, this conventional wisdom is only valid only for limited scenarios. One such scenario is when the policy variable is binary, the focus of early theoretical work. Another scenario generating attenuation bias is when policy variable transitions are uniformly distributed. Even in such cases, the magnitude of the bias is not well understood. For example, under plausible parameterizations, binary assignment and uniformly-distributed assignment results in attenuation bias exceeding 50% even if the expected duration of policy regimes is relatively long (10 years).

As we show, if one considers more realistic empirical settings, with non-binary policy variables, the set of econometric issues becomes far more pernicious. Here treatment responses can understate, overstate, and even have signs opposite to causal effects. More importantly, the quantitative implications are both surprising and troubling. Consider for example, an economy featuring a 5% discount rate, an expected regime life of five years, with a future permanent policy variable change drawn from the uniform distribution on [0,1]. Here, the econometrician will face attenuation bias for 60% of the possible random treatments, those on [0.40,1]. For the remaining 40% of possible treatments, those on [0,0.40), the treatment response is actually opposite in sign to the causal effect. If instead the treatments are drawn from the uniform distribution on [-0.20, 0.80], the treatment response will overshoot the causal effect with probability 20%, have the wrong sign with probability 24%, and undershoot with probability 56%. With skewed policy variable changes, the majority of experiments can have the wrong sign. It is hard to understand the sense in which estimators with such properties can be justifiably described as "credible."

After diagnosing the inference problem arising from dynamics, the majority of the paper is devoted to a constructive analysis of how the problems identified can be addressed. We begin by deriving a necessary and sufficient condition, beyond random assignment, which ensures equality of treatment responses and causal effects: martingale policy variable. This result holds in the presence of convex adjustment costs, wedges between the buy and sell price of capital, and fixed costs of investment. The intuition for the result is as follows. If the policy variable is a martingale, firms invest as if the current policy state will last forever, before and after each policy transition. This allows the empiricist to directly recover causal effects from measured responses to policy variable transitions.

Empiricists sometimes invoke the strong assumption that policy variable changes are completely unexpected and permanent. Setting aside plausibility concerns, we show it is only valid as an identifying assumption in the absence of fixed investment costs, which Abel and Eberly (1994) and Chetty (2012) have flagged as an important friction. In particular, we show that with infinitesimal transition rates, the shadow value of capital under transient policy assignment *approaches* that under permanent policy assignment. This implies treatment responses approach causal effects, but only in the absence of fixed costs, since fixed costs generate investment discontinuities.

In certain cases, empiricists are less concerned about magnitudes, caring only to avoid incorrect estimation of causal effect signs. Here we derive a battery of sign identification assumptions that empiricists can invoke: binary assignment; uniformly distributed policy transitions; stochastic monotonicity of the policy variable; permanent random assignment opposite in sign to the expected treatment; or permanent random assignment equal in sign but with absolute value in excess of the expected treatment.

Finally, we show how problems arising from policy transience can be addressed when the identifying assumptions stated above are not satisfied. For example, we show how causal effects can be recovered from treatment responses for arbitrary policy transition rates. We also show how treatment responses can be extrapolated across different transitions under the same policy generating process, or across different policy processes, overcoming barriers to valid extrapolation. The proposed correction works as follows. Rather than focus on the change in the policy variable, with policy transience one should instead focus on firm responses to changes in the shadow value accounting for expectations. With such estimates in-hand, one can then infer causal effects or forecast responses to future shadow value changes.

Taken together, our results cast doubt on the interpretation and utilization of elasticity estimates shaping policy. For example, implicit in much empirical work is the view that "more change is better" since more observations drives down standard errors and facilitates control for fixed-effects. Slemrod (1992) writes, "Fortunately (for the progress of our knowledge, not for policy), since 1978 the taxation of capital gains has been changed several times, providing much new evidence on the tax responsiveness of realizations." Similarly, Greenstone (2001) writes "The [Clean Air Act] Amendments introduce substantial cross-sectional and longitudinal variation in regulatory intensity at the county level." Cummins, Hassett and Hubbard (1994) examine responses to 13 changes in the U.S. corporate income tax from 1962 until 1986. Romer and Romer (2010) identify 54 exogenous tax shocks during the post-war period.

Lucas (1976) points out the perils of extrapolating macro-econometric evidence across policy rules. Our focus is on the inadequacy of the random assignment assumption invoked by micro-econometricians in estimating causal effects. Naturally, Lucas does not analyze the relationship between treatment responses under dynamic random assignment and causal effects, nor derive sufficient conditions for correct identification of causal effects.

Abel (1983) analyzes effects of permanent versus temporary tax policies under perfect foresight. Auerbach (1986) and Auerbach and Hines (1988) present investment Euler equations under stochastic tax rates. Hassett and Metcalf (1999) present a real options model with a one-time investment which they use to assess whether uncertainty regarding tax credits encourages investment. Gourio and Miao (2008) numerically compare effects of permanent and temporary dividend tax cuts. Keane and Wolpin (2002) show accounting for policy duration is important if households are forward-looking. For example, in their model, a soon-to-expire child benefit should have no effect on fertility absent income effects. Their analysis is granular, accounting for welfare program detail, and numerical.

Our work is in the spirit of a paper by Chetty (2012) who writes, "The identification of structural parameters of stylized models is one of the central tasks of applied economics." He analyzes how to recover structural elasticity parameters if there are transaction costs or inattention. We use a structural model to understand and correct empirical estimates derived from random assignment in dynamic settings, with recovery of adjustment cost parameters being an interim step. Cummins, Hassett and Hubbard (1994) estimate adjustment cost parameters based on tax experiments. They assume each tax change is a complete surprise and viewed as permanent.

The rest of the paper is as follows. Section 2 presents the neoclassical theory and derives causal effects. Section 3 discusses the econometric inference problem. Section 4 examines the relationship between causal effects and measured treatment responses. Section 5 offers bias corrections. Section 6 derives identifying assumptions for dynamic economies. Section 7 discusses extrapolation. Section 8 provides a numerical example. Section 9 concludes, and the appendix contains all proofs.

2. Neoclassical Theory of Capital Demand

This section derives causal effects of government policies based on the neoclassical q-theory of capital demand for a firm facing taxation and regulation. We follow the tractable model of Abel and Eberly (1997).

2.1. Technology

Time is continuous and horizons are infinite. Agents are risk-neutral and discount cash payments at a constant rate $r > 0$. The price-taking firm sells an output flow each instant at a positive stochastic price $(\rho_t)_{t\geq 0}$. The production function is $\zeta_t n_t^{\alpha} s_t^{1-\alpha}$, with the factor share $0 < \alpha < 1$. The productivity process $(\zeta_t)_{t\geq 0}$ is positive and stochastic. The production input s represents physical capital which can be adjusted at a cost. Capital depreciates at a constant rate $\delta \geq 0$. The production input n is costlessly adjustable and has a constant unit price $p > 0$.

Government intervention in the economy takes two forms. First, the government imposes a linear tax at rate $\tau \geq 0$ on gross operating profits.[1] Second, government intervention in the input market determines p. For example, minimum wage laws affect labor input costs. Environmental laws affect fuel input prices. Tariffs and state-level taxes are often imposed on production inputs. To fix ideas, throughout we consider an econometrician interested in estimating the causal effects of these forms of commonly-studied government interventions.

[1] Other tax bases, e.g. income tax with interest deductions, can be modeled but with considerable added complexity such as the need to model leverage policy. See Hennessy, Kasahara and Strebulaev (2015).

Uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, \mathfrak{M})$. For any process $(w_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathfrak{M})$, $\mathcal{F}^w = (\mathcal{F}_t^w)_{t \geq 0}$ denotes the augmented filtration generated by w. We know any Lévy process has a càdlàg version (right continuous with left limits) and attention is confined to such. With this in mind, let $y_t \equiv \rho_t \zeta_t$ and $(z_t)_{t \geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathfrak{M})$. The following law of motion is assumed:

$$\begin{aligned} dy_t &= m y_t dt + \nu y_t dz_t \\ y_0 &> 0. \end{aligned} \tag{1}$$

Gross operating profits, net of tax, are denoted π. We have:

$$\pi \equiv \max_n \quad [y n^\alpha s^{1-\alpha} - pn](1-\tau). \tag{2}$$

It follows that utilization of the flexible input is increasing in y and decreasing in p. Specifically, the optimal flexible input is:

$$n^* = \left(\frac{\alpha y}{p} \right)^{\frac{1}{1-\alpha}} s. \tag{3}$$

Although the focus of our analysis is real investment, throughout the paper effects of government policy on employment can be determined using equation (3).

Gross operating profits, net of tax, can be expressed as:

$$\pi(s, x, p, \tau) = x s \kappa(p, \tau), \tag{4}$$

with

$$\begin{aligned} \kappa(p, \tau) &\equiv (1-\tau)(1-\alpha)\alpha^{\alpha/(1-\alpha)} p^{-\alpha/(1-\alpha)} \\ x &\equiv y^{\frac{1}{1-\alpha}}. \end{aligned} \tag{5}$$

Equation (5) shows how government policy influences profitability, with κ decreasing in p and τ. Rather than analyze the effects of each policy lever separately, for the remainder of the paper we treat the government as determining κ directly. Thus, we shall speak of κ as the *policy variable.*

From Ito's lemma it follows that x evolves as a geometric Brownian motion, with

$$\begin{aligned} dx_t &= \mu x_t dt + \sigma x_t dz_t \\ x_0 &> 0 \end{aligned} \tag{6}$$

where

$$\mu \equiv \frac{m}{1-\alpha} + \frac{1}{2}\frac{\alpha\nu^2}{(1-\alpha)^2} \tag{7}$$
$$\sigma \equiv \frac{\nu}{1-\alpha}.$$

To rule out unbounded valuations, it is assumed $r > 2\mu + \sigma^2$.

The capital stock evolves according to:

$$ds_t = (a_t - \delta s_t)dt \tag{8}$$
$$s_0 > 0.$$

The instantaneous capital accumulation control $(a_t)_{t\geq 0}$ is right-continuous and progressively measurable with respect to $(\mathcal{F}_t^x)_{t\geq 0}$. Since the underlying stochastic structure is Markovian and time-homogeneous we consider Markov control policies of the form $a_t = a(x_t)$. Capital can be purchased and sold at a constant price $\psi \geq 0$. The firm faces quadratic costs γa^2, where γ is a positive constant. Given our objective is to provide insight into the empirical literature, the posited investment cost function is useful in that, as shown below, it maps to the linear causal effect model common in empirical work. A general class of investment cost functions will be considered as an extension in Section 6.

2.2. Optimal Accumulation and Causal Effects

The firm's objective is to maximize the discounted expected cash flow. The performance functional is:

$$J(a) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\kappa x_t s_t - \psi a_t - \gamma a_t^2\right) dt \mid \mathcal{F}_0\right]. \tag{9}$$

The Hamilton-Jacobi-Bellman (HJB) equation for the control problem is:

$$rV(s,x) = \max_a \quad (a-\delta s)V_s(s,x) + \mu x V_x(s,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(s,x) + \kappa x s - \psi a - \gamma a^2. \tag{10}$$

We conjecture a classical solution to the HJB equation, with V being C^∞.[2]

The optimal instantaneous control policy solves:

$$\max_a \quad aV_s(s,x) - \psi a - \gamma a^2. \tag{11}$$

[2] For a verification theorem, see Fleming and Soner (1993), Theorem 9.1.

We conjecture and then verify the value function is of the form

$$V(s,x) = sq(x) + G(x) \tag{12}$$

where q and G are posited to be C^∞. The term $sq(x)$ measures the value of the accumulated capital stock while G measures growth option value.

Under the conjectured value function, $V_s(s,x) = q(x)$ and the optimal accumulation policy is:

$$a(x) = \frac{q(x) - \psi}{2\gamma}. \tag{13}$$

The instantaneous net gain attributable to accumulation is:

$$\begin{aligned} aq - \psi a - \gamma a^2 &= (q-\psi)^2\Gamma \\ \Gamma &\equiv \frac{1}{4\gamma}. \end{aligned} \tag{14}$$

Substituting into the Bellman equation the conjectured value function, and the instantaneous gain under optimal accumulation from equation (14), we obtain:

$$\begin{aligned} rsq(x) + rG(x) &= -\delta sq(x) + \mu x s q_x(x) + \mu x G_x(x) + \frac{1}{2}\sigma^2 x^2 s q_{xx}(x) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x) \\ &\quad + \kappa x s + [q(x) - \psi]^2 \Gamma. \end{aligned} \tag{15}$$

Since the Bellman equation must hold point-wise on the state space, the terms scaled by s in the preceding equation must equate. This implies the following ordinary differential equation describing the evolution of the shadow value of capital:

$$(r+\delta)q(x) = \mu x q_x(x) + \frac{1}{2}\sigma^2 x^2 q_{xx}(x) + \kappa x. \tag{16}$$

From the preceding equation and the Feynman-Kac formula it follows:

$$q(x_0) = \mathbb{E}\left[\int_0^\infty e^{-(r+\delta)t} \kappa x_t dt \mid \mathcal{F}_0\right]. \tag{17}$$

That is, q is the discounted value of the marginal product of capital.

As in Abel and Eberly (1997), we can rule out bubbles causing unbounded valuations as x goes to zero or infinity. We obtain the following solution to equation (16):

$$q^{**}(x;\kappa) = \frac{\kappa x}{r+\delta-\mu}. \tag{18}$$

The optimal accumulation policy is:

$$a^{**}(x,\kappa) = \frac{1}{2\gamma}\left[\frac{\kappa x}{r+\delta-\mu} - \psi\right]. \tag{19}$$

Note, in the preceding two equations, and throughout the remainder of the paper, double-stars are used to denote variables obtained under the present section's neoclassical theory of investment.

Heckman (2000) writes, "Comparative statics exercises formalize Marshall's notion of a ceteris paribus change, which is what economists mean by a causal effect." In the posited neoclassical theory of capital accumulation, we have the following causal effect:

$$\frac{\partial}{\partial\kappa}a^{**}(x,\kappa) = \frac{x}{2\gamma(r+\delta-\mu)}. \tag{20}$$

Note, the causal effect is linear in the government policy variable κ.

A complete model solution requires computing the growth option value function (G). However, since our objective is to analyze causal effects, the policy function (19) is sufficient. The growth option value is presented in the appendix.

3. The Econometric Setting

One would like to determine the conditions under which natural experiments correctly recover the magnitude and/or sign of causal effects. The remainder of the paper treats the model itself as a laboratory. The laboratory is ideal in two respects. First, in contrast to the real-world environment confronting empiricists, we here know the theory-implied causal effects, as shown in equation (20). Second, we consider an econometrician inhabiting an economy where government policy evolves as an independent stochastic process. This independence assumption allows us to abstract from endogeneity bias– the primary focus of contemporary empirical work. We consider an econometrician who uses measured treatment responses in attempting to infer causal effects.

3.1. The Policy Generating Process

The econometrician measures firm responses to exogenous changes in the government policy process $(\kappa_t)_{t\geq 0}$. This process evolves as an N-state continuous-time Markov chain defined on $(\Omega, \mathcal{F}, \mathfrak{M})$ which is independent of $(x_t)_{t\geq 0}$. The instantaneous capital accumulation control $(a_t)_{t\geq 0}$ is right-continuous, Markovian, and progressively measurable with respect to $(\mathcal{F}_t^{x,\kappa})_{t\geq 0}$.

To conserve notation we omit time subscripts where obvious. With this in mind, let government policy in regime i be denoted κ_i, with $i = 1, ..., N$, and $N \geq 2$. The indexing convention is:

$$\kappa_1 > ... > \kappa_N > 0.$$

If the current regime is i, the amount of time policy will remain in regime i before transitioning is denoted T_i. The random variable T_i is exponentially distributed with parameter Λ_i. It follows that, when in regime i, the expected remaining duration of regime i is Λ_i^{-1}.

The conditional probability of transitioning into state j, given a transition out of i, is denoted P_{ij}. The parameter $\lambda_{ij} \geq 0$ denotes the *transition rate* from regime i to regime j, and is defined as follows:

$$\lambda_{ij} \equiv \Lambda_i P_{ij}.$$

From the preceding equation it follows that:

$$\Lambda_i = \sum_{j \neq i} \lambda_{ij}.$$

If $\Lambda_i = 0$, then state i is *absorbing*. If Λ_i tends to ∞, then state i is an *instantaneous state*, using the terminology of Ross (1996).

3.2. Model Solution

We turn next to a characterization of optimal accumulation. The performance functional is:

$$J(a) = \mathbb{E}\left[\int_0^\infty e^{-rt}\kappa_t x_t s_t - \psi a_t - \gamma a_t^2 dt \mid \mathcal{F}_0\right]. \tag{21}$$

The value of the firm in policy state i is denoted V^i. Accounting for regime changes, and concomitant capital gains, we have the following system of N Hamilton-Jacobi-Bellman equations:

$$\begin{aligned} rV^1(s,x) &= \max_a \quad (a - \delta s)V_s^1(s,x) + \mu x V_x^1(s,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}^1(s,x) \\ &\quad + \sum_{j \neq 1} \lambda_{1j}[V^j(s,x) - V^1(s,x)] + \kappa_1 xs - \psi a - \gamma a^2 \\ &\quad ... \\ rV^N(s,x) &= \max_a \quad (a - \delta s)V_s^N(s,x) + \mu x V_x^N(s,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}^N(s,x) \\ &\quad + \sum_{j \neq N} \lambda_{Nj}[V^j(s,x) - V^N(s,x)] + \kappa_N xs - \psi a - \gamma a^2. \end{aligned} \tag{22}$$

The optimal state-contingent accumulation policy solves:

$$a^i(x) \in \arg\max_a \quad aV_s^i(s,x) - \psi a - \gamma a^2;\ i = 1, ..., N. \tag{23}$$

We conjecture the solution to the Bellman system (22) has the following functional form

$$V^1(s,x) = q^1(x)s + G^1(x); ...; V^N(s,x) = q^N(x)s + G^N(x). \tag{24}$$

where each q^i and G^i is posited to be C^∞. Under the value function conjectured in equation (24), $V_s^i(s,x) = q^i(x)$ and the optimal state-contingent accumulation policy is:

$$a^i(x) = \frac{q^i(x) - \psi}{2\gamma};\ i = 1, ..., N. \tag{25}$$

From equation (25) it follows accumulation will exhibit a jump each time there is a policy regime transition. Each jump represents a treatment response measured by the econometrician.

Evaluated at the optimal policy, the instantaneous net gain attributable to accumulation is:

$$a^i(x)q^i(x) - \psi a^i(x) - \gamma[a^i(x)]^2 = [q^i(x) - \psi]^2\Gamma. \tag{26}$$

Substituting the accumulation gain from equation (26) and the conjectured value functions into the original system of Bellman equations, we can rewrite the Bellman system as:

$$\begin{aligned}
&\left(r + \delta + \sum_{j \neq 1} \lambda_{1j}\right) q^1(x)s + \left(r + \sum_{j \neq 1} \lambda_{1j}\right) G^1(x) \\
= \ & \mu x[sq_x^1(x) + G_x^1(x)] + \frac{1}{2}\sigma^2 x^2 [sq_{xx}^1(x) + G_{xx}^1(x)] \\
&+ \sum_{j \neq 1} \lambda_{1j}[q^j(x)s + G^j(x)] + \kappa_1 xs + [q^1(x) - \psi]^2\Gamma \\
&... \\
&\left(r + \delta + \sum_{j \neq N} \lambda_{Nj}\right) q^N(x)s + \left(r + \sum_{j \neq N} \lambda_{Nj}\right) G^N(x) \\
= \ & \mu x[sq_x^N(x) + G_x^N(x)] + \frac{1}{2}\sigma^2 x^2 [sq_{xx}^N(x) + G_{xx}^N(x)] \\
&+ \sum_{j \neq N} \lambda_{Nj}[q^j(x)s + G^j(x)] + \kappa_N xs + [q^N(x) - \psi]^2\Gamma.
\end{aligned} \tag{27}$$

Since the Bellman equations must be satisfied at all points in the state space, the terms scaled by s in each of the preceding equations must equate. Thus, the following system of N equations must be satisfied:

$$
\begin{aligned}
\left(r+\delta+\sum_{j\neq 1}\lambda_{1j}\right)q^{1}(x) &= \mu x q^{1}_{x}(x)+\frac{1}{2}\sigma^{2}x^{2}q^{1}_{xx}(x)+\sum_{j\neq 1}\lambda_{1j}q^{j}(x)+\kappa_{1}x \\
&\ldots \\
\left(r+\delta+\sum_{j\neq N}\lambda_{Nj}\right)q^{N}(x) &= \mu x q^{N}_{x}(x)+\frac{1}{2}\sigma^{2}x^{2}q^{N}_{xx}(x)+\sum_{j\neq N}\lambda_{Nj}q^{j}(x)+\kappa_{N}x.
\end{aligned}
\tag{28}
$$

Applying the Feynman-Kac formula to an arbitrary differential equation in the preceding system, it follows:

$$
\begin{aligned}
q^{i}(x_0) &= \mathbb{E}\left[\int_0^\infty e^{-(r+\delta+\Lambda_i)t}\left(\kappa_i x_t+\Lambda_i\sum_{j\neq i}P_{ij}q^{j}(x_t)\right)dt \mid \mathcal{F}_0\right] \\
&= \mathbb{E}\left[\int_0^\infty\left(\int_0^T e^{-(r+\delta)t}\kappa_i x_t dt+e^{-(r+\delta)T}\sum_{j\neq i}P_{ij}q^{j}(x_T)\right)(\Lambda_i e^{-\Lambda_i T})dT \mid \mathcal{F}_0\right].
\end{aligned}
\tag{29}
$$

The second expression for the shadow value offered above follows from the first via integration by parts. It states that the shadow value is the expectation over the current policy regime life, which is exponentially distributed, of the net marginal product of capital up to the regime change plus the expectation of the shadow value of capital after the regime change.

We conjecture the following solution to system (28):

$$
q^{i}(x)=xc_i;\ \ i=1,...,N. \tag{30}
$$

Substituting the conjectured linear solutions into the system of equations (28), it follows that the shadow value of capital is:

$$
\begin{bmatrix} q^{1}(x) \\ \ldots \\ q^{N}(x) \end{bmatrix} = x\underbrace{[\mathbf{T}(r+\delta-\mu)]^{-1}\begin{bmatrix}\kappa_1 \\ \ldots \\ \kappa_N\end{bmatrix}}_{\equiv\mathbf{c}} \tag{31}
$$

where $\mathbf{T}(R)$ denotes the following *augmented transition matrix*:

$$
\mathbf{T}(R)\equiv\begin{bmatrix}
R+\sum_{j\neq 1}\lambda_{1j} & -\lambda_{12} & \ldots & -\lambda_{1N} \\
-\lambda_{21} & R+\sum_{j\neq 2}\lambda_{2j} & \ldots & -\lambda_{2N} \\
\ldots & \ldots & \ldots & \ldots \\
-\lambda_{N1} & -\lambda_{N2} & \ldots & R+\sum_{j\neq N}\lambda_{Nj}
\end{bmatrix}. \tag{32}
$$

We know from the Lévy-Desplanques Theorem (Horn and Johnson (1985)) that a Strictly Diagonal Dominant matrix is non-singular. In the present context it follows that there is a unique solution $\mathbf{c}$ to equation (31) since:

$$R > 0 \Rightarrow |\mathbf{T}(R)| \neq 0. \tag{33}$$

If there are only two possible policy states, the preceding shadow value expression can be written as:

$$q^i(x) = \left(\frac{\kappa_i}{r+\delta-\mu} + \frac{\lambda_{ij}(\kappa_j - \kappa_i)}{(r+\delta-\mu)(r+\delta-\mu+\lambda_{ij}+\lambda_{ji})} \right) x. \tag{34}$$

To complete the model solution, we must also compute the state-contingent growth option value (G^i). However, since our objective is to analyze the relationship between measured treatment responses and causal effects, the policy function in equation (25) and the shadow value vector (31) are sufficient. Derivation of the growth option value is provided in the appendix.

4. Treatment Responses and Causal Effects

This section analyzes the relationship between measured treatment responses and causal effects. From equation (25) it follows that the optimal accumulation policy is

$$\begin{bmatrix} a^1(x) \\ ... \\ a^N(x) \end{bmatrix} = \frac{1}{2\gamma} \left[\begin{bmatrix} q^1(x) \\ ... \\ q^N(x) \end{bmatrix} - \begin{bmatrix} \psi \\ ... \\ \psi \end{bmatrix} \right] \tag{35}$$

with equation (31) determining the state-contingent shadow value of capital.

Consider now the treatment responses that will be observed by the econometrician. The treatment response associated with a transition from state i to j is denoted TR_{ij}. We have:

$$\begin{aligned} TR_{ij}(x) &\equiv a^j(x) - a^i(x) \\ &= \frac{1}{2\gamma}[q^j(x) - q^i(x)]. \end{aligned} \tag{36}$$

Transitions across policy states result in jumps in accumulation proportional to the jumps in the shadow value.

Recall, causal effects were shown to be linear (equation (20)). Therefore, the causal effect associated with a discrete change in the policy variable is here correctly computed by multiplying the infinitesimal causal effect by the discrete policy variable change (see Heckman (2000)). We have:

$$\begin{aligned} CE_{ij} &\equiv a^{**}(x;\kappa_j) - a^{**}(x;\kappa_i) \\ &= \frac{1}{2\gamma}[q^{**}(x;\kappa_j) - q^{**}(x;\kappa_i)] \\ &= \frac{1}{2\gamma}\left[\frac{x(\kappa_j-\kappa_i)}{r+\delta-\mu}\right]. \end{aligned} \tag{37}$$

4.1. Attenuation Bias

This subsection considers the problem of attenuation bias. We first consider the simplest possible setting, one in which the policy variable can only take on two possible values. The following proposition follows directly from the formula for the shadow value under binary assignment given in equation (34), with treatment responses equal to $\Delta q/2\gamma$.

Proposition 1 *If there are only two possible policy states, the treatment response is*

$$TR_{ij}(x) = \left(1 - \underbrace{\frac{\lambda_{ij}+\lambda_{ji}}{r+\delta-\mu+\lambda_{ij}+\lambda_{ji}}}_{Attenuation}\right)\underbrace{\left(\frac{1}{2\gamma}\right)\left(\frac{x(\kappa_j-\kappa_i)}{r+\delta-\mu}\right)}_{CE_{ij}}.$$

With binary assignment there is always attenuation bias. Further, the bias is quite severe even in settings with long expected policy regime durations. To illustrate, suppose the econometrician inhabits an economy with long expected policy durations, say 10 years ($\lambda_{ij} = \lambda_{ji} = .10$). In this case, attenuation bias exceeds one-half of causal effects under reasonable parameterizations featuring $r + \delta - \mu < .20$. Similarly, the proposition shows that the size of the attenuation bias exceeds one-half under such parameterizations if one considers, say, a permanent transition ($\lambda_{ji} = 0$) out of a policy regime with an expected duration of 5 years ($\lambda_{ij} = .20$) or an unexpected transition (λ_{ij} infinitesimal) to a new regime with an expected 5 year duration ($\lambda_{ji} = .20$). For example, in reviewing the extensive empirical analysis of the Tax Reform Act of 1986, Slemrod (1990) points to surprisingly small estimated tax-elasticities. Our calculations suggest that small responses to the legislation might well have been expected.

Two other implications of Proposition 1 are worth noting. First, it is apparent that if either regime is instantaneous, the treatment response will be infinitesimal and the attenuation bias is nearly 100% of the causal effect. Conversely, if both transition rates are infinitesimal, treatment responses approximate causal effects.

Consider next a setting in which all transition rates are equal, with $\lambda_{ij} = \lambda > 0$ for all $i \neq j$. In this case, each policy regime has an expected life equal to $1/(N-1)\lambda$. Here, if a transition out of an arbitrary state i occurs, the conditional probability of transitioning to each of the remaining states is uniform, with $P_{ij} = 1/(N-1)$. As shown in the appendix, the following proposition follows from equation (31) and the fact that $\Delta a = \Delta q/2\gamma$.

Proposition 2 *If all transition rates are equal to $\lambda > 0$, with N policy states the treatment response is*

$$TR_{ij}(x) = \left(1 - \underbrace{\frac{N\lambda}{r+\delta-\mu+N\lambda}}_{Attenuation}\right) \underbrace{\left(\frac{1}{2\gamma}\right)\left(\frac{x(\kappa_j - \kappa_i)}{r+\delta-\mu}\right)}_{CE_{ij}}.$$

The preceding proposition implies that if all transition rates are equal, there is attenuation bias. Here too, the attenuation bias is quite severe even if one considers settings with long expected policy durations. For example, suppose the effective discount rate is 5% and that the expected regime life is ten years, with $N = 11$ and $\lambda = .01$. The attenuation bias is equal to 69% of the causal effect. Consider instead an expected regime life of twenty years, with $N = 11$ and $\lambda = .005$. Although it would be tempting to treat such a long regime as equivalent to permanent, the attenuation bias here amounts to 52% of the causal effect.

4.2. Overshooting and Sign Reversals

The settings described in the previous subsection featuring binary assignment or equality of all transition rates are comforting inasmuch as an empiricist can claim treatment responses measured in such environments are conservative estimates of causal effects. Of course, it is not clear why one would want conservative estimates. After all, conservative estimates of elasticities lead to downward bias in the estimated deadweight loss arising from government interventions. Further, magnitudes, as distinct from signs, are often used as the basis for falsifying underlying theories. Finally, the settings analyzed in the previous subsection might be viewed as atypical. After all, it is seldom the case that a policy variable is binary, and it is seldom the case that policy transitions are uniformly distributed.

These arguments notwithstanding, one might hope that the settings considered in the preceding subsection are still instructive regarding the nature of the wedge between treatment responses and causal effects. After all, conventional wisdom holds that firms will invest less aggressively in response to transient government policies, so that attenuation bias may be expected to be a general feature. The following proposition demonstrates this is not the case.

Proposition 3 *If there are more than two policy states, there exists a continuum of transition rates such that a proper subset of treatment responses are opposite in sign to their respective causal effects, as well as a continuum of transition rates such that a proper subset of treatment responses are equal in sign but larger in absolute value than their respective causal effects.*

We defer for now discussion of the intuition behind the preceding proposition, since it is most readily understood in the context of experiments featuring permanent assignment, the subject of the next subsection.

4.3. Transitions to Absorbing States

In light of Proposition 3, empiricists might be interested in better understanding the nature of the bias they face, as well as the probabilities of each bias form. This subsection provides a practical guide for random assignment experiments in settings such that each of the states to which the policy variable may possibly transition ("transition-to states") are absorbing. That is, we characterize bias when the upcoming random assignment is permanent.

To begin, when transition-to states are absorbing, there is a simple relationship between treatment responses and causal effects, with:

$$TR_{ij} = CE_{ij} + \frac{1}{2\gamma}\left[q^{**}(x;\kappa_i) - q^i(x)\right] = CE_{ij} + \frac{1}{2\gamma}\left[\frac{\kappa_i x}{r+\delta-\mu} - q^i(x)\right]. \quad (38)$$

If the current regime i has the feature that all potential transition-to regimes are absorbing, the Bellman equation is:

$$\begin{aligned} rV^i(s,x) &= \max_a \quad (a-\delta s)V^i_s(s,x) + \mu x V^i_x(s,x) + \frac{1}{2}\sigma^2 x^2 V^i_{xx}(s,x) \\ &\quad + \sum_{j\neq i}\lambda_{ij}[V^j(s,x) - V^i(s,x)] + \kappa_i xs - \psi a - \gamma a^2. \end{aligned} \quad (39)$$

In the preceding equation, the absorbing transition-to state value functions (V^j) can be obtained from the constant policy model of Section 2. Solving the preceding equation for the shadow value we obtain:

$$\begin{aligned} q^i(x) &= q^{**}(x;\kappa_i) + \frac{x}{(r+\delta-\mu)[1+(r+\delta-\mu)\mathbb{E}(T_i)]} \sum_{j\neq i} P_{ij}(\kappa_j - \kappa_i) \\ \mathbb{E}(T_i) &= \Lambda_i^{-1}. \end{aligned} \tag{40}$$

Substituting equation (40) into equation (38) and rearranging terms, we obtain the following proposition.

Proposition 4 *Consider initial state i such that each potential transition-to state j is absorbing. If the realized transition-to state is k, the ratio of treatment response to causal effect is:*

$$\frac{TR_{ik}}{CE_{ik}} = 1 - \frac{1}{[1+(r+\delta-\mu)\mathbb{E}(T_i)]} \frac{\overbrace{\sum_{j\neq i} P_{ij}(\kappa_j - \kappa_i)}^{\textit{Expected Treatment}}}{\underbrace{\kappa_k - \kappa_i}_{\textit{Realized Treatment}}}.$$

It follows from the proposition that if each possible transition-to state is absorbing and the conditional expectation of the policy variable change is zero, then every treatment response is equal to its respective causal effect. If instead the conditional expectation of the policy variable change is not zero, every treatment response is biased. The ratio of measured treatment response to causal effect approaches one for realized policy variable changes large in absolute value. Conversely, the ratio of treatment response to causal effect becomes unboundedly large in absolute value for infinitesimal realized changes. Thus, the proposition reveals that treatment responses derived from small policy variable changes are especially unreliable as guides to causal effects. Notice this is true absent any notion of rational or irrational response inertia (e.g. Chetty (2012)).

From Proposition 4 we have the following corollary delineating bias regions.

Corollary *If each possible transition-to state is absorbing and the conditional expectation of the change in the policy variable is not zero, the sign and magnitude of the treatment response bias depends on the realized assignment category k. If the conditional expectation of the change in κ is* *positive, then*

$$\begin{aligned}
\kappa_k - \kappa_i &< 0 \Rightarrow TR_{ik} < CE_{ik} < 0 \\
\kappa_k - \kappa_i &\in \left(0, \frac{\sum_{j \neq i} P_{ij}(\kappa_j - \kappa_i)}{1 + (r + \delta - \mu)\mathbb{E}(T_i)}\right) \Rightarrow TR_{ik} < 0 < CE_{ik} \\
\kappa_k - \kappa_i &\geq \frac{\sum_{j \neq i} P_{ij}(\kappa_j - \kappa_i)}{1 + (r + \delta - \mu)\mathbb{E}(T_i)} \Rightarrow 0 \leq TR_{ik} < CE_{ik}.
\end{aligned}$$

If the conditional expectation of the change in κ is negative, then

$$\begin{aligned}
\kappa_k - \kappa_i &> 0 \Rightarrow TR_{ik} > CE_{ik} > 0 \\
\kappa_k - \kappa_i &\in \left(\frac{\sum_{j \neq i} P_{ij}(\kappa_j - \kappa_i)}{1 + (r + \delta - \mu)\mathbb{E}(T_i)}, 0\right) \Rightarrow TR_{ik} > 0 > CE_{ik} \\
\kappa_k - \kappa_i &\leq \frac{\sum_{j \neq i} P_{ij}(\kappa_j - \kappa_i)}{1 + (r + \delta - \mu)\mathbb{E}(T_i)} \Rightarrow 0 \geq TR_{ik} > CE_{ik}.
\end{aligned}$$

The intuition for the corollary is as follows. Consider a setting in which the conditional expectation of the change in κ is positive. If the realized treatment is negative ($\kappa_k < \kappa_i$), the treatment response will have the correct negative sign but will overshoot the causal effect in absolute value terms. Objectively bad news becomes very bad news if arriving news was expected to be positive. If instead the realized treatment features a large increase in κ, the treatment response will have the correct positive sign but be biased downwards. After all, if the conditional expectation of the change in κ is positive, firms already invest at a higher rate prior to treatment. Finally, if the realized treatment entails a sufficiently small increase in κ, the treatment response will actually have the wrong sign. Objectively good news becomes bad news if even better news had been expected.

The corollary allows one to readily compute the probability of the various biases, attenuation, overshooting, and sign reversals, depending on the distribution of the treatments, expected regime life, and discount rates. One simply needs to compute the probability of the treatment random variable falling into the respective bias regions. In general, the results are not comforting. To illustrate, consider an economy in which $r + \delta - \mu = .05$, with an expected current regime life of five years, and a future permanent change in κ being drawn from a granular set of points uniformly distributed on [0,1]. From the proposition it follows that the treatment response is negative for realized changes less than 0.5/[1+.05(5)]=0.40. That is, evaluated ex ante, the probability of the realized treatment response having the wrong sign is 40%. If the effective discount rate is increased from 5% to 10%, or the expected regime life increases from 5 to 10 years, the sign reversal probability is equal to 33%.

Figure 1 offers a quantitative example illustrating the bias regions for the case of absorbing transition-to states. The example assumes: the change in κ is uniformly distributed on [-0.20, 0.80]; the expected regime life is five years; and $r + \delta - \mu = .10$. The figure plots the ratio of treatment response to causal effects for all possible realized policy treatments. The figure shows treatment responses overshoot causal effects for realized $\Delta\kappa < 0$. Since the expected change in κ is positive (0.30) bad news becomes very bad news. Treatment responses overshoot causal effects by a very wide margin in the case of small negative treatments. There is also a region where treatment responses are opposite in sign to causal effects. In particular, firms cut investment in reaction to increases in κ on the interval (0, 0.20). The percentage downward bias is extremely large for small increases in κ.

The preceding examples illustrate that sign reversals and overshooting are not anomalies requiring pathological policy generating processes. Rather, one can readily compute bias probabilities depending on the statistical properties of the randomized treatment under consideration. In the preceding example, the probability of the experiment yielding an incorrect sign was 20%, while the probability of overshooting was also 20%. The remaining 60% of potential experimental treatments would result in treatment responses attenuated relative to causal effects.

Taken together, the results of this subsection imply that for a natural experiment involving permanent random assignment, the ideal experiment features a mean zero policy variable change. However, if the policy variable change is not mean zero, there is inevitably a bias. Nevertheless, as shown, there is an analytical argument for seeking out experiments in which the realized change in the policy variable is large. Further, if the empiricist is pursuing the more limited goal of estimating a causal effect sign, realized policy changes opposite in sign to expected changes are sign-robust, as are experiments in which the realized treatment has the expected sign but exceeds in absolute value the expected treatment.

4.4. Bias Bounds

This subsection demonstrates some inherent limitations on the nature and severity of biases that can arise in natural policy experiments. Consider first the problem of attenuation bias. A natural question to ask is whether there exists any policy generating process such that all treatment responses are equal to zero. The answer is no, as shown in the following lemma.

Lemma 1 *There is no set of policy transition rates such that all treatment responses are equal to zero.*

Of course, the preceding lemma provides only a modicum of comfort in relation to the problem of attenuation bias since it still allows for the possibility of severe attenuation. For example, it follows from Propositions 1 and 2 that treatment responses tend to zero when each of the regimes becomes highly transient (all λ_{ij} tending to infinity).

Consider next the problem of treatment responses overshooting causal effects. The following lemma shows that regardless of the assumed transition rates, overshooting cannot possibly occur in the case of worst-to-best state transitions.

Lemma 2 *The treatment response associated with a transition from the worst state* (N) *to the best state* (1) *cannot overshoot its causal effect.*

To motivate the next lemma, suppose for there are three policy states with low, medium and high values of κ, with the low and high states being absorbing. If the shadow value of capital in the medium state is less than its value under permanent assignment, there will be overshooting in the event of a transition to the high state, but undershooting in the event of a transition to the low state. That is, overshooting for one transition implies undershooting for another transition. The following lemma reveals this to be a more general phenomenon.

Lemma 3 *If there exists a transition from state* j *to a (better) state* $i < j$ *with treatment response exceeding its respective causal effect by* $k > 0$*, then the treatment response for the transition from the worst state* (N) *to* j *plus the treatment response for the transition from state* i *to the best state* (1) *must fall below the sum of their respective causal effects by at least* k*.*

Consider finally the problem of sign reversals. The following lemma shows sign reversals cannot be universal.

Lemma 4 *There is at least one state such that the treatment response associated with a transition from it to the best state* (1) *is positive, and there is at least one state such that the treatment response associated with a transition to it from the worst state* (N) *is positive.*

5. Bias Corrections

This section analyzes how to infer causal effects based upon measured treatment responses. To fix ideas, suppose the econometrician has a panel dataset featuring investment responses to a transition from κ_i to κ_j. Based upon this evidence, she would like to infer the cross-section of causal effects.

As shown below, the causal effect for firm m is determined by the policy-invariant structural parameter γ_m which is assumed to be unobserved. The inference problem confronting the econometrician can be viewed as an application of the heterogeneous treatment effects model commonly used in the fields of medicine, education and labor. In particular, the treatment response of firm m is equal to:

$$\begin{aligned} TR_{ij}^m &= (q_j - q_i)\Theta_m \\ \Theta_m &\equiv \frac{1}{2\gamma_m}. \end{aligned} \tag{41}$$

Heterogeneous effects is a situation that commonly confronts empiricists. Under dynamic random assignment there is an additional source of complexity relative to static experiments since the treatment size is not directly observable. Rather, the treatment, Δq, must be determined via the policy transition matrix and equation (31).

Using equation (31) to compute the relevant state-contingent shadow values, it follows from equation (41) that:

$$\Theta_m = \frac{TR_{ij}^m}{q_j - q_i} = \frac{TR_{ij}^m}{(c_j - c_i)x} \Rightarrow \gamma_m = \frac{1}{2}\frac{(c_j - c_i)x}{TR_{ij}^m}. \tag{42}$$

Therefore, the causal effect implied by an observed treatment response is:

$$CE_{ij}^m = [q_j^{**} - q_i^{**}]\Theta_m = \left[\frac{q_j^{**} - q_i^{**}}{q_j - q_i}\right] TR_{ij}^m = \left[\frac{(\kappa_j - \kappa_i)/(r + \delta - \mu)}{c_j - c_i}\right] TR_{ij}^m \tag{43}$$

with $c_j - c_i$ determined by equation (31). Effectively, recovery of causal effects here demands a rescaling of the observed treatment response to account for the shadow value wedge under permanent versus transient government policies.

Cummins, Hassett and Hubbard (1994) use firm responses to imputed changes in the shadow value of capital to infer adjustment cost parameters. Their imputation assumes each tax code change is unanticipated and expected to be permanent. They argue this assumption overstates the true change in the shadow value of capital and overstates adjustment costs. However, the analysis of Section 4 shows that their imputation method may overstate, understate, or have sign that is opposite to the true change in shadow values. It follows from equation (42) that their inferred adjustment costs can be overstated, understated, or have the wrong sign.

Finally, it is worth noting that in the interest of brevity this section has confined attention to bias correction for the case of linear treatment effects, which arise under the posited smooth linear-quadratic investment cost function. More generally, with correctly imputed shadow values derived from equation (31), one can use observed treatment responses to recover policy-invariant parameters for alternative investment cost functions. In turn, the adjustment cost parameters can then be used to compute implied causal effects.

6. Identifying Assumptions

Section 4 demonstrates random assignment is not generally sufficient for equality of treatment responses and causal effects in dynamic settings. The objective of this section is to provide empiricists with a set of identifying assumptions ensuring treatment responses are unbiased estimators of causal effects. Conditions for sign identification are also derived.

In order to broaden the range of circumstances under which the identifying assumptions can be utilized, it will be useful to consider a general investment cost technology. The literature on capital accumulation has considered realistic real frictions creating lumpy investment and/or inaction regions. For example, Abel and Eberly (1994) consider that there can be fixed costs to nonzero investment, and that the firm may not be able to sell its installed capital for the same price at which it was originally purchased. Chetty (2012) has argued that such frictions and associated inaction regions can cloud the interpretation of empirical evidence. To account for these empirical concerns, we consider the following General Investment Cost Function.

Definition 1 *General Investment Cost Function: The firm can buy capital at price ψ^+ and sell capital at price $\psi^- \leq \psi^+$. Adjustment costs are ξ, where ξ is a strictly convex twice differentiable function of accumulation attaining a minimum value of zero at $a = 0$. The fixed cost to non-zero investment is $\varphi \geq 0$.*

Two points are worth noting at this stage. First, since the General Investment Cost Function shares with the initially-posited investment cost function ($\psi a + \gamma a^2$) the property of being invariant to s, it follows that the shadow value formulae derived in the preceding sections remain valid. Second, Abel and Eberly (1994) show that under the General Adjustment Cost Function, instantaneous investment is weakly monotone increasing in q. If $\varphi = 0$, optimal accumulation is continuous in q, with $a = 0$ optimal for all $q \in [\psi^-, \psi^+]$. If $\varphi > 0$, optimal accumulation is zero over a wider interval of q values, and exhibits a discontinuity at optimal thresholds for switching from inaction to action.

With these points in mind, we begin by deriving a restriction on transition rates such that the treatment responses converge to causal effects absent fixed costs ($\varphi = 0$). To begin, note that in performing comparative statics, e.g. equation (20), it is as if agents expect the initial policy to be permanent, with nature perturbing the policy variable and agents then expecting the new policy to be permanent. One might thus expect treatment responses to approximate causal effects if the policy generating process features "almost completely unanticipated" policy changes that are "nearly-permanent." Indeed, it follows from equation (31) that as all transition rates tend to zero, all state-contingent shadow values ($q^i(x)$) under transient policies approach their value under permanent assignment to their respective states ($q^{**}(x; \kappa_i)$). Such shadow value convergence is sufficient for treatment responses to approximate causal effects absent discontinuity in the investment policy function arising from fixed costs. We have the following proposition.

Proposition 5 *In the limit as all transition rates tend to zero, each treatment response converges to its respective causal effect if fixed costs of investment are equal to zero.*

Given the limited availability of rare-event natural experiments, and the potential for fixed costs to cloud inference, it would be useful to derive alternative identifying assumptions. Conveniently, we have the following result.

Proposition 6 *Necessary and sufficient conditions for treatment responses to equal causal effects for each policy transition are that the best state* (1) *and worst state* (N) *are absorbing, with remaining states being absorbing or featuring mean-zero policy variable changes.*

To provide intuition for the preceding proposition it is convenient to express the shadow value as the discounted marginal product of capital. Given the econometrician's assumption that κ_t and x_t are independent stochastic processes, the shadow value (equation (29)) can be rewritten as:

$$q^i(x_0) = \int_0^\infty \{\mathbb{E}_0[\kappa_t]\mathbb{E}_0[x_t]\}e^{-(r+\delta)t}dt. \tag{44}$$

Under the conditions described in Proposition 6, $\mathbb{E}_0[\kappa_t] = \kappa_i$ which implies $q^i(x_0) = q^{**}(x_0; \kappa_i)$. That is, the shadow value capitalizes the current government policy state as if lasting forever. And this holds the instant before and the instant after each policy variable change. Thus, at each instant the firm acts as if the current policy state will last forever despite knowing it will not. In such benign environments the econometrician can directly extract causal effects from treatment responses even if firms face highly transient policies. From this result it follows that reliance on rare-events is overly-restrictive.

In some cases, an empiricist may be interested in the more limited objective of avoiding an incorrect sign estimate. A description of identifying assumptions ensuring against sign reversals is therefore worthwhile. Since investment is weakly monotone increasing in q, a sufficient condition for the absence of sign reversals is $q^j \geq q^i$ for all $j < i$. It is readily verified that this condition is satisfied under binary assignment (Proposition 1) or if all transitions rates are equal (Proposition 2).

In order to derive another sign identification condition, return attention to equation (44). Since the current instantaneous marginal product of capital is strictly decreasing in the current state index j, it follows that q^j is strictly decreasing in j if the policy state is stochastically monotone. Letting the superscript j index the initial state, we recall from Kolokoltsov (2011) the continuous-time Markov chain for the policy state ι_t^j is said to be *stochastically monotone* if:

$$\Pr[\iota_t^j \leq k] \geq \Pr[\iota_t^{j+1} \leq k] \quad \forall\ k \in \{1, ..., N\},\ j \in \{1, ..., N-1\},\ t \in \mathbb{R}_+. \tag{45}$$

That is, if the policy state is stochastically monotone, then at all future dates t, ι_t^j is first-order stochastic dominant to ι_t^{j+1}.

Keilson and Kester (1977) derive necessary and sufficient conditions for stochastic monotonicity (Theorem 2.1). Applying their theorem, we have the following lemma.

Lemma 5 *No treatment response has sign opposite to its respective causal effect if the following matrix has non-negative off-diagonal elements:*

$$\begin{bmatrix} 1 & 0 & ... & 0 \\ 1 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\sum_{j\neq 1}\lambda_{1j} & \lambda_{12} & ... & \lambda_{1N} \\ \lambda_{21} & -\sum_{j\neq 2}\lambda_{2j} & ... & \lambda_{2N} \\ ... & ... & ... & ... \\ \lambda_{N1} & \lambda_{N2} & ... & -\sum_{j\neq N}\lambda_{Nj} \end{bmatrix} \begin{bmatrix} 1 & 0 & ... & 0 \\ 1 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Stochastic monotonicity thus represents another possible identifying assumption for the more limited goal of sign identification. A number of commonly-utilized processes are stochastically monotone, such as random walks with absorbing or impenetrable boundaries, as shown by Keilson and Kester (1977).

The following proposition summarizes the conditions derived for sign identification.

Proposition 7 *No treatment response has sign opposite its respective causal effect if the policy variable is binary; transitions are uniformly distributed with $\lambda_{ij} = \lambda$ for all i and j; or the policy variable is stochastically monotone.*

7. The Lucas Critique: Application, Extension and Correction

Up to this point attention has been confined to determining the relationship between measured treatment responses and causal effects. This section considers a related but distinct question: whether and how some historic treatment response in a given economy can be used to forecast future treatment responses in that same economy or in different economies.

7.1. Limits to External and Internal Validity

An observation following directly from the model is that it is not generally valid to use the treatment response in one economy to forecast the treatment response in another economy. This argument holds even if the change in the policy variable is the same and the economies have identical production and investment technologies. To see this, note that it follows from equation (31) that if economies A and B have different policy transition rates, then $\mathbf{T}^A \neq \mathbf{T}^B$, implying differences in shadow values and treatment responses. In other words, as argued by Lucas (1976), if the policy generating process is changed (here represented by $\mathbf{T}$), decision rules (here the TR_{ij}) will change. This is an important limitation on the external validity.

In fact, the Lucas Critique is just one limit on extrapolation. The model also illustrates important limits to the *internal validity* of measured treatment responses arising from natural experiments. Specifically, even if one considers a single economy and holds fixed the policy transition matrix $\mathbf{T}$, so that the Lucas Critique does not apply, it is generally not valid to directly use a historic treatment response to forecast future treatment responses.

To illustrate limits to internal validity, we consider three numerical examples. Each example approximates a different policy environment. All three examples consider variation in the κ policy variable across ten states, with jumps across neighboring states of equal size. Each example assumes that only nearest-neighbor transitions are possible. The effective discount rate is set to 10%.

In the first example it is assumed that the transition rate for an up move is $\lambda_u = .15$ and the transition rate for a down move is $\lambda_d = .05$. The implied expected regime duration is five years. The results are shown in Table 1. As shown, the fact that $\lambda_u > \lambda_d$ results in asymmetric treatment responses. Since an up move is more likely than a down move, shadow values are pushed upward, implying relatively large (small) treatment responses for downward (upward) transitions. For example, starting at $\kappa = 1.9$ the treatment response for an up move is .449, while that for a down move is -.697, a 55% difference. In addition, treatment responses vary across initial states, despite the fact that the transition rates for each state are identical and causal effects are linear. For example, the minimum treatment response is .449 and the maximum treatment response is .978, a difference of 118%.

Consider next Table 2. This example again assumes jumps in κ across neighboring states are of equal size and that only nearest-neighbor transitions are possible. In contrast to the first example, we now assume that for states 1 through 5, there is a bias toward further increases in κ with $\lambda_u = .15$ and $\lambda_d = .05$. Conversely, if the initial state is 6-10, there is a bias toward further decreases in κ with $\lambda_u = .05$ and $\lambda_d = .15$. One can think of this example as approximating a polarized political system. As in the prior example, the expected policy regime duration is 5 years. As shown in the table, here one sees even larger differences between treatment responses across states. For example, the minimum treatment response is .451 and the maximum treatment response is 2.208, a striking difference of 390%. Further, under this policy generating process the highest treatment responses involve transitions between the interim states 5 and 6.

Consider finally Table 3. As in the preceding two examples, jumps in κ across neighboring states are of equal size and only nearest-neighbor transitions are possible. However, it is now assumed that for states 1 through 5 there is a bias toward further decreases in κ with $\lambda_d = .15$ and $\lambda_u = .05$. Conversely, for states 6 through 10, there is a bias toward further increases in κ with $\lambda_d = .05$ and $\lambda_u = .15$. One can think of this example as approximating a political system with a tendency toward centrism. As in the prior two examples, the expected policy regime duration is 5 years. As shown in the table, here again one sees large differences in treatment responses across the states. In contrast to the preceding example, the lowest treatment response is for transitions between states 5 and 6, with the highest treatment responses near the two boundaries.

7.2. Overcoming Limits to External and Internal Validity

A natural question to ask is how the limits to internal and external validity detailed in the previous subsection can be overcome. For simplicity, assume all firms are driven by a common geometric Brownian motion x and employ potentially-heterogeneous linear-quadratic investment cost functions $\psi a + \gamma_m a^2$. A similar inversion procedure would apply for alternative adjustment cost functions.

Consider the problem of internal validity. To fix ideas, suppose the empiricist has measured a historic treatment response associated with κ jumping from state i to state j. She is now interested in using this historical evidence, specifically the sample average treatment response $\overline{TR}_{ij}$, to generate a forecast $\widehat{TR}$ regarding how firms in this same economy will respond to a future transition from say κ_h to κ_k. From equation (42) we have:

$$\widehat{TR}_{hk} = \widehat{\mu}_{\Theta}(q_k - q_h) = \left(\frac{\overline{TR}_{ij}}{q_j - q_i}\right)(q_k - q_h) = \left[\frac{c_k - c_h}{c_j - c_i}\right]\overline{TR}_{ij}. \tag{46}$$

Extrapolation can be understood as a two-step process. In the first step, one computes the accumulation response per unit of shadow value $\widehat{\mu}_{\Theta}$ by normalizing the historical treatment response by the change in shadow value at that transition date. This is then re-scaled by the change in shadow value associated with the future-date transition. The final term in square brackets, the ratio of shadow value changes, is computed via equation (31).

Given its apparent brevity, it is worth emphasizing that the last term in squared brackets in equation (46) generally depends upon all transition rates and all possible realizations of the policy variable, not just those in the directly-relevant set $\{h, i, j, k\}$. However, in exceptional cases the preceding formula simplifies. For example, if the identifying assumptions described in Propositions 5 or 6 are satisfied, we have:

$$\frac{c_k - c_h}{c_j - c_i} = \frac{\kappa_k - \kappa_h}{\kappa_j - \kappa_i} \Rightarrow \widehat{TR}_{hk} = \left[\frac{\kappa_k - \kappa_h}{\kappa_j - \kappa_i}\right] \overline{TR}_{ij}. \tag{47}$$

That is, if the underlying policy generating process meets the respective identifying assumptions for equality of treatment responses and causal effects, the extrapolation of treatment responses within a policy generating process requires a straightforward adjustment for the relative size of changes in the policy variable. Although imputations along the lines of formula (47) are commonplace, it is apparent that such imputations are not generally valid.

Consider next the problem of external validity (Lucas (1976)). Continuing with the preceding example, suppose the econometrician would like to forecast how firms in Economy B will respond to a change in κ from κ_h^B to κ_k^B based upon the evidence provided by the observed response of firms in Economy A to a change in κ from κ_i^A to κ_j^A. Applying equation (31) we have:

$$\widehat{TR}_{hk}^B = \widehat{\mu}_\Theta(q_k^B - q_h^B) = \left(\frac{\overline{TR}_{ij}^A}{q_j^A - q_i^A}\right)(q_k^B - q_h^B) = \left[\frac{c_k^B - c_h^B}{c_j^A - c_i^A}\right] \overline{TR}_{ij}^A. \tag{48}$$

That is, correct extrapolation across two economies requires taking explicit account of their respective policy generating processes via equation (31).

8. Numerical Example: Greenstone (2001)

This section illustrates how one can extract causal effects from observed treatment responses in real-world dynamic settings. We consider the empirical environment analyzed by Greenstone (2001) since this paper is rare in actually providing sufficient detail allowing one to estimate transition rates. The primary emphasis here is not on specific numerical results but rather to outline the basic method in a real-world setting. Since our model was not tailor-made for the policy setting considered by Greenstone, some approximations are necessary. For example, we set our analysis in continuous time in order to facilitate analytical solutions whereas the regulations Greenstone considers might be better approximated using a discrete-time setup.

Greenstone (2001) considers the effect of environmental regulations, exploiting quasi-random regulatory assignment arising from the Clean Air Act (CAA). Under the CAA, counties are designated as achieving attainment or non-attainment status in four pollutant categories: Carbon Monoxide, Ozone, Sulfur Dioxide, and Total Suspended Particulates (TSP). Firms emitting the relevant pollutant in non-attainment counties face surcharges and taxes. The Pulp, Paper, Iron and Steel sectors emit all 4 pollutants, implying 16 possible combinations of attainment versus non-attainment status over all pollutants. For brevity, we will confine our attention to these sectors.

Table 4 provides detail on the possible regulation categories and assumed κ values. The numerical examples assume each non-attainment infraction results in a reduction in κ. The baseline κ for a fully compliant firm is normalized at 1. The following κ reductions are assumed for non-attainment for each pollutant: Carbon Monoxide=0.30; Ozone=0.25; Sulfur Dioxide=0.20; and TSP=0.10. The lowest possible κ value is 0.15 for a firm emitting all four pollutants and residing in a county that has non-attainment status for all four pollutants.

The key input into the analysis are the transitions rates. Greenstone (2001) reports summary statistics allowing one to infer for each pollutant the conditional probability of starting a given (five-year) observation period in Attainment/Non-Attainment status and ending the period in Attainment/Non-Attainment status. Using the Kolmogorov forward equations, we calibrate the continuous-time transition rates in order to match these conditional probabilities.[3] Table 5 reports transition rates across categories. The average regime life across the 16 possible categories is 1.4 periods or 7 years. We set $r + \delta - \mu = 0.50$ given that each observation period is five years.

Table 6 reports the ratio of treatment response to causal effect for the 240 possible regulation category transitions for firms in the Pulp, Paper, Iron and Steel sectors. As shown in the table, the most common form of bias is attenuation. In many cases, the attenuation bias is substantial. Further, there are 8 transitions (3%) for which treatment response is opposite in sign to the causal effect. And there are 24 transitions (10%) for which treatment response overshoots the causal effect.

Apparently, under the data generating process exploited by Greenstone (2001), sign reversals are uncommon. This is because the most common and probable transition is to a neighboring state. In contrast, we recall that sign reversals tend to occur when a large positive or negative change is expected, while the realized change is equal in sign but relatively small in absolute value.

[3] See Ross (1996) pages 242-250.

Thus, for those interested in understanding the causal effect sign of environmental regulations, the policy generating process exploited by Greenstone is attractive. However, for those wishing to estimate deadweight losses of future environmental regulations, naïve substitution of his estimates into standard deadweight formulae would generally result in a severe understatement of welfare losses.

Setting aside the particular application considered in this section, it is apparent that only by way of opening up the black-box of the policy generating process, as done here, can one hope to understand the sign and magnitude of biases arising in applied empirical exercises in dynamic settings. This can be done by the empiricist or following Greenstone (2001) the empiricist can provide enough detail on the underlying process such that others can perform independent back-of-the-envelope bias estimates.

8. Concluding Remarks

This paper analyzes the relationship between theory-implied causal effects and measured treatment responses in dynamic economies. We show that independent policy assignment is insufficient for valid inference. With binary assignment, or uniformly distributed transitions, treatment responses understate causal effects, and substantially so under plausible parameterizations. With more than two policy regimes, treatment responses can understate, overstate, or have a sign that is opposite to causal effects. Moreover, the probability of overshooting and sign reversals can be high under plausible distributional assumptions. We also show important limits to the generalizability of historic treatment responses. It is not valid, in general, to extrapolate treatment responses within or across policy generating processes. Together, these results call into question the economic meaning and utilization of a broad range of elasticity estimates shaping policy decisions.

As constructive steps, we derive identifying assumptions required to ensure equality of treatment responses and causal effects in dynamic settings. We show that if all possible policy changes are rare events, treatment responses approximate causal effects absent fixed costs. However, we offer a more widely applicable identifying condition, showing that even with fixed costs, treatment responses equal causal effects if the policy variable is a martingale. In the special case of permanent assignment, each possible treatment response is equal to its respective causal effect if, and only if, the expected policy variable change has mean zero.

In certain cases, empiricists are willing to settle for the more limited objective of avoiding incorrect sign inference. We derive a battery of identifying assumptions that empiricists can invoke to avoid incorrect signs: binary assignment; uniformly distributed policy transitions; permanent assignment opposite in sign to the expected assignment; permanent assignment with the ratio of treatment to expected treatment exceeding unity; or stochastic monotonicity of the policy variable. As final constructive steps, we show how observed treatment responses can be mapped backed to causal effects, or extrapolated within or across policy generating processes.

It follows from our analysis that an apparent falsification of a given theory based on an incorrect sign prediction may be a false falsification. In particular, in addition to "*the* identifying assumption" of random assignment, natural policy experiments must be understood as predicated upon equally-important identifying assumptions about policy generating processes. Of course, this is a special example of the arguments of Willard Van Orman Quine who pointed out that an empirical falsification only allows one to either reject the underlying theory or to reject an experimenter's auxiliary assumptions. Unfortunately, these auxiliary assumptions have gone unrecognized, unstated and unsatisfied in much of the quasi-experimental literature.

In the view of many, random assignment constitutes the ideal empirical setting for establishing causality. Instead, we have shown that random assignment is properly seen as just one ingredient in the making of an ideal policy experiment in dynamic economies.

Appendix: Proofs and Derivations

For brevity, let:

$$\begin{aligned} R &\equiv r+\delta-\mu \\ c &\equiv \frac{\kappa}{r+\delta-\mu}. \end{aligned}$$

Growth Option Value with Constant Government Policies

Since the terms in the Bellman equation scaled by s cancel each other, satisfaction of the equation demands the growth option value satisfies the following ODE:

$$rG(x) = \mu x G_x(x) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x) + \underbrace{(\Gamma c^2)x^2 - 2\Gamma c\psi x + \Gamma\psi^2}_{Dividend}. \tag{49}$$

It follows from the preceding equation and the Feynman-Kac formula that the growth option value G is equal to the value of a claim to a dividend stream that is linear-quadratic in x. To value this linear-quadratic claim, conjecture a linear-quadratic value function G with unknown constants and substitute this function into the preceding differential equation. One obtains:

$$G(x) = \frac{1}{4\gamma}\left[\left(\frac{c^2}{r-2\mu-\sigma^2}\right)x^2 - \left(\frac{2c\psi}{r-\mu}\right)x + \frac{\psi^2}{r}\right]. \tag{50}$$

Growth Option Value with Regime Shifts

To derive the growth option value, return to the Bellman system (27) and confine attention to the remaining terms not scaled by s. The following system remains:

$$\begin{aligned} \left(r+\sum_{j\neq 1}\lambda_{1j}\right)G^1(x) &= \mu x G^1_x(x) + \frac{1}{2}\sigma^2 x^2 G^1_{xx}(x) + \sum_{j\neq 1}\lambda_{1j}G^j(x) + \Gamma c_1^2 x^2 - 2\Gamma c_1\psi x + \Gamma\psi^2 \quad (51) \\ &\cdots \\ \left(r+\sum_{j\neq N}\lambda_{Nj}\right)G^N(x) &= \mu x G^N_x(x) + \frac{1}{2}\sigma^2 x^2 G^N_{xx}(x) + \sum_{j\neq N}\lambda_{Nj}G^j(x) + \Gamma c_N^2 x^2 - 2\Gamma c_N\psi x + \Gamma\psi^2. \end{aligned}$$

We conjecture a growth option value that is linear-quadratic in x, with regime shifts. The following lemma follows directly by substituting the conjectured linear-quadratic solution into the system of ODEs in equation (51).

Lemma: *The no-bubbles solution to the differential equations*

$$\left(r+\sum_{j\neq 1}\lambda_{1j}\right)J^1(x) = \mu x J^1_x(x)+\frac{1}{2}\sigma^2x^2J^1_{xx}(x)+\sum_{j\neq 1}\lambda_{1j}J^j(x)+\phi_1x^2+\widetilde{\phi}_1x+\widehat{\phi} \quad (52)$$

$$...$$

$$\left(r+\sum_{j\neq N}\lambda_{Nj}\right)J^N(x) = \mu x J^N_x(x)+\frac{1}{2}\sigma^2x^2J^N_{xx}(x)+\sum_{j\neq N}\lambda_{Nj}J^j(x)+\phi_Nx^2+\widetilde{\phi}_Nx+\widehat{\phi}$$

is

$$\begin{bmatrix} J^1(x) \\ ... \\ J^N(x)\end{bmatrix} = x^2[\mathbf{T}(r-2\mu-\sigma^2)]^{-1}\begin{bmatrix}\phi_1 \\ ... \\ \phi_N\end{bmatrix} + x[\mathbf{T}(r-\mu)]^{-1}\begin{bmatrix}\widetilde{\phi}_1 \\ ... \\ \widetilde{\phi}_N\end{bmatrix} + \frac{\widehat{\phi}}{r}\begin{bmatrix}1 \\ ... \\ 1\end{bmatrix}. \quad (53)$$

Proposition 2

We have the following $N\times N$ augmented transition matrix:

$$\mathbf{T}(R)\equiv\begin{bmatrix} R+(N-1)\lambda & -\lambda & ... & -\lambda \\ -\lambda & R+(N-1)\lambda & ... & -\lambda \\ ... & ... & ... & ... \\ -\lambda & -\lambda & ... & R+(N-1)\lambda\end{bmatrix} \Rightarrow \mathbf{T}^{-1}=\frac{1}{R(R+N\lambda)}\begin{bmatrix} R+\lambda & \lambda & ... & \lambda \\ \lambda & R+\lambda & ... & \lambda \\ ... & ... & ... & ... \\ \lambda & \lambda & ... & R+\lambda\end{bmatrix} \quad (54)$$

The shadow values are:

$$\begin{bmatrix} q^1(x) \\ q^2(x) \\ ... \\ q^N(x)\end{bmatrix} = \frac{x}{R(R+N\lambda)}\begin{bmatrix} R+\lambda & \lambda & ... & \lambda \\ \lambda & R+\lambda & & \lambda \\ ... & ... & ... & ... \\ \lambda & \lambda & ... & R+\lambda\end{bmatrix}\begin{bmatrix}\kappa_1 \\ \kappa_2 \\ ... \\ \kappa_N\end{bmatrix}. \quad (55)$$

Taking differences across rows, the transition response of the shadow value is:

$$q^j(x)-q^i(x)=\left[1-\frac{N\lambda}{R+N\lambda}\right]\frac{1}{R}(\kappa_j-\kappa_i)x. \quad (56)$$

And the result follows from $\Delta a=\Delta q/2\gamma$.■

Proposition 3

Suppose there are $N\geq 3$ states and assume there are three states $(1,2,3)$ that are absorbing as a system in that once the policy process enters any one of these three states, there cannot be a transition to any state $i \notin \{1,2,3\}$. Transition rates from states outside $\{1,2,3\}$ can be set arbitrarily. It follows that the shadow value in any of these three states can be computed by focusing on the system confined to $\{1,2,3\}$. Following our indexing convention, let $\kappa_3 < \kappa_2 < \kappa_1$.

To prove the first part of the proposition it is sufficient to find transition rates for the three state system with sign reversals. To this end, assume states 1 and 2 are absorbing and let us identify conditions such that $q^3 > q^2$ so that $TR_{32} < 0$ while $CE_{32} > 0$. We have:

$$\mathbf{T} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ -\lambda_{31} & -\lambda_{32} & R+\lambda_{31}+\lambda_{32} \end{bmatrix} \Rightarrow \mathbf{T}^{-1} = \frac{1}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\lambda_{31}}{R+\lambda_{31}+\lambda_{32}} & \frac{\lambda_{32}}{R+\lambda_{31}+\lambda_{32}} & \frac{R}{R+\lambda_{31}+\lambda_{32}} \end{bmatrix}. \tag{57}$$

This implies:

$$\begin{aligned} q^2(x) &= \left(\frac{x}{R}\right)\kappa_2 \\ q^3(x) &= \frac{x}{R\left(R+\lambda_{31}+\lambda_{32}\right)}\left[\lambda_{31}\kappa_1 + \lambda_{31}\kappa_2 + R\kappa_3\right]. \end{aligned} \tag{58}$$

It follows that

$$\lambda_{31} > \frac{R(\kappa_2-\kappa_3)}{\kappa_1-\kappa_2} \Rightarrow q^3(x) > q^2(x).$$

To prove the second part of the proposition it is sufficient to find transition rates for the three state system such that, say, $TR_{21} > CE_{21}$. To this end, assume states 1 and 3 are absorbing. The transition matrix is:

$$\mathbf{T} = \begin{bmatrix} R & 0 & 0 \\ -\lambda_{21} & R+\lambda_{21}+\lambda_{23} & -\lambda_{23} \\ 0 & 0 & R \end{bmatrix} \Rightarrow \mathbf{T}^{-1} = \frac{1}{R}\begin{bmatrix} 1 & 0 & 0 \\ \frac{\lambda_{21}}{R+\lambda_{21}+\lambda_{23}} & \frac{R}{R+\lambda_{21}+\lambda_{23}} & \frac{\lambda_{23}}{R+\lambda_{21}+\lambda_{23}} \\ 0 & 0 & 1 \end{bmatrix}. \tag{59}$$

We have:

$$\begin{bmatrix} q^1(x) \\ q^2(x) \\ q^3(x) \end{bmatrix} = \frac{x}{R}\begin{bmatrix} 1 & 0 & 0 \\ \frac{\lambda_{21}}{R+\lambda_{21}+\lambda_{23}} & \frac{R}{R+\lambda_{21}+\lambda_{23}} & \frac{\lambda_{23}}{R+\lambda_{21}+\lambda_{23}} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix}. \tag{60}$$

Since treatment responses and causal effects are computed as $\Delta q/2\gamma$, we know:

$$q^1(x) - q^2(x) > \frac{x(\kappa_1-\kappa_2)}{R} \Rightarrow TR_{21} > CE_{21}. \tag{61}$$

The first inequality is satisfied if:

$$\lambda_{21}(\kappa_1 - \kappa_2) < \lambda_{23}(\kappa_2 - \kappa_3).\blacksquare$$

Lemma 1

The treatment response is zero across all regime changes if and only if the shadow value is constant across regimes. This holds if and only if there is some constant k such that the shadow value multipliers $\mathbf{c} = \mathbf{1}k$. But this implies the existence of an augmented transition matrix $\mathbf{T}$ satisfying:

$$k\mathbf{1} = \mathbf{T}^{-1}(R)\begin{bmatrix}\kappa_1 \\ ... \\ \kappa_N\end{bmatrix} \Rightarrow \mathbf{T}(R)\mathbf{1} = \frac{1}{k}\begin{bmatrix}\kappa_1 \\ ... \\ \kappa_N\end{bmatrix} \tag{62}$$

which is contradicted by the fact that

$$\mathbf{T}(R)\mathbf{1} = \begin{bmatrix}R \\ ... \\ R\end{bmatrix}.\blacksquare$$

Lemma 2

The result follows from:

$$\begin{aligned} q^N(x) &\geq q^{**}(x;\kappa_N) \\ q^1(x) &\leq q^{**}(x;\kappa_1)\blacksquare. \end{aligned}$$

Lemma 3

From Lemma 2 it follows:

$$q^1(x) - q^N(x) \leq q^{**}(x;\kappa_1) - q^{**}(x;\kappa_N). \tag{63}$$

The preceding inequality may be rewritten as:

$$\begin{aligned} [q^1(x) - q^i(x)] + [q^j(x) - q^N(x)] + [q^i(x) - q^j(x)] \quad &\leq \quad [q^{**}(x;\kappa_1) - q^{**}(x;\kappa_i)] + [q^{**}(x;\kappa_j) - q^{**}(x;\kappa_N)] \\ &\quad +[q^{**}(x;\kappa_i) - q^{**}(x;\kappa_j)]. \end{aligned} \tag{64}$$

Rearranging terms, it follows:

$$\begin{aligned}[q^1(x) - q^i(x)] + [q^j(x) - q^N(x)] \quad \leq \quad & [q^{**}(x;\kappa_1) - q^{**}(x;\kappa_i)] + [q^{**}(x;\kappa_j) - q^{**}(x;\kappa_N)] \\ & -\{[q^i(x) - q^j(x)] - [q^{**}(x;\kappa_i) - q^{**}(x;\kappa_j)]\}.\end{aligned} \tag{65}$$

Under the conditions stipulated in the lemma, the final term in {} brackets in the preceding equation is equal to $2\gamma k$ and so it follows:

$$[q^1(x) - q^i(x)] + [q^j(x) - q^N(x)] \leq [q^{**}(x;\kappa_1) - q^{**}(x;\kappa_i)] + [q^{**}(x;\kappa_j) - q^{**}(x;\kappa_N)] - 2\gamma k. \tag{66}$$

The result follows by dividing the preceding equation by 2γ.■

Lemma 4

For the first part of the lemma, consider a transition to state 1. Let $q^n(x_0)$ be the minimum shadow value amongst $i > 1$. And let

$$\widehat{q}^1(x_0, \Lambda) \equiv \mathbb{E}\left[\int_0^\infty \left(\int_0^T e^{-(r+\delta)t}\kappa_1 x_t dt + e^{-(r+\delta)T}\sum_{j\neq i} P_{ij} q^j(x_T)\right)(\Lambda e^{-\Lambda T})dT \mid \mathfrak{F}_0\right]. \tag{67}$$

Since κ_1 is maximal it follows that $\widehat{q}^1(x_0, \Lambda)$ is decreasing in its second argument. Further:

$$q^1(x_0) \equiv \widehat{q}^1(x_0, \Lambda_1) > \lim_{\Lambda \uparrow \infty} \widehat{q}^1(x_0, \Lambda) = \sum_{j\neq i} P_{ij} q^j(x_T) \geq q^n(x_0).$$

An analogous argument applies for a transition to state N.■

Proposition 5

Since T is Strictly Diagonal Dominant it is invertible. Moreover, each entry in the inverse is the quotient of a polynomial and the determinant. Thus, fixing, say, the Euclidean norm, the inverse function is continuous on the set of invertible matrices. It follows:

$$\lim_{\boldsymbol{\lambda} \downarrow \mathbf{0}} [\mathbf{T}(R;\boldsymbol{\lambda})]^{-1} = [\mathbf{T}(R;\mathbf{0})]^{-1} = R^{-1}\mathbf{I}_N.$$

This implies $q^i(x)$ converges to $\kappa_i x/R$ implying that absent fixed investment costs each TR_{ij} converges to CE_{ij}.■

Proposition 6

Consider an array of distinct policies 1 to N with the indexing convention being that the shadow value under permanent policies is decreasing in the index. A necessary and sufficient condition for each treatment response to equal its respective causal effect is that the difference between state-contingent shadow values is equal to their respective difference in shadow values under permanent policies. Thus, there must exist some k such that for all i:

$$q^i(x) = \frac{\kappa_i x}{r+\delta-\mu} + k. \tag{68}$$

Therefore, we must identify conditions such that the following equilibrium conditions can be met under this functional form:

$$\begin{aligned} \left(r+\delta+\sum_{j\neq 1}\lambda_{1j}\right) q^1(x) &= \mu x q_x^1(x) + \frac{1}{2}\sigma^2 x^2 q_{xx}^1(x) + \sum_{j\neq 1}\lambda_{1j} q^j(x) + \kappa_1 x \\ &\ldots \\ \left(r+\delta+\sum_{j\neq N}\lambda_{Nj}\right) q^N(x) &= \mu x q_x^N(x) + \frac{1}{2}\sigma^2 x^2 q_{xx}^N(x) + \sum_{j\neq N}\lambda_{Nj} q^j(x) + \kappa_N x. \end{aligned} \tag{69}$$

Substituting in the required functional form and canceling terms we obtain:

$$\begin{aligned} (r+\delta)\,k &= \sum_{j\neq 1}\lambda_{1j}\left[\frac{(\kappa_j-\kappa_1)x}{r+\delta-\mu}\right] \\ (r+\delta)\,k &= \sum_{j\neq 2}\lambda_{2j}\left[\frac{(\kappa_j-\kappa_2)x}{r+\delta-\mu}\right] \\ &\ldots \\ (r+\delta)\,k &= \sum_{j\neq N-1}\lambda_{N-1j}\left[\frac{(\kappa_j-\kappa_{N-1})x}{r+\delta-\mu}\right] \\ (r+\delta)\,k &= \sum_{j\neq N}\lambda_{Nj}\left[\frac{(\kappa_j-\kappa_N)x}{r+\delta-\mu}\right]. \end{aligned} \tag{70}$$

First note that any solution to the preceding system entails $k=0$ since the right-side of the first equation is non-positive while the right-side of the last equation is non-negative. It follows that any candidate solution to the system entails:

$$\begin{aligned} \lambda_{1j} &= 0 : j = 2, ..., N \\ \lambda_{Nj} &= 0 : j = 1, ..., N-1. \end{aligned}$$

Further, it must be the case that:

$$0 = \sum_{j\neq i}\lambda_{ij}\left[\frac{(\kappa_j-\kappa_i)x}{r+\delta-\mu}\right] : i = 2, ..., N-1 \tag{71}$$

The preceding equation can be rewritten as:

$$\sum_{j \neq i} \left(\frac{\lambda_{ij}}{\sum_{j \neq i} \lambda_{ij}} \right) (\kappa_j - \kappa_i) = 0 : i = 2, ..., N-1. \blacksquare \tag{72}$$

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Table 1: Treatment Responses Constant Transition Rates

Initial Kappa	Transition Up	Transition Down
2.0	NA	-0.449
1.9	0.449	-0.697
1.8	0.697	-0.833
1.7	0.833	-0.908
1.6	0.908	-0.950
1.5	0.950	-0.971
1.4	0.971	-0.978
1.3	0.978	-0.958
1.2	0.958	-0.813
1.1	0.813	NA

Table 2: Treatment Responses Polarized Political System

Initial Kappa	Transition Up	Transition Down
2.0	NA	-0.451
1.9	0.451	-0.705
1.8	0.705	-0.875
1.7	0.875	-1.140
1.6	1.140	-2.208
1.5	2.208	-1.140
1.4	1.140	-0.875
1.3	0.875	-0.705
1.2	0.705	-0.451
1.1	0.451	NA

Table 3: Treatment Responses Centrist Political System

Initial Kappa	Transition Up	Transition Down
2.0	NA	-0.768
1.9	0.768	-0.871
1.8	0.871	-0.818
1.7	0.818	-0.680
1.6	0.680	-0.420
1.5	0.420	-0.680
1.4	0.680	-0.818
1.3	0.818	-0.870
1.2	0.870	-0.769
1.1	0.769	NA

Table 4: Regulatory Attainment Categories and Profitability Factors

Numerical Category	Carbon Monoxide	Ozone	Sulfur Dioxide	Total Suspended Particulates	Profitabiilty Factor
1	Attain	Attain	Attain	Attain	1.00
2	Attain	Attain	Attain	Non	0.85
3	Attain	Attain	Non	Attain	0.80
4	Attain	Non	Attain	Attain	0.75
5	Non	Attain	Attain	Attain	0.70
6	Attain	Attain	Non	Non	0.65
7	Attain	Non	Attain	Non	0.60
8	Attain	Non	Non	Attain	0.55
9	Non	Attain	Attain	Non	0.55
10	Non	Attain	Non	Attain	0.50
11	Non	Non	Attain	Attain	0.45
12	Attain	Non	Non	Non	0.40
13	Non	Attain	Non	Non	0.35
14	Non	Non	Attain	Non	0.30
15	Non	Non	Non	Attain	0.25
16	Non	Non	Non	Non	0.10

Table 5: Transition Rate Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0.0000	0.0671	0.0182	0.0946	0.0238	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.6463	0.0000	0.0000	0.0001	0.0000	0.0181	0.0944	0.0000	0.0238	0.0000	0.0001	0.0002	0.0000	0.0000	0.0000	0.0000
3	0.7403	0.0000	0.0000	0.0000	0.0000	0.0671	0.0000	0.0944	0.0001	0.0238	0.0000	0.0002	0.0000	0.0000	0.0001	0.0000
4	0.1725	0.0000	0.0000	0.0000	0.0000	0.0000	0.0670	0.0182	0.0000	0.0000	0.0239	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.2821	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0671	0.0181	0.0944	0.0000	0.0000	0.0001	0.0001	0.0000
6	0.0000	0.7405	0.6464	0.0000	0.0000	0.0000	0.0001	0.0002	0.0000	0.0000	0.0000	0.0944	0.0241	0.0002	0.0001	0.0000
7	0.0000	0.1727	0.0001	0.6464	0.0000	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0182	0.0000	0.0240	0.0000	0.0000
8	0.0001	0.0001	0.1725	0.7402	0.0000	0.0000	0.0001	0.0000	0.0001	0.0000	0.0000	0.0672	0.0001	0.0000	0.0239	0.0000
9	0.0000	0.2822	0.0000	0.0000	0.6462	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0180	0.0944	0.0001	0.0000
10	0.0000	0.0000	0.2822	0.0002	0.7405	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0671	0.0002	0.0946	0.0000
11	0.0001	0.0000	0.0000	0.2821	0.1725	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0670	0.0182	0.0000
12	0.0001	0.0000	0.0000	0.0000	0.0000	0.1729	0.7409	0.6468	0.0001	0.0001	0.0000	0.0000	0.0000	0.0001	0.0001	0.0239
13	0.0002	0.0000	0.0000	0.0000	0.0000	0.2824	0.0000	0.0002	0.7401	0.6462	0.0003	0.0000	0.0000	0.0000	0.0000	0.0945
14	0.0001	0.0000	0.0000	0.0000	0.0000	0.0002	0.2821	0.0000	0.1726	0.0001	0.6463	0.0000	0.0000	0.0000	0.0000	0.0182
15	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0000	0.2822	0.0003	0.1727	0.7403	0.0002	0.0000	0.0001	0.0000	0.0671
16	0.0000	0.0004	0.0003	0.0004	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2827	0.1730	0.7409	0.6468	0.0000

Table 6: Ratio of Treatment Response to Causal Effect
Pulp, Paper, Iron and Steel

CATEGORY	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	NA	0.4131	0.3966	0.6522	0.6202	0.4038	0.5622	0.5388	0.5510	0.5308	0.6348	0.5069	0.5031	0.5870	0.5712	0.5440
2	0.4131	NA	0.3471	1.0109	0.8273	0.3968	0.6517	0.6017	0.6199	0.5813	0.7179	0.5382	0.5301	0.6345	0.6107	0.5702
3	0.3966	0.3471	NA	1.6747	1.0674	0.4134	0.7279	0.6526	0.6745	0.6203	0.7709	0.5621	0.5504	0.6632	0.6347	0.5861
4	0.6522	1.0109	1.6747	NA	0.4601	-0.2172	0.4122	0.3970	0.4245	0.4094	0.6203	0.4031	0.4098	0.5508	0.5307	0.5024
5	0.6202	0.8273	1.0674	0.4601	NA	-0.8946	0.3883	0.3760	0.4126	0.3968	0.6523	0.3936	0.4027	0.5622	0.5385	0.5059
6	0.4038	0.3968	0.4134	-0.2172	-0.8946	NA	1.6712	1.0113	1.0662	0.8272	1.0390	0.6512	0.6189	0.7703	0.7176	0.6332
7	0.5622	0.6517	0.7279	0.4122	0.3883	1.6712	NA	0.3514	0.4611	0.4053	0.8283	0.3962	0.4084	0.6201	0.5814	0.5294
8	0.5388	0.6017	0.6526	0.3970	0.3760	1.0113	0.3514	NA	NA	0.4591	1.0667	0.4112	0.4226	0.6739	0.6198	0.5492
9	0.5510	0.6199	0.6745	0.4245	0.4126	1.0662	0.4611	NA	NA	0.3494	1.0119	0.3746	0.3952	0.6519	0.6015	0.5370
10	0.5308	0.5813	0.6203	0.4094	0.3968	0.8272	0.4053	0.4591	0.3494	NA	1.6744	0.3872	0.4105	0.7276	0.6519	0.5604
11	0.6348	0.7179	0.7709	0.6203	0.6523	1.0390	0.8283	1.0667	1.0119	1.6744	NA	-0.8999	-0.2214	0.4120	0.3963	0.4013
12	0.5069	0.5382	0.5621	0.4031	0.3936	0.6512	0.3962	0.4112	0.3746	0.3872	-0.8999	NA	0.4571	1.0679	0.8283	0.6182
13	0.5031	0.5301	0.5504	0.4098	0.4027	0.6189	0.4084	0.4226	0.3952	0.4105	-0.2214	0.4571	NA	1.6788	1.0140	0.6504
14	0.5870	0.6345	0.6632	0.5508	0.5622	0.7703	0.6201	0.6739	0.6519	0.7276	0.4120	1.0679	1.6788	NA	0.3492	0.3933
15	0.5712	0.6107	0.6347	0.5307	0.5385	0.7176	0.5814	0.6198	0.6015	0.6519	0.3963	0.8283	1.0140	0.3492	NA	0.4080
16	0.5440	0.5702	0.5861	0.5024	0.5059	0.6332	0.5294	0.5492	0.5370	0.6500	0.4013	0.6182	0.6504	0.3933	0.4080	NA

Figure 1: Treatment Response to Causal Effect Ratio