The Term Structure of Variance Swaps and Risk Premia*

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This version: December 17, 2013
First draft: April 2012

*For helpful comments we thank Peter Carr, Jin-Chuan Duan, Darrell Duffie, Damir Filipovic, Rob Engle, Michael Johannes, Markus Leippold, Ian Martin, Philippe Mueller, Andrew Papanicolaou, Christian Schlag, George Skiadopoulos, Mete Soner, Viktor Todorov, Fabio Trojani, Jules van Binsbergen and participants at the 2013 Econometric Society meetings, 2012 European Finance Association meetings, 2011 Princeton-Lausanne workshop, European University Institute, Imperial College, Goethe University, Tinbergen Institute, University of Piraeus, Banque de France and Morgan Stanley. Financial support from the NSF under grant SES-0850533 (Aït-Sahalia) and the SNSF NCCR-FinRisk (Mancini) is gratefully acknowledged. The views expressed here are not necessarily those of UBS AG. We thank Yan Wang for excellent research assistance. An earlier draft of this paper circulated under the title "The Term Structure of Variance Swaps, Risk Premia and the Expectation Hypothesis."

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Abstract

We study the term structure of variance swaps, equity and variance risk premia. A model-free analysis reveals a significant jump risk component embedded in variance swaps. A model-based analysis shows that the term structure of variance risk premia is negative and downward sloping. Investors' willingness to ensure against volatility risk increases after a market crash. The effect is stronger over short horizons and more persistent over long horizons. Variance risk premia over short horizons mainly reflect investors' fear of a market crash. A simple trading strategy with variance swaps generates significant returns.

Keywords: Variance Swap, Stochastic Volatility, Likelihood Approximation, Term Structure, Equity Risk Premium, Variance Risk Premium.

JEL Codes: C51, G12, G13.

1. Introduction

Over the last decade, the demand for volatility derivative products has grown exponentially, driven in part by the need to hedge volatility risk in portfolio management and derivative pricing. In 1993, the Chicago Board Options Exchange (CBOE) introduced the VIX as a volatility index computed as an average of the implied volatilities of short term, near the money, S&P100 options. Ten years later, the definition of the VIX was amended to become based on the more popular S&P500, itself the underlying of the most liquid index options (SPX), and to be computed in a largely model-free manner as a weighted average of option prices across all strikes at two nearby maturities, instead of relying on the Black–Scholes implied volatilities (e.g., Carr and Wu (2006).) Shortly thereafter, VIX futures and options on VIX were introduced at the CBOE Futures Exchange (CFE). Carr and Lee (2009) provide an excellent history of the market for volatility derivatives and a survey of the relevant methodologies for pricing and hedging volatility derivatives products.

Among volatility derivatives, variance swaps (VS) can be thought of as the basic building block. According to the financial press (e.g., Gangahar (2006)), VS have become the preferred tool by which market practitioners bet on and/or hedge volatility movements. VS are in principle simple contracts: the fixed leg agrees at inception that it will pay a fixed amount at maturity, in exchange to receiving a floating amount based on the realized variance of the underlying asset, usually measured as the sum of the squared daily log-returns, over the life of the swap. One potential difficulty lies in the path-dependency introduced by the realized variance.

The payoff of a VS can be replicated by dynamic trading in the underlying asset and a static position in vanilla options on that same underlying and maturity date. This insight, originally due to Neuberger (1994) and Dupire (1993), meant that the path-dependency implicit in VS could be circumvented; it also made possible an important literature devoted to analyzing and exploiting the various hedging errors when attempting to replicate a given VS (e.g., Carr and Madan (1998), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009), Carr and Lee (2010).) Because of the interest in replicating a given contract, VS rates have generally been studied at a single maturity.

But VS rates give rise naturally to a term structure, by varying the maturity at which the exchange of cash flows take place, and it is possible to analyze them in a framework comparable to that employed for the term structure of interest rates, including determining the number of factors necessary to capture the variation of the curve (see Bühler (2006), Gatheral (2008), Amengual (2008) and Egloff et al. (2010).) We continue this line of research with two differences. First, we do not proceed fully by analogy with the term structure of interest rates, i.e., taking either the variances themselves or their latent factors as the primitives: instead, we incorporate the fact that the variance in a VS is that of an underlying asset and explicitly incorporate the presence of that asset in our modeling. This means that we can infer properties of the risk premia associated not just with the variances but also with the asset itself, which in the case of the S&P500 is the classical equity risk premium. Second, and most importantly, we allow for the presence of jumps in asset returns and variance. When studying the term structure of VS rates, we examine how they behave as a function of maturity and the information they convey, particularly about risk premia.

This analysis allows for a better understanding of how volatility and jump risk is perceived by investors, as reflected in VS contracts at different horizons. It also has implications for investing in VS, as the profitability of the investment obviously depends on risk premia. We use actual, rather than synthetic, daily VS rates on the S&P500 index with fixed time to maturity of 2-, 3-, 6-, 12- and 24-month from January 4, 1996 to September 2, 2010.

The analysis reveals clear patterns in the term structure of VS rates. When time to maturity increases, the level and persistence of VS rates increase, while their volatility, skewness and kurtosis decrease. In agreement with Egloff et al. (2010), Gatheral (2008) and Amengual (2008), we find through Principal Component Analysis that two factors, which can be interpreted as level and slope factors, explain 99.8% of the variation in VS rates.

We then use a model-free approach to measure the jump component embedded in VS rates, relying on recent theoretical results for model-free implied volatilities. Specifically, we compare variance swap rates and VIX-type indices extracted from options on the S&P500 index (SPX) for various maturities. We find that a large and time-varying jump risk component is embedded in VS rates, which becomes even more pronounced in the latter part of the sample. A flexible stochastic volatility model cannot fully explain the jump risk component. This suggests that either the jump risk is heavily priced by VS traders or some segmentation between the VS and option markets exits or both.

Various aspects of the VS term structure cannot be studied in a model-free manner, because the necessary data are either insufficient in quantity or simply unavailable. A model-free analysis of the term structure of jump risk in VS would require observations on long lived, out-of-the-money, SPX options with a fixed time to maturity. These options are, unfortunately, unavailable or at least not sufficiently liquid.[1] To further the analysis of the VS term structure, we therefore rely on a parametric stochastic volatility model, namely a two-factor stochastic volatility model with price jumps and variance jumps, which is consistent with the salient empirical features of VS rates documented in the model-free analysis. The model is estimated using maximum-likelihood, combining time series information on stock returns and cross sectional information on the term structure of VS rates.

Our model-based analysis of risk premia uncovers the following phenomena. The integrated variance risk premium (IVRP), i.e., the expected difference between objective and risk neutral integrated variance, is negative and usually exhibits a downward-sloping term structure. As the IVRP is the ex-ante, expected payoff of the variance swap, a negative risk premium implies that the VS holder is willing to pay a “large” premium to get protection against volatility increases, which in turn induces a negative return on average at maturity. As the IVRP increases with the time to maturity, taking short positions in long-term VS contracts is more profitable on average than taking short positions in short-term VS contracts. This term structure finding complements the (model-free) analysis of IVRP for a single, short maturity in Carr and Wu (2009).

The term structure of IVRP due to negative jumps is negative, generally downward sloping in quiet times but upward sloping in turbulent times. The contribution of the jump component is modest in quiet times, but becomes large during market crashes, and mostly impacts the short-end of the IVRP term structure. This suggests that short-term variance risk premia mainly reflect investors’ fear of a market crash, rather than the impact of stochastic volatility on the investment set. It also suggests that investors’ willingness to ensure against future volatility risk over given time horizons increases after a market crash. This effect is stronger for short horizons but more persistent for long horizons. Recently, Bollerslev and Todorov (2011) provided a model-free analysis of the jump component in the IVRP for a single, short time to maturity (with median of 14 days). Todorov (2010) studied the IVRP due to jump risk over a one-month time horizon. Using a model-based approach, we extend such analyses to the term structure of IVRP.

Regression analysis shows that the term structure of IVRP responds nearly monotonically to variables proxying for equity, option, corporate and Treasury bond market conditions. Not surprisingly, a drop in the S&P500 index induces a more negative IVRP, but this effect “quickly

[1] Available options have discrete strike prices and fixed maturities, rather than fixed time to maturities. To carry out such a model-free analysis, interpolation and extrapolation schemes across strike prices and time to maturities are necessary with the potential to introduce significant approximation errors.

dies out" in the term structure of the IVRP, becoming statistically insignificant beyond a 6-month horizon. The VIX index, despite being a 30-day volatility index, has a fairly uniform and strong impact throughout the term structure of the IVRP, acting more like a "level" factor, than a short term factor, for variance risk premia.

We also study the term structure of the (integrated) equity risk premium, defined as the expected excess return from a buy-and-hold position in the S&P500 index, over various time horizons. Given our affine jump stochastic volatility model, equity risk premia are available in semi-closed form, up to the solution of nonlinear ordinary differential equations, using the transform analysis in Duffie et al. (2000). Equity risk premia exhibit significant countercyclical dynamics. The term structure of risk premia is slightly upward sloping in quiet times but steeply downward sloping during market crashes. This suggests that during, a financial crisis investors demand a large risk premium to hold risky stocks, but the risk premium largely depends and strongly decreases with the holding horizon. Indeed, in Fall 2008, after Lehman Brothers' bankruptcy, 2-month equity risk premia reached historically high values, around 50%. During low volatility periods, equity risk premia are about 6.5%, in line with historical estimates. Recently, Martin (2013) and van Binsbergen et al. (2013) provided related studies on equity risk premia, using different datasets and methods, and they also document large swings in equity risk premia, comparable to those we document here. We complement these studies by analyzing the term structure of equity risk premia and potential drivers.

Finally, as for the IVRP, we conduct regression analysis to understand which economic variables may drive the term structure of the IERP. We find that an increase in the VIX index increases the IERP, but the longer the horizon of the equity risk premium the smaller the effect. Hence, in contrast to the IVRP, the VIX index does not behave like a level factor for the IERP. An indicator of credit riskiness within the corporate sector (the difference between Moody's BAA and AAA corporate bond yields) has a positive and decreasing impact on the term structure of the IERP, amplifying the countercyclical variation of the IERP. Other variables have opposite impact on the term structure of the IERP. For example, the slope of the term structure of Treasury yields (the difference between the yields on 10-year and 2-year Treasury securities) has a positive impact on the short-end and a negative impact on the long-end of the IERP term structure. In Fall 2008 such a difference increased significantly, it amplified the downward slope of the IERP term structure. All in all, these empirical findings point to a rich impact of economic indicators on the term structure of equity and variance risk premia.

The structure of the paper is as follows. Section 2 briefly describes variance swaps and their properties. Section 3 introduces the model and estimation methodology. Section 4 presents the actual estimates. Section 5 reports risk premium estimates. Section 6 concludes. The Appendix contains technical derivations.

2. Variance Swaps

We introduce the general setup we will work with in order to analyze the term structure of variance swap contracts. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying usual conditions (e.g., Protter (2004)), with P denoting the objective or historical probability measure. Let S be a semimartingale modeling the stock (or index) price process with dynamics

$$dS_t/S_{t_-} = \mu_t\, dt + \sqrt{v_t}\, d\tilde{W}_t^P + (\exp(J_t^{s,P}) - 1)\, dN_t^P - \nu_t^P\, dt \tag{1}$$

where μ_t is the drift, v_t the spot variance, $\tilde{W}_t^P$ a Brownian motion, N_t^P a counting jump process with stochastic intensity λ_t^P, $J_t^{s,P}$ the random price jump size, and $\nu_t^P = g_t^P \lambda_t^P$ the compensator with $g_t^P = E_t^P[\exp(J^s) - 1]$ and E_t^P the time-t conditional expectation under P. When a jump occurs, the induced price change is $(S_t - S_{t_-})/S_{t_-} = \exp(J_t^{s,P}) - 1$, which implies that $\log(S_t/S_{t_-}) = J_t^{s,P}$. Thus, $J_t^{s,P}$ is the random jump size of the log-price under P. When no confusion arises superscripts are omitted. The dynamics of the drift, variance, and jump component are left unspecified and in this sense the first part of the analysis of VS contracts will be model-free. Indeed, the Model (1) subsumes virtually all models used in finance with finite jump activity.

Let $t = t_0 < t_1 < \cdots < t_n = t + \tau$ denote the trading days over a given time period $[t, t+\tau]$, for e.g., six months. The typical convention employed in the market is for the floating leg of the swap to pay at $t+\tau$ the annualized realized variance defined as the annualized sum of daily squared log-returns (typically closing prices) over the time horizon $[t, t+\tau]$:

$$\mathrm{RV}_{t,t+\tau} = \frac{252}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2. \tag{2}$$

Like any swap, no cash flow changes hands at inception of the contract at time t; the fixed leg of the variance swap agrees to pay an amount fixed at time t, defined as the variance swap rate, $\mathrm{VS}_{t,t+\tau}$. Any payment takes place in arrears. Unlike many other swaps, such as interest rates or currency swaps, a variance swap does not lead to a repeated exchange of cash flows, but rather to a single one at expiration, at time $t+\tau$. Therefore, at maturity, $t+\tau$, the long position in a variance swap contract receives the difference between the realized variance between times t and $t+\tau$, $\mathrm{RV}_{t,t+\tau}$, and the variance swap rate, $\mathrm{VS}_{t,t+\tau}$, which was fixed at time t. The difference is multiplied by a fixed notional amount to convert the payoff to dollar terms:

$$(\mathrm{RV}_{t,t+\tau} - \mathrm{VS}_{t,t+\tau}) \times (\text{notional amount}).$$

Variance swaps tend to provide positive payoffs in high volatility periods. If the period $[t, t+\tau]$ will be an unexpected high volatility period, the realized variance $\mathrm{RV}_{t,t+\tau}$ will be higher than the variance swap rate $\mathrm{VS}_{t,t+\tau}$ set at time t, which will trigger a positive payoff to the long side of the contract. Typically investors regard volatility increases as unfavorable events, because volatility increases imply high uncertainty and are usually associated to market crashes, e.g., Bekaert and Wu (2000). Thus, variance swaps are effectively insurance contracts against such negative events.

The analysis of variance swap contracts is simplified when the realized variance is replaced by the quadratic variation of the log-price process. It is well-known that when $\sup_{i=1,\dots,n}(t_i - t_{i-1}) \to 0$ the realized variance in Equation (2) converges in probability to the annualized quadratic variation of the log-price, $\mathrm{QV}_{t,t+\tau}$, (e.g., Jacod and Protter (1998)):

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \longrightarrow \frac{1}{\tau}\int_t^{t+\tau} v_u\,du + \frac{1}{\tau}\sum_{u=N_t}^{N_{t+\tau}} (J_u^s)^2 = \mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau} = \mathrm{QV}_{t,t+\tau} \qquad (3)$$

which is itself the sum of two terms, one due to the continuous part of the Model (1), $\mathrm{QV}^c_{t,t+\tau}$, and one to its discontinuous or jump part, $\mathrm{QV}^j_{t,t+\tau}$. This approximation is commonly adopted in practice and is quite accurate at the daily sampling frequency (e.g., Broadie and Jain (2008) and Jarrow et al. (2013)), as is the case in our dataset. Market microstructure noise, while generally an important concern in high frequency inference, is largely a non-issue at the level of daily returns. Note that, if the spot variance includes a jump component, the convergence above still holds and such variance jumps are accommodated in the time integral of v_u.

As usual, we assume absence of arbitrage, which implies the existence of an equivalent risk neutral measure Q. By convention, the variance swap contract has zero value at inception. Assuming that the interest rate does not depend on the quadratic variation, which is certainly a tenuous assumption and one commonly made when valuing these contracts, no arbitrage implies that the variance swap rate is

$$\mathrm{VS}_{t,t+\tau} = E_t^Q[\mathrm{QV}_{t,t+\tau}] = \overline{v}^Q_{t,t+\tau} + E_t^Q[(J^s)^2]\,\overline{\lambda}^Q_{t,t+\tau} \tag{4}$$

where E_t^Q denotes the time-t conditional expectation under Q, $\overline{v}^Q_{t,t+\tau} = E_t^Q[\mathrm{QV}^c_{t,t+\tau}]$, and $\overline{\lambda}^Q_{t,t+\tau} = E_t^Q \int_t^{t+\tau} \lambda_u^Q \, du/\tau$, i.e., the average risk neutral jump intensity.

The variance swap rate depends, of course, on the information available at time t. It also depends on the time to maturity, τ. The latter dependence produces the term structure we are interested in.

2.1. Preliminary Data Analysis

Our dataset consists of over the counter quotes on variance swap rates on the S&P500 index provided by a major broker-dealer in New York City. The data are daily closing quotes on variance swap rates with fixed time to maturities of 2, 3, 6, 12, and 24 months from January 4, 1996 to September 2, 2010, resulting in 3,624 observations for each maturity. Standard statistical tests do not detect any day-of-the-week effect, so we use all available daily data.

We start by identifying some of the main features of the VS rates data. These salient features are important not only because allow us to understand the dynamics of the VS rates, but also because they single out model-free characteristics of VS rates that any parametric model should be able to reproduce. Figure 1 shows the term structure of VS rates over time and suggests that VS rates are mean-reverting, volatile, with spikes and clustering during the major financial crises over the last 15 years, and historically high values during the acute phase of the recent financial crisis in Fall 2008. While most term structures are upward sloping (53% of our sample), they are often $\cup$-shape too (23% of our sample). The remaining term structures are roughly split in downward sloping and $\cap$-shape term structures.[2] The bottom and peak of the $\cup$- and $\cap$-shape term structures, respectively, can be anywhere at 3 or 6 or 12 months to maturity VS rate. The slope of the term structure (measured as the difference between the 24 and 2 months VS rates) shows a strong negative association with the contemporaneous volatility level. Thus, in high volatility periods or turbulent times, the short-end of the term structure (VS rates with 2 or 3 months to maturity) rises more than the long-end, producing downward sloping term structures.

[2]On some occasions, the term structure is $\sim$-shape, but the differences between, for e.g., the 2 and 3 months VS rates are virtually zero and these term structures are nearly $\cup$-shape.

Table 1 provides summary statistics of our data. For the sake of interpretability, we follow market practice and report variance swap rates in volatility percentage units, i.e., $\sqrt{\text{VS}_{t,t+\tau}} \times 100$. Various patterns emerge from these statistics. The mean level and first order autocorrelation of swap rates are slightly but strictly increasing with time to maturity. The standard deviation, skewness and kurtosis of swap rates are strictly decreasing with time to maturity. Ljung–Box tests strongly reject the hypothesis of zero autocorrelations, while generally Dickey–Fuller tests do not detect unit roots,[3] except for longest maturities – it is well-known that the outcome of standard unit root tests should be carefully interpreted with slowly decaying memory processes; e.g., Schwert (1987). First order autocorrelations of swap rates range between 0.982 and 0.995, confirming mean reversion in these series. As these coefficients increase with time to maturity, the longer the maturity the higher the persistence of VS rates with mean half-life[4] of shocks between 38 and 138 days.

Principal Component Analysis (PCA) shows that the first principal component explains about 95.4% of the total variance of VS rates and can be interpreted as a level factor, while the second principal component explains an additional 4.4% and can be interpreted as a slope factor.[5] This finding is somehow expected because PCA of several other term structures, such as bond yields, produce qualitatively similar results. Less expected is that two factors explain nearly all the variance of VS rates, i.e., 99.8%. Repeating the PCA for various subsamples produces little variation in the first two factors and explained total variance. Overall, PCA suggests that at most two factors are driving VS rates. When compared to typical term structures of bond yields, the one of VS rates appears to be simpler, as a third principal component capturing the curvature of the term structure is largely nonexistent here.

Table 1, Panel D, also shows summary statistics of ex-post realized variance of S&P500 index returns for various time to maturities. Realized variances are substantially lower on average than VS rates, which implies that shorting variance swaps is profitable on average. However, realized variances are also more volatile, positively skewed and leptokurtic than VS rates, which highlights the riskiness of shorting VS contracts. The large variability and in particular the positive skewness of ex-post realized variances can induce large losses to the short side of the contract. The ex-post variance risk premium, i.e., the difference between average realized variance and VS rate, is negative and increasing with time to maturities. Shorting VS

[3]Under the null hypothesis of unit root the Dickey–Fuller test statistic has zero expectation.

[4]The half-life H is defined as the time necessary to halve a unit shock and solves $\varrho^H = 0.5$, where ϱ is the first order autocorrelation coefficient.

[5]To save space, factor loadings are not reported, but are available from the authors upon request.

contracts with different time to maturities allows to earn such variance risk premia.

2.2. Model-free Jump Component in Variance Swap Rates

We start with a model-free method to quantify the price jump component in VS rates. We take advantage of recent theoretical advances collectively described as "model-free implied volatilities" (see Neuberger (1994), Dupire (1993), Carr and Madan (1998), Demeterfi et al. (1999), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009) and Carr and Lee (2010).)[6] The main result in this literature is that, under some conditions, if the stock price process is continuous, the variance swap payoff can be replicated by dynamic trading in futures contracts (or in the underlying asset) and a static position in a continuum of European options with different strikes and same maturity. The replication is model-free in the sense that the stock price can follow the general Model (1), but with the restriction $\lambda_t^P = 0$ and/or $J_t^{s,P} = 0$.

If the stock price has a jump component, this replication no longer holds. This observation makes it possible to assess whether VS rates embed a priced jump component and to quantify how large it is, in a model-free manner. Specifically, we compare the variance swap rate and the cost of the replicating portfolio using options. If the difference between the two is zero, then the stock price has no jump component and the VS rate cannot embed a priced jump component. If the difference is not zero, a priced jump component is likely to be reflected in such a difference and thus in the VS rate.

In practice, of course, only a typically small number of options is available to construct the replicating portfolio for a given horizon τ. Moreover, options are available only for a few maturities that typically do not match the horizon τ. An interpolation across maturities is therefore necessary. Jiang and Tian (2005) provide a detailed discussion of these issues that likely introduce approximation errors.

Our procedure to detect the price jump component in VS rates is as follows. Model (1) implies the following risk neutral dynamic for the futures price F_t

$$d \log F_t = -\frac{1}{2} v_t \, dt + \sqrt{v_t} dW_t^Q + J_t^{s,Q} \, dN_t^Q - E_t^Q[\exp(J^s) - 1] \lambda_t^Q \, dt.$$

[6] Recently, Fuertes and Papanicolaou (2011) developed a method to extract the probability distribution of stochastic volatility from observed option prices.

The VIX contract is priced from an options portfolio that replicates a log contract[7]

$$\text{VIX}_{t,t+\tau} = -\frac{2}{\tau}E_t^Q\left[\log\frac{F_{t+\tau}}{F_t}\right] = -\frac{2}{\tau}E_t^Q\int_t^{t+\tau} d\log F_u = \overline{v}^Q_{t,t+\tau} + 2E_t^Q[\exp(J^s)-1-J^s]\,\overline{\lambda}^Q_{t,t+\tau}.$$

The difference between the VS rate in (4) and $\text{VIX}_{t,t+\tau}$ is

$$\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau} = 2E_t^Q\left[\frac{(J^s)^2}{2} + J + 1 - \exp(J^s)\right]\overline{\lambda}^Q_{t,t+\tau} \tag{5}$$

which provides a model-free assessment of the price jump term. Thus, up to a discretization error, $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is a model-free measure of a price jump component in VS rates. If the jump component is zero, i.e., $J^s = 0$ and/or the intensity $\lambda_t^Q = 0$, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is zero as well, and the VIX index is indeed a VS rate. If the jump component is not zero, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is expected to be positive. The reason is that the function in the square brackets in Equation (5) is downward sloping and passing through the origin. If the jump distribution under Q is mainly concentrated on negative values, suggesting that jump risk is priced, the expectation in Equation (5) tends to be positive. The average risk neutral jump intensity $\overline{\lambda}^Q_{t,t+\tau}$ is, of course, always nonnegative. Note that if the price jump risk is not priced, i.e., the jump size distributions under P and Q are the same, the difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ could be nonzero, depending on the expectation in Equation (5).

Following the revised post-2003 VIX methodology, we calculate daily VIX-type indices, $\text{VIX}_{t,t+\tau}$, for $\tau = 2$, 3, and 6 months to maturity from January 4, 1996 to September 2, 2010 and compute the difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$. SPX option prices are obtained from OptionMetrics. Although it is straightforward to calculate VIX-type indices for longer maturities, the interpolation of existing maturities straddling 12 and 24 months is likely to introduce significant approximation errors. Table 1, Panel B, shows summary statistics of calculated VIX-type indices. These indices have the same term structure features as VS rates, qualitatively. However, on average, VS rates are higher, more volatile, skewed, and leptokurtic than VIX-type indices for each maturity. Moreover, the difference $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ increases with time to maturity. Figure 2 shows time series plots of $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ for the various times to maturity.

[7] The identity

$$\frac{F_{t+\tau}}{F_t} - 1 - \log\frac{F_{t+\tau}}{F_t} = \int_0^{F_t}\frac{(K-F_{t+\tau})^+}{K^2}dK + \int_{F_t}^{\infty}\frac{(F_{t+\tau}-K)^+}{K^2}dK$$

leads to computing the VIX index using forward prices of the out-of-the-money put and call options on the S&P500 index with maturity $t+\tau$. The VIX index is based on a calendar day counting convention and linear interpolation of options whose maturities straddle 30 days (e.g., Carr and Wu (2006) provide a description of the VIX calculation.)

Such differences are mostly positive, statistically significant, larger during market turmoils but sizeable also in quiet times. In volatility units, they easily exceed 2% suggesting that they are economically important when compared to an average volatility level of about 20%. A positive difference is not a crisis-only phenomenon, when jumps in stock price are more likely to occur and investors may care more about jump risk. These findings are consistent with the presence of a significant priced jump component embedded in VS rates.

A few reasons are conceivable for a non-zero difference of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. The first reason is that, since European options on the S&P500 index (SPX) are likely to be more liquid than VS contracts, a larger liquidity risk premium could be embedded in VS rates than in SPX options. Everything else equal, the higher the illiquidity of VS the higher the return of a long position in VS should be, reflecting a liquidity risk premium. However, this would imply that the higher the liquidity risk premium, the lower the VS rate. Thus, if anything, liquidity issues should bias downward, an otherwise larger and positive difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$.

A second reason for the non-zero difference in (5) could be that the SPX and VS are segmented or disconnected markets. In that case, comparing asset prices from the two markets could easily generate large gaps between $\mathrm{VS}_{t,t+\tau}$ and $\mathrm{VIX}_{t,t+\tau}$. On one hand, there is anecdotal evidence that VS contracts are typically hedged with SPX options and vice versa.[8] Thus, it is unlikely that the two markets are completely segmented. On the other hand, Bardgett et al. (2013) provide evidence that VIX derivatives and SPX options carry different information about volatility dynamics, which could be interpreted as a form of segmentation between volatility and option markets. A temporary disconnection between the two markets could explain the negative difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ observed on a few occasions in Fall 2008. For example if the SPX market reacts more quickly than the VS market to negative news, option prices increase faster than VS rates, inducing a negative difference.

A third reason for the non-zero difference in (5) could be that VS sellers price heavily jump risk, and VS buyers are ready to pay such high premiums. Indeed, the trading strategy of taking at day t a short position in a VS and a long, static position in SPX options generates a random payoff at day $t + \tau$, which includes a fixed cash flow given by $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. If the difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ is positive, it is then cashed by the trader shorting VS and hedging the position with SPX options, and can be interpreted as compensation for the imperfect hedging due to jumps in the underlying asset.

[8] The difficulties involved in carrying out such hedging strategies became prominent in October 2008 when volatility reached historically high values (see Schultes (2008).)

While an average positive difference in (5) is economically sensible, the remaining question is whether quantitatively the difference documented in Table 1 is economically "fair." To tackle this issue, we computed the difference in (5) using the general stochastic volatility Model (6)–(7) introduced in the next section and fitted to VS rates and S&P500 returns. Although the model produces a positive and time-varying difference, it cannot match the observed large time-variation of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. Therefore, based on this metric, the positive difference appears to be excessively high, hinting to some segmentation between the VS and SPX markets.

The CBOE methodology to select options for the VIX calculation is to include all out-of-the-money options, far in the moneyness range, until two consecutive zero bid prices are found. The rationale is to exclude illiquid options from the VIX calculation. Unfortunately, this procedure implies that the actual number of options used in the VIX calculation can change substantially from one day to the next, for example if options with zero bid price are suddenly traded and deeper out-of-the-money options had non-zero bid prices. This may produce some instabilities in the calculated VIX-type indices.[9]

As a robustness check of the findings above, we also calculated the VIX-type indices using the Carr and Wu (2009) methodology.[10] Table 1, Panel C, shows that the corresponding VIX-type indices are on average rather constant across maturities and closer to the VS rates than VIX-type indices based on the CBOE methodology. VIX-type indices based on the Carr–Wu methodology are still less volatile and somewhat smaller than VS rates for the 6-month time to maturity (and even more so for the unreported 12-month time to maturity). The corresponding time series of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$, for $\tau = 2$, 3, 6 months, are similar to the trajectories shown in Figure 2 and thus exhibit a significant time variation. This suggests that when the VIX-type indices are calculated more accurately the jump risk premium embedded in VS rates appears to be smaller. In other words, based on the Carr–Wu methodology, the VS market appears

[9]Andersen et al. (2012) argue that the CBOE rule for selecting liquid options induces significant instabilities in the intraday calculation of the VIX index, especially during periods of market turmoil, when an accurate assessment of volatility risk is most needed. We use the CBOE methodology to compute VIX-type indices on a daily basis. These instabilities should be less severe than on an intraday basis.

[10]The Carr–Wu methodology is as follows. For a given day t and time to maturity τ, implied volatilities at different moneyness levels are linearly interpolated to obtain 2,000 implied volatility points. The strike range is ± 8 standard deviations from the current stock price. The standard deviation is approximated by the average implied volatility. For moneyness below (above) the lowest (highest) available moneyness level in the market, the implied volatility at the lowest (highest) strike price is used. Given the interpolated implied volatilities, the forward price at day t of out-of-the-money options with different strikes K and time to maturity τ, $O_t(K,\tau)$, are computed using the Black–Scholes formula. The VIX-type index is then given by a discretization of $2/\tau \int_0^\infty O_t(K,\tau)/K^2\,dK$. This procedure is repeated for each day t in our sample and for the two time to maturities available in the market, say $\underline{\tau}$ and $\overline{\tau}$, straddling the time to maturity τ (which may not be available in the market), i.e., $\underline{\tau} \leq \tau \leq \overline{\tau}$, where $\tau = 2$, 3, 6 months. Finally, the linear interpolation across time to maturities of $2/\underline{\tau} \int_0^\infty O_t(K,\underline{\tau})/K^2\,dK$ and $2/\overline{\tau} \int_0^\infty O_t(K,\overline{\tau})/K^2\,dK$ gives the (squared) VIX-type index for the time to maturity τ.

to set VS rates at levels which are roughly in line with option market's expectations of future quadratic variations, at least over short time horizons. However, there is an important difference between the CBOE and Carr–Wu methodologies, namely that only the former is associated to the trading strategy of shorting variance swaps and hedging this position with SPX options. Given the available SPX options, the compensation for jump risk premium embedded in VS appears to be substantial.

2.3. A Parametric Stochastic Volatility Model

The limitations of the data available make it necessary to adopt a parametric structure, with a specification informed by the model-free analysis above, in order to go further. So we now parameterize the Model (1). Given the data analysis above, as well as the evidence in Gatheral (2008) and Egloff et al. (2010) that two factors are both necessary and sufficient to accurately capture the dynamics of the VS rates, we adopt under the objective probability measure P, the following model for the ex-dividend stock price and its variance:

$$
\begin{aligned}
dS_t/S_{t_-} &= \mu_t\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \rho\sqrt{v_t}\,dW_{2t}^P + (\exp(J_t^{s,P}) - 1)\,dN_t - \nu_t^P\,dt \\
dv_t &= k_v^P(m_t\,k_v^Q/k_v^P - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^P + J_t^{v,P}\,dN_t \\
dm_t &= k_m^P(\theta_m^P - m_t)\,dt + \sigma_m\sqrt{m_t}\,dW_{3t}^P
\end{aligned}
\tag{6}
$$

where $\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t$, r is the risk free rate and δ the dividend yield, both taken to be constant for simplicity only. The instantaneous correlation between stock returns and spot variance changes, ρ, captures the so-called leverage effect. The base Brownian increments, $dW_{it}^P, i = 1,2,3$, are uncorrelated.[11]

The random price jump size, $J_t^{s,P}$, is independent of the filtration generated by the Brownian motions and jump process, and normally distributed with mean μ_j^P and variance σ_j^2. Hence, $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$ is the Laplace transform of the random jump size. Similarly, $g^Q = \exp(\mu_j^Q + \sigma_j^2/2) - 1$. The jump intensity is the same under the P and Q measures and it is given by $\lambda_t = \lambda_0 + \lambda_1 v_t$, where λ_0 and λ_1 are positive constants. This specification allows for more jumps to occur during more volatile periods, with the intensity bounded away from 0 by λ_0. Bates (2006) provides time series evidence that the jump intensity is stochastic. Besides the empirical evidence on jumps in stock returns, the main motivation for introducing such a jump component in stock returns is to account for the jump component in VS rates, as suggested by

[11]Under this model specification, $d\tilde{W}_t^P$ in Model (1) becomes $\sqrt{(1-\rho^2)}\,dW_{1t}^P + \rho\,dW_{2t}^P$ in Model (6).

our model-free analysis in Section 2.2.

The spot variance, v_t, follows a two-factor model where m_t controls its stochastic long-run mean or central tendency. The speed of mean reversion is k_v^P under P, k_v^Q under Q and $k_v^P = k_v^Q - \gamma_2 \sigma_v$, where γ_2 is the market price of risk for W_{2t}^P; Section 2.4 discusses the last equality. The process m_t controlling the stochastic long run mean follows its own stochastic mean reverting process and mean reverts to a positive constant θ_m^P, when the speed of mean reversion k_m^P is positive. Typically, v_t is fast mean reverting and volatile to capture sudden movements in volatility, while m_t is more persistent and less volatile to capture long term movements in volatility. Andersen et al. (2002), Alizadeh et al. (2002), and others, provide evidence that two factors are necessary to describe variance dynamics.[12] The square-root specification of the diffusion components, $\sigma_v \sqrt{v_t}$ and $\sigma_m \sqrt{m_t}$, is adopted to keep Model (6) close to commonly used models, e.g., Chernov and Ghysels (2000), Pan (2002), Broadie et al. (2007, 2009), Egloff et al. (2010), and Todorov (2010).

The random jump size of the spot variance, $J_t^{v,P}$, is independent of W_t^P and $J_t^{s,P}$, and exponentially distributed with parameter μ_v^P, i.e., $E[J_t^{v,P}] = \mu_v^P$, ensuring that v_t stays positive. Thus, the variance jump $J_t^{v,P}$ captures quick upward movements of v_t. The Model (6) features contemporaneous jumps both in returns and variance, that is the double-jump model introduced by Duffie et al. (2000). Eraker et al. (2003) fit models with contemporaneous and independent jumps in returns and variance to S&P500 data. They find that the two models perform similarly, but the model with contemporaneous jumps is estimated more precisely. Eraker (2004), Broadie et al. (2007), Chernov et al. (2003), and Todorov (2010) provide further evidence for contemporaneous jumps in returns and variance.

Model (6) covers existing stochastic volatility models along most dimensions. For example, none of the studies cited above allow at the same time for stochastic long run mean, stochastic jump intensity and jumps in returns and variance. Bakshi et al. (1997), Bates (2000, 2006), Pan (2002), Eraker et al. (2003), Eraker (2004), Broadie et al. (2007, 2009) set m_t to a constant, positive value. Almost all studies assume either constant jump intensities (e.g., Eraker et al. (2003) and Broadie et al. (2007)) or jumps in returns but not in variance (e.g., Pan (2002) and Broadie et al. (2009)).

[12] Using alternative approaches, Adrian and Rosenberg (2008), Engle and Rangel (2008), Christoffersen et al. (2009) and Corradi et al. (2013) provide additional evidence for a two-factor volatility structure.

Under Q, the ex-dividend price process evolves as

$$
\begin{aligned}
dS_t/S_{t_-} &= (r-\delta)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^Q + \rho\sqrt{v_t}\,dW_{2t}^Q + (\exp(J_t^{s,Q})-1)\,dN_t - \nu_t^Q\,dt \\
dv_t &= k_v^Q(m_t - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^Q + J_t^{v,Q}\,dN_t \\
dm_t &= k_m^Q(\theta_m^Q - m_t)\,dt + \sigma_m\sqrt{m_t}\,dW_{3t}^Q
\end{aligned} \tag{7}
$$

where the Brownian motions $W_i^Q, i = 1,2,3$, price jump size $J^{s,Q}$, counting jump process N, its compensator ν^Q, and variance jump size $J^{v,Q}$ are governed by the measure Q.

Given the stochastic volatility model above, the VS rate is available in closed form. We first calculate $\overline{v}_{t,t+\tau}^Q$ in Equation (4). Interchanging expectation and integration (justified by Tonelli's theorem)

$$
\overline{v}_{t,t+\tau}^Q = \frac{1}{\tau}\int_t^{t+\tau} E_t^Q[v_u]\,du = (1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_v^Q(\tau)v_t + \phi_m^Q(\tau)\tilde{m}_t \tag{8}
$$

where $\tilde{m}_t = (k_v^Q m_t + \mu_v^Q\lambda_0)/\tilde{k}_v^Q$, $\tilde{k}_v^Q = k_v^Q - \mu_v^Q\lambda_1$, and

$$
\begin{aligned}
\phi_v^Q(\tau) &= \left(1-\exp(-\tilde{k}_v^Q\tau)\right)/(\tilde{k}_v^Q\tau) \\
\phi_m^Q(\tau) &= \left(1+\exp(-\tilde{k}_v^Q\tau)k_m^Q/(\tilde{k}_v^Q-k_m^Q) - \exp(-k_m^Q\tau)\tilde{k}_v^Q/(\tilde{k}_v^Q-k_m^Q)\right)/(k_m^Q\tau).
\end{aligned}
$$

Equation (8) is obtained using the risk neutral jump-compensated dynamic of v_t.[13] Finally, using independence among $J^{s,Q}$, $J^{v,Q}$ and N

$$
\mathrm{VS}_{t,t+\tau} = \overline{v}_{t,t+\tau}^Q + E_t^Q[(J^s)^2]\,\overline{\lambda}_{t,t+\tau}^Q \tag{9}
$$

where $E_t^Q[(J^s)^2] = E^Q[(J^s)^2] = (\mu_j^Q)^2 + \sigma_j^2$, as the return jump size is time-homogeneous, and $\overline{\lambda}_{t,t+\tau}^Q = \lambda_0 + \lambda_1\overline{v}_{t,t+\tau}^Q$. Note that if the variance jump component was absent, i.e., $J_t^{v,Q} = 0$, then $\mu_v^Q = 0$ and $\overline{v}_{t,t+\tau}^Q$ had the same analytical expression as in (8) with $\tilde{m}_t = m_t$ and $\tilde{k}_v^Q = k_v^Q$.

Given the linearity of the variance swap payoff in the spot variance, only the drift of v_t

[13]The risk neutral jump-compensated dynamic is $dv_t = k_v^Q(m_t - v_t)\,dt + \mu_v^Q(\lambda_0 + \lambda_1\,v_t)dt + dM_t^Q$, where the martingale increment $dM_t^Q = \sigma_v\sqrt{v_t}\,dW_{2t}^Q + J_t^{v,Q}\,dN_t - \mu_v^Q(\lambda_0+\lambda_1\,v_t)dt$. Rewriting the dynamic as $dv_t = \tilde{k}_v^Q(\tilde{m}_t - v_t)\,dt + dM_t^Q$ gives the expressions for $\tilde{k}_v^Q$ and $\tilde{m}_t$. Applying Itô's Lemma to $e^{\tilde{k}_v^Q\,t}v_t$, integrating between time t and s, and rearranging terms, as usual, give

$$
v_s = v_t e^{-\tilde{k}_v^Q(s-t)} + \int_t^s e^{-\tilde{k}_v^Q(s-u)}\tilde{k}_v^Q\tilde{m}_u\,du + \int_t^s e^{-\tilde{k}_v^Q(s-u)}dM_u^Q.
$$

Taking E_t^Q, the last term above vanishes. The expectation $E_t^Q[\tilde{m}_u]$ can be computed following similar steps. Calculating all integrals gives Equation (8).

enters the variance swap rate. The martingale part of v_t (diffusion and jump compensated parts) affects only the dynamic of $\mathrm{VS}_{t,t+\tau}$. The Q-expectation of the stochastic jump intensity provides a time-varying contribution to $\mathrm{VS}_{t,t+\tau}$, given by $\overline{\lambda}^Q_{t,t+\tau}$, which depends on the time to maturity of the contract.

According to our model estimates in Section 4, when $\tau \to 0$, $\phi_v^Q(\tau) \to 1$ and $\phi_m^Q(\tau) \to 0$. Thus, short maturities VS rates are mainly determined by v_t. In contrast, when $\tau \to \infty$, $\phi_v^Q(\tau) \to 0$ and $\phi_m^Q(\tau) \to 0$, and long maturities VS rates are mainly determined by θ_m^Q. As $\phi_m^Q(\tau)$ is slower than $\phi_v^Q(\tau)$ in approaching zero when $\tau \to \infty$, m_t has also a relatively large impact on long maturity VS rates.

The two-factor model for the spot variance is key to reproduce the variety of shapes of VS term structures described in Section 2.1. Egloff et al. (2010) also made this observation. In Equation (8), the right hand side is a weighted average of θ_m^Q, v_t and $\tilde{m}_t$. The relative level of the three components controls the shape of the term structure. For example, the term structure is monotonically increasing in τ when $v_t < \tilde{m}_t = \theta_m^Q$, or it is hump-shape when $v_t < \tilde{m}_t$ and $\tilde{m}_t > \theta_m^Q$. Moreover, the two-factor model can produce level, persistency, volatility and higher order moments of VS rates which are broadly consistent with the observed empirical features. For example, as v_t is less persistent, more volatile and positively skewed than m_t, according to our estimates, the shorter the time to maturity, the more the VS rates inherit such properties, as observed empirically. Indeed, Section 4 shows that Model (6)–(7) matches such features quite well. As shown in Section 2.1, two principal components virtually explain all the variation in VS rates. Thus, PCA supports the two-factor model as well. All in all, Model (6)–(7) appears to be a parsimonious parametric model consistent with the model-free analysis of actual VS rates.

2.4. *Market Prices of Risk*

As in Pan (2002), Aït-Sahalia and Kimmel (2010), and others, we specify the market price of risks for the Brownian motions as

$$\Lambda_t' = [\gamma_1 \sqrt{(1-\rho^2)v_t}, \quad \gamma_2 \sqrt{v_t}, \quad \gamma_3 \sqrt{m_t}] \tag{10}$$

where $'$ denotes transposition. Thus, P and Q parameters controlling v_t and m_t are related as follows

$$k_v^P = k_v^Q - \gamma_2 \sigma_v, \quad k_m^P = k_m^Q - \gamma_3 \sigma_m, \quad \theta_m^P = \theta_m^Q \, k_m^Q / k_m^P.$$

More flexible specifications of the market price of risks for the Brownian motions have been suggested (e.g., Cheridito et al. (2007).) In the present application, there does not appear to be a strong need for an extension of (10), given the tradeoffs between the benefits of a more richly parameterized model and the costs involved in its estimation and out-of-sample performance.

The price jump size risk premium is $(g^P - g^Q) = \exp(\mu_j^P + \sigma_j^2/2) - \exp(\mu_j^Q + \sigma_j^2/2)$. The variance of the price jump size is the same under P and Q, implying that the jump distribution has the same shape but potentially different location under P and Q. As, e.g., in Pan (2002), Eraker (2004), and Broadie et al. (2007), we assume that the jump intensity is the same under both measures. The main motivation for this assumption is the well-known limited ability to estimate jump components in stock returns and the corresponding risk premium using daily data. Thus, all price jump risk premium is absorbed by the price jump size risk premium, $(g^P - g^Q)$. The total price jump risk premium is time-varying and given by $(g^P - g^Q)(\lambda_0 + \lambda_1 v_t)$. Similarly, the variance jump size risk premium is $(\mu_v^P - \mu_v^Q)$, and the total variance jump premium is $(\mu_v^P - \mu_v^Q)(\lambda_0 + \lambda_1 v_t)$.

The jump component makes the market incomplete with respect to the risk free bank account, the stock and any finite number of derivatives. Hence, the state price density is not unique. The specification we adopt is

$$\begin{aligned}\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} &= \exp\left(-\int_0^t \Lambda_s' \, dW_s^P - \frac{1}{2}\int_0^t \Lambda_s'\Lambda_s \, ds\right) \\ &\quad \prod_{u=1}^{N_t} \exp\left(\frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2} J_u^{s,P} + \frac{\mu_v^Q - \mu_v^P}{\mu_v^P \mu_v^Q} J_u^{v,P}\right). \qquad (11)\end{aligned}$$

Appendix A shows that Equation (11) is a valid state price density. The first exponential function is the usual Girsanov change of measure of the Brownian motions. The remaining part is the change of measure for the jump component in the stock price and variance. Equation (11) shows that, in the economy described by this model, price and variance jumps are priced because when a jump occurs the state price density jumps as well. Bad states of the economy, in which marginal utility is high, can be reached when a negative price jump and/or a positive variance jump occur. When the risk neutral mean of the price jump size is lower than the objective mean, i.e., $\mu_j^Q < \mu_j^P$, and a negative price jump occurs ($J^{s,P} < 0$), the state price density jumps up giving high prices to (Arrow–Debreu) securities with positive payoffs in these bad states of the economy, namely when the stock price falls. Similarly, when the risk neutral mean of the variance jump size is larger than the objective mean, i.e., $\mu_v^Q > \mu_v^P$, and a positive variance jump occurs ($J^{v,P} > 0$), the state price density jumps up in these bad states of the economy, namely when volatility is high. In our empirical estimates, we do find that $\mu_j^Q < \mu_j^P$ and $\mu_v^Q > \mu_v^P$.

3. Likelihood-Based Estimation Method

Model (6)–(7) is estimated using the general approach in Aït-Sahalia (2002, 2008). The procedure we employ then combines time series information on the S&P500 returns and cross sectional information on the term structures of VS rates in the same spirit as in other derivative pricing contexts, e.g., Chernov and Ghysels (2000) and Pan (2002). Hence, P and Q parameters, including risk premia, are estimated jointly by exploiting the internal consistency of the model, thereby making the inference procedure theoretically sound.

Let $X_t' = [\log(S_t), Y_t']$ denote the state vector, where $Y_t = [v_t, m_t]'$. The spot variance and its stochastic long run mean, collected in Y_t, are not observed and will be extracted from actual VS rates. Likelihood-based estimation requires evaluation of the likelihood function of index returns and term structures of variance swap rates for each parameter vector during a likelihood search. The procedure for evaluating the likelihood function consists of four steps. First, we extract the unobserved state vector Y_t from a set of benchmark variance swap rates, assumed to be observed without error.[14] Second, we evaluate the joint likelihood of the stock returns and extracted time series of latent states, using an approximation to the likelihood function. Third, we multiply this joint likelihood by a Jacobian determinant to compute the likelihood of observed data, namely index returns and term structures of VS rates. Finally, for the remaining VS rates assumed to be observed with error, we calculate the likelihood of the observation errors induced by the previously extracted state variables. The product of the two likelihoods gives the joint likelihood of the term structures of all variance swap rates and index returns. We then maximize the joint likelihood over the parameter vector to produce the estimator.

3.1. *Extracting State Variables from Variance Swap Rates*

Model (6)–(7) implies that the VS rates are affine in the unobserved state variables. This feature suggests a natural procedure to extract latent states and motivates our likelihood-based approach.

[14]This assumption makes the filtering of the latent variables Y_t unnecessary and is often adopted in the term structure literature, e.g., Pearson and Sun (1994) and Aït-Sahalia and Kimmel (2010). Alternatively, one could assume that all VS rates are observed with errors, which would require filtering of the latent variables Y_t, as, e.g., in Eraker (2004) and Wu (2011). The latter approach is more computationally intensive and not pursued here.

The unobserved part in the state vector, Y_t, is ℓ dimensional, where $\ell = 2$ in Model (6)–(7). As the method can be applied for $\ell \geq 1$, we describe the procedure for a generic ℓ. At each day t, ℓ variance swap rates are observed without error, with times to maturities $\tau_1, \ldots, \tau_\ell$. The state vector Y_t is exactly identified by the ℓ variance swap rates, $\text{VS}_{t,t+\tau_1}, \ldots, \text{VS}_{t,t+\tau_\ell}$. These VS rates jointly follow a Markov process and satisfy

$$\begin{bmatrix} \text{VS}_{t,t+\tau_1} \\ \vdots \\ \text{VS}_{t,t+\tau_\ell} \end{bmatrix} = \begin{bmatrix} a(\tau_1;\Theta) \\ \vdots \\ a(\tau_\ell;\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_1;\Theta)' \\ \vdots \\ b(\tau_\ell;\Theta)' \end{bmatrix} Y_t \tag{12}$$

where Θ denotes the model parameters. Rearranging Equation (9) gives $\text{VS}_{t,t+\tau} = a(\tau;\Theta) + b(\tau;\Theta)'[v_t, m_t]'$, where

$$\begin{aligned} a(\tau;\Theta) &= E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2])\left((1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_m^Q(\tau)\mu_v^Q\lambda_0/\tilde{k}_v^Q\right) \\ b(\tau;\Theta)' &= (1+\lambda_1 E^Q[J^2])\,[\phi_v^Q(\tau), \quad \phi_m^Q(\tau)k_v^Q/\tilde{k}_v^Q]. \end{aligned}$$

Equation (12) in vector form reads $\text{VS}_{t,\cdot} = a(\Theta)+b(\Theta)Y_t$, with obvious notation. The current value of the unobserved state vector Y_t can easily be found by solving the equation for Y_t, i.e., $Y_t = b(\Theta)^{-1}[\text{VS}_{t,\cdot} - a(\Theta)]$. The affine relation between VS rates and latent variables makes recovering the latter numerically costless, especially compared to recovering latent variables from standard call and put options as, for e.g., in Pan (2002).

3.2. Likelihood of Stock Returns and Variance Swap Rates Observed Without Error

The extracted time series values of the unobserved state vector Y_t at dates $t_0, t_1, \ldots, t_n$ allows to infer the dynamics of the state variables $X_t' = [\log(S_t), Y_t']$ under the objective probability P. Since the relationship between the unobserved state vector Y_t and variance swap rates is affine, the transition density of variance swap rates can be derived from the transition density of Y_t by a change of variables and multiplication by a Jacobian determinant which depends, in this setting, on model parameters but not on the state vector.

Let $p_X(x_\Delta|x_0;\Theta)$ denote the transition density of the state vector X_t under the measure P, i.e., the conditional density of $X_{t+\Delta} = x_\Delta$, given $X_t = x_0$. Let $A_t = [\log(S_t), \text{VS}_{t,t+\tau_1}, \ldots, \text{VS}_{t,t+\tau_\ell}]'$ be the vector of observed asset prices and $p_A(a_\Delta|a_0;\Theta)$ the corresponding transition density.

Observed asset prices, A_t, are given by an affine transformation of X_t

$$A_t = \begin{bmatrix} \log(S_t) \\ \mathrm{VS}_{t,\cdot} \end{bmatrix} = \begin{bmatrix} \log(S_t) \\ a(\Theta) + b(\Theta) Y_t \end{bmatrix} = \begin{bmatrix} 0 \\ a(\Theta) \end{bmatrix} + \begin{bmatrix} 1 & 0' \\ 0 & b(\Theta) \end{bmatrix} X_t$$

and rewritten in matrix form reads $A_t = \tilde{a}(\Theta) + \tilde{b}(\Theta) X_t$, with obvious notation. The Jacobian term of the transformation from X_t to A_t is therefore

$$\det \left| \frac{\partial A_t}{\partial X_t'} \right| = \det \left| \tilde{b}(\Theta) \right| = \det \left| b(\Theta) \right| .$$

In Model (6)–(7), $\det |b(\Theta)| = |(1 + \lambda_1 E^Q[J^2])^2 \left(\phi_v^Q(\tau_1) \phi_m^Q(\tau_2) - \phi_v^Q(\tau_2) \phi_m^Q(\tau_1) \right) k_v^Q / \tilde{k}_v^Q|$. Since $X_t = \tilde{b}(\Theta)^{-1}[A_t - \tilde{a}(\Theta)]$,

$$p_A(A_\Delta | A_0; \Theta) = \det \left| b(\Theta)^{-1} \right| \; p_X(\tilde{b}(\Theta)^{-1}[A_\Delta - \tilde{a}(\Theta)] | \tilde{b}(\Theta)^{-1}[A_0 - \tilde{a}(\Theta)]; \Theta). \tag{13}$$

As the vector of asset prices is Markovian, applying Bayes' Rule, the log-likelihood function of the asset price vector A_t sampled at dates $t_0, t_1, \ldots, t_n$ has the simple form

$$l_n(\Theta) = \sum_{i=1}^{n} l_A(A_{t_i} | A_{t_{i-1}}; \Theta) \tag{14}$$

where $l_A = \ln p_A$. As usual in likelihood estimation, we discard the unconditional distribution of the first observation since it is asymptotically irrelevant.

In our applications below, models are estimated using daily data, hence the sampling process is deterministic and $t_i - t_{i-1} = \Delta = 1/252$; see Aït-Sahalia and Mykland (2003) for a treatment of maximum likelihood estimation in the case of randomly spaced sampling times.

3.3. Likelihood of Stock Returns and All Variance Swap Rates

From the coefficients $a(\tau;\Theta)$ and $b(\tau;\Theta)$ and the values of the state vector X_t found in the first step, we can calculate the implied values of the variance swap rates which are assumed to be observed with error and whose time to maturities are denoted by $\tau_{\ell+1}, \ldots, \tau_{\ell+h}$

$$\begin{bmatrix} \mathrm{VS}_{t,t+\tau_{\ell+1}} \\ \vdots \\ \mathrm{VS}_{t,t+\tau_{\ell+h}} \end{bmatrix} = \begin{bmatrix} a(\tau_{\ell+1};\Theta) \\ \vdots \\ a(\tau_{\ell+h};\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_{\ell+1};\Theta)' \\ \vdots \\ b(\tau_{\ell+h};\Theta)' \end{bmatrix} Y_t.$$

The observation errors, denoted by $\varepsilon(t, \tau_{\ell+i})$, $i = 1, \ldots, h$, are the differences between such model-based implied VS rates and actual VS rates from the data. By assumption, these errors are Gaussian with zero mean and constant variance, independent of the state process and across time, but possibly correlated across maturities. The joint likelihood of the observation errors can be calculated from the h dimensional Gaussian density function. Since the observation errors are independent of the state variable process, the joint likelihood of stock returns and all observed variance swap rates is simply the product of the likelihood of stock returns and variance swap rates observed without error, multiplied by the likelihood of the observation errors. Equivalently, the two log-likelihoods can simply be added to obtain the joint log-likelihood of stock returns and all variance swap rates.

3.4. Likelihood Approximation

Since the state vector X is a continuous-time multivariate jump diffusion process, its transition density is unknown. Since jumps are by nature rare events in a model with finite jump activity, it is unlikely that more than one jump occurs on a single day Δ. This observation motivates the following Bayes' approximation of p_X

$$p_X(x_\Delta|x_0) = p_X(x_\Delta|x_0, N_\Delta = 0)\,\Pr(N_\Delta = 0) + p_X(x_\Delta|x_0, N_\Delta = 1)\,\Pr(N_\Delta = 1) + o(\Delta)$$

where $\Pr(N_\Delta = j)$ is the probability that j jumps occur at day Δ, omitting the dependence on the parameter Θ for brevity. An extension of the method due to Yu (2007) for jump-diffusion models can provide higher order terms if necessary.

In Model (6)–(7), the largest contribution to the transition density of X (hence to the likelihood) comes from the conditional density that no jump occurs at day Δ. The reason is that the probability of such an event, $\Pr(N_\Delta = 0)$, is typically large and of the order $1-(\lambda_0+\lambda_1 v_0)\,\Delta$. The contribution of the second term is only of the order $(\lambda_0 + \lambda_1 v_0)\,\Delta$. As Δ is one day in our setting, the contribution of higher order terms appears to be quite modest. The main advantage of this approximation is that the leading term, $p_X(x_\Delta|x_0, N_\Delta = 0)$, can be accurately computed using the likelihood expansion method. The expansion for the transition density of X conditioning on no jump has the form of a Taylor series in Δ at order K, with each coefficient $C^{(k)}$ in a Taylor series in $(x - x_0)$ at order $j_k = 2(K - k)$. Denoting $C^{(j_k,k)}$ such expansions, the transition density expansion is

$$\tilde{p}^{(K)}(x|x_0;\Theta) = \Delta^{-(\ell+1)/2} \exp\left[-\frac{C^{(j_{-1},-1)}(x|x_0;\Theta)}{\Delta}\right] \sum_{k=0}^{K} C^{(j_k,k)}(x|x_0;\theta)\frac{\Delta^k}{k!}. \tag{15}$$

Coefficients $C^{(j_k,k)}$ are computed by forcing the Equation (15) to satisfy, to order Δ^K, the forward and backward Kolmogorov equations. A key feature of the method is that the coefficients are obtained in closed form by solving a system of linear equations. This holds true for arbitrary specifications of the dynamics of the state vector X. Moreover, the coefficients need to be computed only once and not at each iteration of the likelihood search. Equation (15) provides a very accurate approximation of the transition density of X already when $K = 2$; e.g., Jensen and Poulsen (2002). In our empirical application below, we use expansions at order $K = 2$.

4. Fitting Variance Swap Rates

4.1. In-Sample Estimation

Table 2 reports parameter estimates for Model (6)–(7), based on the in-sample period January 4, 1996 to April 2, 2007. The spot variance is relatively fast mean reverting as k_v^P implies a half-life[15] of 33 days. Its stochastic long run mean is slowly mean reverting with a half-life of about 1.5 years. The instantaneous volatility of v_t is about twice that of m_t. The correlation between stock returns and variance changes, ρ, is -69%, confirming the so-called leverage effect. The long-run average volatility, $\sqrt{\theta_m^P}$, is 20%, in line with the summary statistics in Table 1. Both γ_2 and γ_3 are negative, implying negative instantaneous variance risk premia. The correlation parameter for the VS pricing errors, ρ_e, is slightly negative suggesting that the model does not produce any systematic pricing error.[16]

The expected jump size is slightly negative under the objective probability measure, μ_j^P, and even more negative under the risk neutral measure, μ_j^Q, which implies a positive price jump risk premium. Estimates of jump intensity implies about 2.5 jumps per year (i.e., $\lambda_0 + \lambda_1(k_v^Q\theta_m^P + \mu_v^P\lambda_0)/(\kappa_v^P - \mu_v^P\lambda_1)$), which is roughly in line with previous estimates reported in the literature.

Table 2 also reports estimates of three nested models: (i) a two-factor model with price jumps only (labeled SV2F-PJ) obtained setting $\mu_v^P = \mu_v^Q = 0$, (ii) a two-factor model with no

[15]The half-life is defined as the time necessary to halve a unit shock and is given by $-\log(0.5)/k_v^P \times 252$ in number of days.
[16]The determinant of the 3×3 error term correlation matrix is $2\rho_e^3 - 3\rho_e^2 + 1$, which is strictly positive when $\rho_e > -0.5$.

jump component (labeled SV2F) imposing the additional restriction $\lambda_0 = \lambda_1 = 0$, and (iii) the Heston model (labeled SV1F) imposing the additional restriction $m_t = \theta_m^P$ for all t. Imposing each additional restriction significantly deteriorates the fitting of VS rates and S&P500 returns, according to likelihood ratio tests. Thus, Model (6)–(7) outperforms all nested models.

4.2. Out-of-Sample Robustness Checks

We conduct all subsequent analyses using two subsamples. Data from January 4, 1996 to April 2, 2007 are used for in-sample analysis, as Model (6)–(7) is estimated using these data. The remaining sample data, from April 3, 2007 to September 2, 2010, are used for out-of-sample analysis and robustness checks. Such out-of-sample analysis appears to be particularly interesting because this subsample covers the financial crisis of Fall 2008, a period of unprecedented market turmoil, which was not experienced in the prior fitting sample.

Table 3 shows the pricing errors of Model (6)–(7) when fitting VS rates, for the in- and out-of-sample periods. Pricing errors of the Heston model are also reported for comparison.[17] The pricing error is defined as the model-based VS rate minus the observed VS rate. Model (6)–(7) fits VS rates well both in- and out-of-sample and significantly outperforms the Heston model. For example, its root mean square error is 6 times smaller than that of the Heston model when fitting 24-month to maturity VS rates. The small pricing errors imply that Model (6)–(7) captures well the empirical features of VS rates documented in Table 1.

In terms of likelihood, Model (6)–(7) significantly outperforms the two-factor price-jump model, obtained by setting $\mu_v^P = \mu_v^Q = 0$ in Model (6)–(7). However, unreported pricing errors of the restricted model, especially out-of-sample, are only slightly larger than those of the general model. This suggests that variance jumps are less important than the two-factor variance structure to price variance swaps. Several studies have shown that variance (and price) jumps are important for pricing short term, out-of-the-money put options. Compared to options data, available variance swap rates are long term (the shortest maturity is already 2 months) and have no strike price dimension (there is a single variance swap rate for each maturity). These features of variance swap rates may explain why variance jumps do not appear to be so important for fitting these data. The main difference between Model (6)–(7) and the restricted model (with $\mu_v^P = \mu_v^Q = 0$) is that the former induces lower estimates of the jump intensity, which are more in line with previous studies. This difference may be explained by the more

[17]Pricing errors of other nested models are in most cases somewhere between the pricing errors of the Heston model and Model (6)–(7), and are not reported.

flexible dynamic of v_t in the general model.

Below, we explore the ability of the model, fitted in-sample, to explain the in-sample realized risk premia and predict the out-of-sample risk premia.

5. Risk Premia: Equity Premium and Volatility Premium

One advantage of modeling the underlying asset returns jointly with the VS rates is that the resulting model produces estimates of risk premia for both sets of variables, including in particular estimates of the classical equity premium. We distinguish between the spot or instantaneous risk premia at each instant t and the integrated ones, defined over each horizon τ. In each case, the model provides a natural breakdown between the continuous and jump components of the respective risk premia.

What have we learned about risk premia that we did not know before? The term structure of integrated equity and variance risk premia, which is largely unexplored in the finance literature, exhibits significant time variation throughout our sample period and large swings during crisis periods. Integrated equity risk premia are countercyclical but the slope of the term structure is procyclical. This suggests that after a financial crisis investors demand a large risk premium to hold risky stocks, but the risk premium largely depends and strongly decreases with the holding horizon. Integrated variance risk premia become more negative as the horizon increases, especially during turbulent times. This suggests that, to insure against volatility risk, investors are ready not only to pay a large premium (variance swap rates are high) but also to suffer a large expected loss (variance risk premia are negative and large). Market crashes impact and propagate differently throughout the term structure of equity and variance risk premia, with the short-end being more affected, but the long-end exhibiting more persistency. Both term structures respond nearly monotonically to various economic variables, such as credit spreads, VIX index, and slopes of the interest rate term structure.

5.1. Spot Risk Premia

Model (6)–(7) features four main instantaneous or spot risk premia: A Diffusive Risk Premium (DRP), a Jump Risk Premium (JRP), a Variance Risk Premium (VRP), and a Long-run Mean Risk Premium (LRMRP) which are defined as

$$\begin{aligned} \mathrm{DRP}_t &= (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t, & \mathrm{JRP}_t &= (g^P-g^Q)(\lambda_0+\lambda_1 v_t) \\ \mathrm{VRP}_t &= \gamma_2\sigma_v v_t, & \mathrm{LRMRP}_t &= \gamma_3\sigma_m m_t. \end{aligned}$$

DRP is the remuneration for diffusive-type risk only (due to Brownian motions driving stock prices). JRP is the remuneration for the jump component in stock price. The instantaneous Equity Risk Premium (ERP) is the sum of the two, i.e., $\mathrm{ERP}_t = \mathrm{DRP}_t + \mathrm{JRP}_t$.

The mean growth rates of v_t and m_t are different under the probability measures P and Q, and such differences are given by VRP_t and LRMRP_t, respectively. As γ_2 and γ_3 are estimated to be negative (Table 2), VRP and LRMRP are both negative, and on average v_t and m_t are higher under Q than under P. The negative sign of the variance risk premium is not abnormal. The risk premium for return risk is positive, because investors require a higher rate of return as compensation for return risk. On the other hand, investors require a lower level of variance as compensation for variance risk, hence the negative variance risk premium. Risk-averse investors dislike both higher return variance, and higher variance of the return variance.

Table 4 reports the estimated risk premia. During our in-sample period, January 1996 to April 2007, the average ERP is 7%. Notably, about 1/3 of the ERP is due to the jump risk premium, which thus accounts for a large fraction of the equity risk premium. Jump prices are quite rare events (about 2.5 jumps per year), but arguably jump risk is important as it cannot be hedged with any finite number of securities. The average VRP is also substantial and around −8%, while the LRMRP is much lower and around −0.8%. During the out-of-sample period, April 2007 to September 2010, all risk premia almost doubled reflecting the unprecedented turmoil in financial markets around the Lehman Brothers' bankruptcy.

Unreported results show that VRP estimated using the Heston model is only −1.7%, but it increases to about −4% for all other nested models with reacher variance dynamics. Heston and two-factor models without jump component imply an ERP of 7%, which is roughly the sum of the DRP and JRP based on Model (6)–(7). This suggests that, in nested models without jump component, all ERP is artificially absorbed by DRP.

As discussed above, Model (6)–(7) also features a variance jump risk premium, $(\mu_v^P - \mu_v^Q)(\lambda_0 + \lambda_1 v_t)$, which is estimated to be negative but small, as estimates of μ_v^P and μ_v^Q are rather close, and hence it is not reported. This means that setting $\mu_v^P = \mu_v^Q$ as, e.g., in Eraker et al. (2003) and Eraker (2004), does not materially change estimates of risk premia based on Model (6)–(7).

5.2. *Integrated Risk Premia*

5.2.1 Integrated Equity Risk Premium

The annualized integrated Equity Risk Premium (IERP) is defined as

$$\mathrm{IERP}_{t,t+\tau} = E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau \tag{16}$$

and represents the ex-ante expected (or forward looking) excess return from buying and holding the S&P500 index from t to $t+\tau$.[18] Extensive research has been devoted to study levels and dynamics of the IERP for a single maturity (often one year, using ex-post measures of the IERP), in particular investigating the so-called equity premium puzzle. Surprisingly, much less attention has been devoted to study the term structure of the IERP, which is the focus of this section.

The IERP can be decomposed in the continuous and jump part, i.e., $\mathrm{IERP}_{t,t+\tau} = \mathrm{IERP}^c_{t,t+\tau} + \mathrm{IERP}^j_{t,t+\tau}$, where the continuous part $\mathrm{IERP}^c_{t,t+\tau}$ is the IERP when the jump component is absent, i.e., the jump intensity $\lambda_t = 0$ in the drift μ_t of Model (6), and the jump part $\mathrm{IERP}^j_{t,t+\tau} = \mathrm{IERP}_{t,t+\tau} - \mathrm{IERP}^c_{t,t+\tau}$. This decomposition allows us to quantify how the various risks contribute to the IERP and the corresponding term structure of risk premia.

An advantage of studying the term structure of IERP in a parametric model is that risk premia and their decompositions are exact. Model-free approaches typically involve options, which in turn require interpolations or moving average schemes to reduce the impact on risk premia of time-varying maturities; see Bollerslev and Todorov (2011) for a discussion of this issue.

The time-t conditional expectations in (16) can be computed using the transform analysis in Duffie et al. (2000), i.e., numerically solving a system of nonlinear ordinary differential equations derived in Appendix B. The IERP is exponentially affine in the state variables, i.e., $\mathrm{IERP}_{t,t+\tau} = \exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$. Our model estimates in Table 2 imply that $A(\tau)$, $B(\tau)$ and $C(\tau)$ are positive coefficients. Therefore, in quiet times, when the spot variance v_t and its stochastic long run mean m_t are low, IERPs are low as well. When asset prices fall and v_t and/or m_t increase, IERPs increase as well, reflecting distressed asset prices. Thus, the IERP is countercyclical.

To compute the IERP, we use the daily term structure of interest rates, downloaded from

[18]The IERP is the familiar equity risk premium. We use the wording "integrated" to distinguish it from the instantaneous equity risk premium discussed in the previous section.

OptionMetrics and linearly interpolated to match the VS time to maturities, rather than a constant interest rate as in the analysis above. Table 5 reports mean and standard deviation of the integrated equity risk premium over 2-, 6-, 12- and 24-month horizons.[19] From January 1996 to April 2007, our in-sample period, IERPs are around 6.5% and the term structure is essentially flat. From April 2007 to September 2010, our out-of-sample period, IERPs are significantly larger and about 10%, reflecting distressed asset prices around the Lehman Brothers' bankruptcy. In this period, the term structure of IERPs is downward sloping on average.

Figure 3 shows the evolution of the IERP over time, along with the S&P500 index. The entire term structure of the IERP exhibits significant variation over time, with the short-end being more volatile than the long-end. When the S&P500 steadily increased, such as in 2005–7, the 2-month IERP dropped at the lowest level, around 4%, during our sample period. The term structure was slightly upward sloping with the 24-month IERP at almost 6%. At the end of 2008 and beginning of 2009, after Lehman Brothers collapsed, the term structure of the IERP became significantly downward sloping with the 2-month IERP reaching the highest values in decades. This implies that at the peak of the crisis, investors required equity risk premia as large as 50% to invest in the S&P500 index over short horizons like 2 months, and required less than half these risk premia for investing over long horizons like 2 years.[20] When volatility is high, equity positions carry a large risk. However, this risk is expected to decrease in the long run, when volatility will revert to lower levels and the positive drift of equities will induce price recoveries on average. On November 20, 2008, the annualized 2-month IERP was as high as 54%, and between October and December 2008, was above 30% on various occasions, somehow mirroring the fall of the index. Indeed, from mid-September to mid-November 2008, the S&P500 index dropped from 1,200 to 750, loosing 37% of its value. On March 9, 2009, it reached the lowest historical value in more than a decade, at 677, and then recovered 35% of its value within the next two months. Such large swings in the S&P500 index suggest that the large model-based estimates of the IERP are quite sensible. Recently, Martin (2013) provides a model-free lower bound on the equity premium that is by construction lower than, but closely mimics, the equity risk premia depicted in Figure 3.[21]

[19] As the IERP for the 2- and 3-month horizons are rather close, the latter is not reported.

[20] It's now obvious in retrospect that Spring 2009 was a great time to go long equities, on the basis of the large equity premium at that point in time, but note that this is here an ex-ante prediction of the model (in fact, made on the basis of the in-sample data only).

[21] van Binsbergen et al. (2013) study the term structure of "equity yields," in analogy to bond yields, extracted from dividends derivatives. The term structure of forward equity yields on the S&P500 has similar dynamics as the term structure of equity risk premia depicted in Figure 3. Lettau and Wachter (2007, 2011) provide related studies on the term structure of equity returns, focusing on value and growth stocks.

Table 5 shows that the jump component, $\text{IERP}^{j}_{t,t+\tau}$, contributes significantly to the IERP and its term structure. For example, during our in-sample period, the one-year IERP is 6.3% and 2.5% is due to jump risk. Using a model-free approach, Bollerslev and Todorov (2011) also find that a large fraction of the equity risk premium, around 5% in their study, is due to (large) jump risk, for a short time horizon τ.[22]

To understand which economic factors may drive the term structure of the IERP we conduct regression analysis. We regress the IERP, for each horizon τ, on variables proxying for overall equity, option, corporate and Treasury bond market conditions, namely daily S&P500 returns, VIX index, the difference between Moody's BAA and AAA corporate bond yields (CScorp, an indicator of credit riskiness within the corporate sector), the difference between Moody's AAA corporate bond yield and 3-month Treasury securities (CSgov, an indicator of credit spread between corporate and Treasury sectors), the difference between the yields on 2-year and 3-month Treasury securities (TermS, the short term slope of the interest rate term structure), the difference between the yields on 10-year and 2-year Treasury securities (TermL, the long term slope of the interest rate term structure). Figure 4 shows the time series plots of the latter four variables.

Panel A in Table 6 summarizes the regression results. Interestingly, these variables have a monotonic (decreasing or increasing) impact on the term structure of IERP, as measured by the slope coefficients. For example, daily S&P500 returns have progressively less negative impact on the IERP as the horizon increases, with the impact becoming insignificant beyond the 6-month horizon. In other words, a negative S&P500 return does increase the IERP but propagates differently throughout the term structure of IERP, with the short-end being more sensitive than the long-end to the shock. An increase of the VIX index has progressively less positive impact on the IERP as the time horizon increases, but the impact remains statistically and economically significant also for the 2-year horizon. CScorp has a positive and decreasing impact on the IERP, amplifying the countercyclical variation of the IERP, especially in the short-end of the term structure. The slope coefficients of other variables change sign throughout the term structure of IERP, for example from positive to negative for TermL. During the market crash in Fall 2008, TermL increased (Figure 4). Consequently, the positive slope coefficients for short term IERP and negative slope coefficients for long term IERP amplified the downward slope of the IERP term structure during those turbulent times.

[22]Bollerslev and Todorov rely on intraday S&P500 data and SPX options to study the equity risk premium over a single time horizon τ, with median of 14 days.

5.2.2 Integrated Variance Risk Premium

The annualized integrated variance risk premium (IVRP) is defined as $\text{IVRP}_{t,t+\tau} = E_t^P[\text{QV}_{t,t+\tau}] - E_t^Q[\text{QV}_{t,t+\tau}]$ and represents the ex-ante expected profit to the long side of a VS contract, which is entered at time t and held till maturity $t+\tau$. In our setting, $E_t^P[\text{QV}_{t,t+\tau}]$ can be computed following similar steps as in Equations (8) and (9).

Besides the studies mentioned above, an important literature has investigated the variance risk premium, albeit almost exclusively for a single maturity. Bakshi and Kapadia (2003) provide early evidence for the variance risk premium using S&P500 and S&P100 index options, while Bakshi et al. (2003) analyze risk premiums for individual stocks. Bollerslev et al. (2009) linked the one-month variance risk premium to time-varying economic uncertainty and show empirically that this premium predicts aggregate market returns. Bekaert and Hoerova (2013) expand the evidence on the predictive power of one-month variance risk premium for stock returns. Mueller et al. (2013) study the term structure of Treasury bond variance risk premia and document a significant negative risk premium, albeit decreasing with the time horizon. Amengual (2008) studies the term structure of S&P500 variance risk premia, under the assumption that the risk premium for the jump component is zero.

Table 5 reports summary statistics of the annualized integrated variance risk premia and Figure 5 shows the dynamic over time. For example, average IVRP for 24-month maturity is -2.9% during our out-of-sample period and can be as high as -5% in variance units. These are large risk premia compared to an average spot variance of 4% in variance units. While Model (6)–(7) is flexible enough to generate positive and negative IVRP, estimated ex-ante IVRP is always negative. This confirms that investors perceive volatility increases as unfavorable events and are willing to suffer large expected losses to insure against such volatility increases.

The longer the time to maturity the higher in absolute value the annualized IVRP. Thus, the term structure of IVRP is on average downward sloping, suggesting that long-term VS contracts carry more risk premium for stochastic variance than short-term contracts. Filipović et al. (2012) find that an optimal investment in VS is to go short in long-term VS and partially hedge the exposure by going long in short-term VS and the underlying stock. Shorting long-term variance swaps allows to earn the large risk premium embedded in such contracts.[23]

As for the IERP, we conduct regression analysis to understand which economic factors

[23] Egloff et al. (2010) also study optimal investment in VS but they reach the opposite conclusion for the optimal allocation. This can be explained by the different stochastic volatility models, investment strategies and market price of risk specifications used in the two studies.

may drive the term structure of the IVRP. Panel B in Table 6 summarizes the regression results. The variables have nearly a monotonic impact, measured by the corresponding slope coefficients, on the term structure of IVRP. For example, a negative S&P500 return induces a more negative IVRP, especially for the short-end of the term structure, and the effect becomes statistically insignificant beyond the 1-year horizon. An increase of the VIX index also induces a more negative IVRP and its impact is quite uniform, statistically and economically significant, throughout the IVRP term structure. Thus, despite being a 30-day index, the VIX index behaves more like a "level" factor than a short term factor for variance risk premia. CScorp has a negative and decreasing impact on the IVRP, amplifying the procyclical variation of the IVRP. The slope coefficients of other variables change sign throughout the term structure of IVRP, for example from negative to positive for TermL, therefore amplifying or reducing the negative IVRP.

As the quadratic variation can be naturally decomposed in the continuous, $\mathrm{QV}^c_{t,t+\tau}$, and discontinuous, $\mathrm{QV}^j_{t,t+\tau}$, part (see Equation (3)), the IVRP can also be decomposed as

$$\begin{aligned}\mathrm{IVRP}_{t,t+\tau} &= E^P_t[\mathrm{QV}_{t,t+\tau}] - E^Q_t[\mathrm{QV}_{t,t+\tau}] \\ &= (E^P_t[\mathrm{QV}^c_{t,t+\tau}] - E^Q_t[\mathrm{QV}^c_{t,t+\tau}]) + (E^P_t[\mathrm{QV}^j_{t,t+\tau}] - E^Q_t[\mathrm{QV}^j_{t,t+\tau}]) \\ &= \mathrm{IVRP}^c_{t,t+\tau} + \mathrm{IVRP}^j_{t,t+\tau}.\end{aligned}$$

We now investigate the impact of negative price jumps and the induced term structure of variance risk premia. As many investors are "long in the market" and the leverage effect is very pronounced, negative price jumps are perceived as unfavorable events and thus can carry particular risk premia. The contribution of negative price jumps to the IVRP is given by

$$\mathrm{IVRP}(k)^j_{t,t+\tau} = E^P_t[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}] - E^Q_t[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}]$$

where $1\{J^s < k\}$ is the indicator function of the event $J^s < k$. We set $k = -1\%$, i.e., we study the contribution of daily jumps below -1% to the IVRP.[24] Similar values of the threshold k produce similar results for $\mathrm{IVRP}(k)^j_{t,t+\tau}$. Given Model (6)–(7), $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is available in closed form.

Table 5 reports summary statistics for $\mathrm{IVRP}(k)^j_{t,t+\tau}$, when $k = -1\%$. Since $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is essentially constant when the time horizon τ increases, its relative contribution to the IVRP

[24] From January 1996 to September 2010, daily S&P500 returns are on average 3 times a month below -1%, having a standard deviation of 1.4%.

is decreasing on average and thus largest for the 2-month IVRP. This suggests that short-term variance risk premia mainly reflect investors' fear of a market crash, rather than the impact of stochastic volatility on the investment opportunity set. Although price jumps below -1% are infrequent events, their contribution to short-term IVRP is substantial. For the 2-month horizon, $\text{IVRP}(k)^j_{t,t+\tau}$ accounts for about 20% of the IVRP.

Figure 5 shows the term structure of $\text{IVRP}(k)^j_{t,t+\tau}$ over time. Similarly to the IVRP, the term structure of $\text{IVRP}(k)^j_{t,t+\tau}$ is generally downward sloping in quiet times. However, in contrast to IVRP, during market crashes the term structure of $\text{IVRP}(k)^j_{t,t+\tau}$ becomes suddenly upward sloping, reflecting the large jump risk due to a price fall. As an example, in Fall 2008 the whole term structure of $\text{IVRP}(k)^j_{t,t+\tau}$ moved downward but the two-month $\text{IVRP}(k)^j_{t,t+\tau}$ exhibited the largest negative drop and took several months to revert to average values. The 12- and 24-month $\text{IVRP}(k)^j_{t,t+\tau}$ took even longer to revert to average values. All in all, these findings suggest that investors' willingness to ensure against a market crash increases after a price fall with a persistent impact on the IVRP. The dynamics of the term structure of $\text{IVRP}(k)^j_{t,t+\tau}$ further show that the price fall has the strongest impact on the short-term IVRP but the persistency is more pronounced for long-term IVRP.

In order to examine the extent to which the large variance risk premia potentially translate into economic gains, we consider a simple but relatively robust trading strategy involving VS. The trading strategy is robust in the sense that Model (6)–(7) and corresponding estimates are used only to decide whether or not to invest in VS, i.e., to extract a trading signal.

Since realized variances are lower on average than VS rates, shorting VS contracts generates a positive return on average. Such a trading strategy can be refined as follows. At each day t, we compute the expected profit from shorting a VS contract, i.e., $\text{VS}_{t,t+\tau} - E^P_t[\text{QV}_{t,t+\tau}]$. Then, the strategy is to short the VS contract only when the expected profit is large enough and precisely n times larger than its expected standard deviation. When $n = 0$, the VS contract is shorted as soon as the expected profit is positive. When $n > 0$, the contract is shorted less often. When at day t the VS contract is shorted, we compute the actual return from the investment by comparing the VS rate and the ex-post realized variance, i.e., $\text{VS}_{t,t+\tau} - \text{RV}_{t,t+\tau}$. Since the strategy is short-and-hold (conditional on a model-based signal), transaction costs are unlikely to affect the results and will not be considered. If at day t the VS is not shorted, the return from t to $t + \tau$ is obviously zero and not considered when assessing the performance of the strategy. We repeat this procedure for each day t in our sample.

As a benchmark, we consider the following trading strategy based on the S&P500 index. If at day t the VS contract with maturity $t + \tau$ is shorted, we invest \$1 in the S&P500 index and liquidate the position at day $t + \tau$. Thus, the investment horizon is the same as the one for the VS strategy. The actual return is computed using S&P500 index prices. This buy-and-hold strategy is repeated for each day t in our sample.

Table 7 compares the trading strategies using the classical Sharpe ratios. We also computed Sortino ratios[25] and results were very similar, and not reported. As the VS is a forward contract, Sharpe ratios of corresponding short-and-hold strategies are calculated simply as the average return throughout our sample divided by its standard deviation. To compute Sharpe ratios of buy-and-hold strategies with the S&P500 index, we use the daily term structure of interest rates, downloaded from OptionMetrics and linearly interpolated to match the various investment horizons. We experimented other values of interest rates, such as a constant rate of zero or 4%, and the results reported in Table 7 change only marginally.

Shorting VS appears to be significantly more profitable than investing in the S&P500 index, over the same time horizons. This suggests that VS contracts offer economically important investment opportunities. It also confirms our model-based finding that investors are ready to pay high “insurance premia” to obtain protection against volatility increases.

When the threshold n increases, the VS is shorted less often.[26] As shown in Table 7, Sharpe ratios from investing in VS are nearly uniformly and significantly increasing in the threshold n. Thus, Model (6)–(7) seems to provide valuable information to generate a trading signal for shorting variance swaps.

Figure 6 shows the returns of the short-and-hold trading strategy based on 12-month VS and the long-and-hold trading strategy based on the S&P500 index. With the exception of 2008, shorting VS tends to provide stable and substantial positive returns. The losses during 2008 reflect jump and volatility risk that short positions are carrying, but they are smaller than the losses from the buy-and-hold S&P500 strategy. Long positions in the S&P500 generate substantial more volatile returns. Interestingly, shorting VS does not appear to suffer from the “picking up nickels in front of steamroller” syndrome during the period we looked at, despite the inclusion out-of-sample of the 2007–2009 financial crisis.

[25]The Sortino ratio is a popular performance measure and defined as the mean return in excess of a minimum acceptable return divided by the downside deviation. This ratio penalizes only returns below the minimum acceptable return, in contrast to the standard deviation which equally penalizes returns below and above the average return. In our computations we set the minimum acceptable return to zero, and the Sortino ratio is $(\sum_{t=1}^{T} r_t/T)/\sigma_D$, where r_t is the time-t return of a given trading strategy, the downside variance $\sigma_D^2 = \sum_{t=1}^{T}(r_t\, 1\{r_t < 0\})^2/T$ and T is the total number of returns.

[26]For example, the 12-month VS contract is shorted 80%, 59% and 23% of the times when $n = 1/4$, $1/2$, 1, respectively.

Finally, does shorting VS provide any diversification benefit? Table 8 shows correlations between daily returns of short positions in VS, long positions in the S&P500 index, and Treasury bond yields over the same time horizons. Short positions in VS are generally positively correlated with long positions in the S&P500 and, consistently with the patterns of the integrated risk premia, more so during turbulent than quiet times. They are also generally negatively correlated with long bond positions.

5.3. Risk Premia: Robustness Checks

To check the robustness of the parametric model, we note that the change of measure in Equation (11) implies that the mean jump size is different, not the jump intensity, under P and Q. Now we let the jump intensity be $\lambda_t^P = \lambda_0^P + \lambda_1^P v_t$ under P and $\lambda_t^Q = \lambda_0^Q + \lambda_1^Q v_t$ under Q. The drift under P of the index price process becomes

$$\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2 \rho v_t + g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t)$$

and jump risk premia become

$$\begin{aligned} \mathrm{JRP}_t &= g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t) \\ \mathrm{IVRP}^j_{t,t+\tau} &= E^P[(J^s)^2](\lambda_0^P + \lambda_1^P E_t^P[\mathrm{QV}^c_{t,t+\tau}]) - E^Q[(J^s)^2](\lambda_0^Q + \lambda_1^Q E_t^Q[\mathrm{QV}^c_{t,t+\tau}]). \end{aligned}$$

Estimation results of this more general model imply nearly the same dynamics for spot variance, stochastic long run mean, instantaneous risk premia, and integrated risk premia due to the continuous part of the quadratic variation. However, the estimated overall risk neutral jump-intensity, λ_t^Q, turns out to be smaller than objective jump-intensity, λ_t^P; Pan (2002) reports the same finding using her stochastic volatility model.[27] These estimates would imply positive jump-timing risk premium, $\lambda_t^P - \lambda_t^Q$, which in turn would induce positive IVRP^j. This finding confirms the limited ability of estimating very flexible change of measures.

[27] Pan considers jump intensities $\lambda_1^P v_t$ under P and $\lambda_1^Q v_t$ under Q, in our notation, and defines the jump-timing risk premium as $\lambda_1^Q - \lambda_1^P$, the opposite of our definition. Note that Pan's specification of jump intensities can be recovered setting $\lambda_0^P = \lambda_0^Q = 0$ in our model.

6. Conclusions

We study the term structure of variance swaps, equity and variance risk premia. Using a model-free approach to compare VS rates and VIX-type indices, we find evidence for a significant and time-varying jump risk component in VS rates. A flexible stochastic volatility model cannot explain such a jump risk component. This suggests that either the jump risk is heavily priced by VS traders or some segmentation between the VS and option markets exists or both.

Based on our model estimates, the term structure of variance risk premia is negative and generally downward sloping. The term structure of variance risk premia due to negative price jumps exhibits similar features in quiet times but is upward sloping in turbulent times. This implies that the short-end of term structure mainly reflects investors' fear of a market crash, rather than the impact of stochastic volatility on the investment set. It also suggests that investors' willingness to ensure against volatility risk over certain horizons increases after a market crash. This effect is stronger for short horizons and more persistent for long horizons.

We find that the term structure of equity risk premia is countercyclical while the slope is procyclical. Thus, at the peak of a crisis, investors appear to demand large risk premia for holding equities over short horizons (like 2 months), but require significantly smaller risk premia for holding equities over long horizons (like 2 years).

Finally, both term structures of equity and variance risk premia respond nearly monotonically to usual economic indicators. For example, an increase in the VIX index has progressively less impact on the equity risk premia as the horizon increases, but has a rather uniform and strong impact throughout the term structure of the variance risk premia.

A. Pricing Kernel

Recall that the market price of risks for the Brownian motions are

$$\Lambda_t' = [\gamma_1 \sqrt{(1-\rho^2)v_t}, \quad \gamma_2 \sqrt{v_t}, \quad \gamma_3 \sqrt{m_t}].$$

The pricing kernel (or Stochastic Discount Factor) is defined as

$$\pi_t = e^{-rt} \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \exp\left(-rt - \int_0^t \Lambda_u' \, dW_u^P - \frac{1}{2} \int_0^t \Lambda_u' \Lambda_u \, du \right) \prod_{u=1}^{N_t} \exp\left(a_j + b_j J_u^{s,P} + c_j J_u^{v,P} \right) \frac{\mu_v^P}{\mu_v^Q}$$

where $a_j = ((\mu_j^P)^2 - (\mu_j^Q)^2)/(2\sigma_j^2)$, $b_j = (\mu_j^Q - \mu_j^P)/\sigma_j^2$, and $c_j = (\mu_v^Q - \mu_v^P)/(\mu_v^P \mu_v^Q)$. The process π_t is a valid pricing kernel when deflated bank account and deflated cum-dividend price processes are P-martingales.

When a jump occurs the pricing kernel jumps from π_{t_-} to $\pi_t = \pi_{t_-} e^{a_j + b_j J_t^{s,P} + c_j J_t^{v,P}} \frac{\mu_v^P}{\mu_v^Q}$, hence

$$\begin{aligned}
\frac{d\pi_t}{\pi_t} &= -r\,dt - \Lambda_t' \, dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P}) \frac{\mu_v^P}{\mu_v^Q} - 1) \, dN_t^P \\
&= -r\,dt - (\gamma_1 \sqrt{(1-\rho^2)v_t} \, dW_{1t}^P + \gamma_2 \sqrt{v_t} \, dW_{2t}^P + \gamma_3 \sqrt{m_t} \, dW_{3t}^P) \\
&\quad + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P}) \frac{\mu_v^P}{\mu_v^Q} - 1) \, dN_t^P.
\end{aligned}$$

Let $B_t = e^{rt}$ denote the bank account level and $B_t^\pi = B_t \, \pi_t$ the deflated bank account. Applying Itô's formula

$$\begin{aligned}
d(B_t^\pi) &= B_t \, d\pi_t + \pi_t \, dB_t \\
&= B_t^\pi(-r\,dt - \Lambda_t' \, dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P}) \frac{\mu_v^P}{\mu_v^Q} - 1) \, dN_t^P) + B_t^\pi r \, dt \\
d(B_t^\pi)/B_t^\pi &= -\Lambda_t' \, dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P}) \frac{\mu_v^P}{\mu_v^Q} - 1) \, dN_t^P.
\end{aligned}$$

Hence, B_t^π is a P-martingale (or has zero drift) when $E^P[\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P}) \frac{\mu_v^P}{\mu_v^Q}] = 1$. As $J^{s,P}$ and $J^{v,P}$ are independent, the last equation holds when $E^P[\exp(a_j + b_j J_t^{s,P})] = 1$ and

$E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = 1$, which is shown in the following calculations:

$$\begin{aligned}
E^P[\exp(a_j + b_j J_t^{s,P})] &= \exp(a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2}) \\
a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} &= \frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\mu_j^P + \left(\frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\right)^2\frac{\sigma_j^2}{2} \\
&= \frac{(\mu_j^P)^2 - (\mu_j^Q)^2 + 2\mu_j^Q\mu_j^P - 2(\mu_j^P)^2 + (\mu_j^Q)^2 + (\mu_j^P)^2 - 2\mu_j^Q\mu_j^P}{2\sigma_j^2} = 0
\end{aligned}$$

where we used $J^{s,P} \sim \mathcal{N}(\mu_j^P, \sigma_j^2)$. As $J^{v,P} \sim Exp(\mu_v^P)$

$$E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = \frac{\mu_v^P}{\mu_v^Q}\int_0^\infty e^{c_j J^v}\frac{e^{-J^v/\mu_v^P}}{\mu_v^P}\,dJ^v = 1.$$

Let $S_{\delta,t} = S_t e^{\delta t}$ denote the cum-dividend stock price, hence

$$\begin{aligned}
\frac{dS_{\delta,t}}{S_{\delta,t}} = \frac{dS_t}{S_t} + \delta\,dt &= (r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \rho\sqrt{v_t}\,dW_{2t}^P \\
&\quad + (\exp(J_t^{s,P}) - 1)\,dN_t^P
\end{aligned}$$

where S_t is the ex-dividend stock price. Let $S_{\delta,t}^\pi$ be the deflated cum-dividend stock price, i.e., $S_{\delta,t}\,\pi_t$. When a jump occurs, both π_t and S_t jump and S_δ^π jumps from S_{δ,t_-}^π to $S_{\delta,t}^\pi = S_{\delta,t_-}^\pi\exp(a_j + b_j J_t^{s,P} + J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}$. Hence, at the jump time, $dS_{\delta,t}^\pi/S_{\delta,t}^\pi = \exp(a_j + (b_j + 1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1$.

Applying Itô's formula, with π_t^c and $S_{\delta,t}^c$ denoting the continuous part of π_t and $S_{\delta,t}$, respectively,

$$\begin{aligned}
dS_{\delta,t}^\pi &= S_{\delta,t}\,d\pi_t^c + \pi_t\,dS_{\delta,t}^c + dS_{\delta,t}^c\,d\pi_t^c + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P \\
&= S_{\delta,t}\,\pi_t(-r\,dt - \gamma_1\sqrt{(1-\rho^2)v_t}\,dW_{1t}^P - \gamma_2\sqrt{v_t}\,dW_{2t}^P - \gamma_3\sqrt{m_t}\,dW_{3t}^P) \\
&\quad + \pi_t\,S_{\delta,t}((r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \rho\sqrt{v_t}\,dW_{2t}^P) \\
&\quad - S_{\delta,t}\pi_t(\gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t)\,dt + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P \\
\frac{dS_{\delta,t}^\pi}{S_{\delta,t}^\pi} &= \sqrt{(1-\rho^2)v_t}(1-\gamma_1)\,dW_{1t}^P + (\rho - \gamma_2)\sqrt{v_t}\,dW_{2t}^P - \gamma_3\sqrt{m_t}\,dW_{3t}^P \\
&\quad + (\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P - g^Q\lambda_t\,dt.
\end{aligned}$$

Hence, $S^{\pi}_{\delta,t}$ is a P-martingale (or has zero drift) when $E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = 1$, which we already showed above, and when $E^P[\exp(a_j + (b_j+1)J_t^{s,P}) - 1] = g^Q$, which is indeed the case as shown in the following calculations:

$$\begin{aligned}
E^P[\exp(a_j + (b_j+1)J_t^{s,P}) - 1] &= g^Q \\
\exp(a_j + (b_j+1)\mu_j^P + (b_j+1)^2\frac{\sigma_j^2}{2}) - 1 &= \exp(\mu_j^Q + \frac{\sigma_j^2}{2}) - 1 \\
a_j + b_j\mu_j^P + \mu_j^P + b_j^2\frac{\sigma_j^2}{2} + 2b_j\frac{\sigma_j^2}{2} &= \mu_j^Q \\
\mu_j^P + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\sigma_j^2 &= \mu_j^Q
\end{aligned}$$

where we used $a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} = 0$, which is implied by the martingale property of the deflated bank account.

Finally, the relation between the pricing kernel π_t and the risk-neutral dynamics is derived as usual. Define the density process $\xi_t = \pi_t e^{rt}$. Under usual technical conditions, applying Itô's formula, $d\xi_t/\xi_t = -\Lambda_t'\, dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\, dN_t^P$, which shows that ξ_t is a P-martingale and hence it uniquely defines an equivalent martingale measure Q. Defining the Q-Brownian motions as $dW_{1t}^Q = dW_{1t}^P + \gamma_1\sqrt{(1-\rho^2)v_t}\, dt$, $dW_{2t}^Q = dW_{2t}^P + \gamma_2\sqrt{v_t}\, dt$ and $dW_{3t}^Q = dW_{3t}^P + \gamma_3\sqrt{m_t}\, dt$, gives the risk-neutral dynamic of the stock price S, spot variance v, and stochastic long run mean m in Equation (7).

B. Integrated Equity Risk Premium

To compute the IERP in (16) we rely on the transform analysis of Duffie et al. (2000), which is often used in finance applications; e.g., Duffie et al. (2003). In this appendix we provide a self-contained application of this theory to the calculation of the IERP in our setting.

The basic step is to compute a conditional expectation of the form $E_t^P[\exp(\zeta\int_t^{t+\tau} v_s\, ds)]$, where ζ is a given constant. The first conditional expectation in (16) is $E_t^P[S_{t+\tau}/S_t] = E_t^P[\exp(\int_t^{t+\tau} \mu_s\, ds)]$, where μ_s is an affine function of v_s, defined after (6).[28]

Define the stochastic process $\psi_t = E_t^P[\exp(\zeta\int_0^T v_s\, ds)]$, which is a P-martingale by construction for all $t \geq 0$, under standard integrability conditions. Guess the functional form $\psi_t = \exp(\zeta\int_0^t v_s\, ds)\exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$, which is exponentially affine in the state

[28]The second conditional expectation is simply $E_t^Q[S_{t+\tau}/S_t] = \exp((r_{t,t+\tau} - \delta)\tau)$, assuming a time varying but deterministic term structure of interest rates.

variables v_t and m_t. Recall $\tau = T - t$. The necessary derivatives to apply Itô's formula to ψ_t are

$$\begin{aligned}
\frac{\partial \psi_t}{\partial t} &= \psi_t(\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t) \\
\frac{\partial \psi_t}{\partial v_t} &= \psi_t B(\tau), \qquad \frac{\partial^2 \psi_t}{\partial v_t^2} = \psi_t B(\tau)^2 \\
\frac{\partial \psi_t}{\partial m_t} &= \psi_t C(\tau), \qquad \frac{\partial^2 \psi_t}{\partial m_t^2} = \psi_t C(\tau)^2
\end{aligned}$$

If a jump occurs at time t, the spot variance jumps from v_{t_-} to $v_t = v_{t_-} + J_t^{v,P}$, and consequently the process ψ jumps from ψ_{t_-} to ψ_t, which implies that

$$\frac{\psi_t}{\psi_{t_-}} - 1 = \frac{e^{\zeta \int_0^t v_s\, ds}\ e^{A(\tau)+B(\tau)v_t}}{e^{\zeta \int_0^t v_s\, ds}\ e^{A(\tau)+B(\tau)v_{t_-}}} - 1 = e^{B(\tau)(v_t - v_{t_-})} - 1 = e^{B(\tau)J_t^{v,P}} - 1.$$

Rewriting the P-dynamic of the spot variance, with obvious notation, as

$$dv_t = (k_v^Q m_t - k_v^P v_t)\, dt + \sigma_v \sqrt{v_t}\, dW_{2t}^P + J_t^{v,P}\, dN_t = dv_t^{cont} + J_t^{v,P}\, dN_t$$

and applying Itô's formula to ψ_t gives

$$\begin{aligned}
\frac{d\psi_t}{\psi_{t_-}} &= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\, dt + B(\tau)(dv_t^{cont}) + \frac{1}{2} B(\tau)^2 (dv_t^{cont})^2 \\
&\quad + C(\tau)(dm_t) + \frac{1}{2} C(\tau)^2 (dm_t)^2 + (\frac{\psi_t}{\psi_{t_-}} - 1) dN_t \\
&= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\, dt + B(\tau)(k_v^Q m_t - k_v^P v_t) dt + \sigma_v \sqrt{v_t} dW_{2t}^P) + \frac{1}{2} B(\tau)^2 \sigma_v^2 v_t\, dt \\
&\quad + C(\tau)(k_m^P(\theta_m^P - m_t) dt + \sigma_m \sqrt{m_t} dW_{3t}^P) + \frac{1}{2} C(\tau)^2 \sigma_m^2 m_t\, dt \\
&\quad + (e^{B(\tau)J_t^{v,P}} - 1) dN_t - E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\, dt + E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\, dt \\
&= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\, dt + B(\tau)(k_v^Q m_t - k_v^P v_t) dt + \frac{1}{2} B(\tau)^2 \sigma_v^2 v_t\, dt \\
&\quad + C(\tau) k_m^P(\theta_m^P - m_t) dt + \frac{1}{2} C(\tau)^2 \sigma_m^2 m_t\, dt \\
&\quad + E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\, dt + dM_t^P
\end{aligned}$$

where $dM_t^P = \sigma_v \sqrt{v_t}\, dW_{2t}^P + \sigma_m \sqrt{m_t}\, dW_{3t}^P + (e^{B(\tau)J_t^{v,P}} - 1) dN_t - E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\, dt$ is a P-martingale increment.

As ψ_t is a P-martingale, the drift must be zero for each time t and level of the state variables v_t and m_t. Collecting terms in dt, $v_t\, dt$ and $m_t\, dt$, respectively, and setting them equal to zero, give three nonlinear ordinary differential equations

$$\begin{aligned}
0 &= -A(\tau)' + C(\tau)k_m^P\theta_m^P + E^P[e^{B(\tau)J_t^v} - 1]\lambda_0 \\
0 &= \zeta - B(\tau)' - B(\tau)k_v^P + \frac{1}{2}B(\tau)^2\sigma_v^2 + E^P[e^{B(\tau)J_t^v} - 1]\lambda_1 \\
0 &= -C(\tau)' + B(\tau)k_v^Q - C(\tau)k_m^P + \frac{1}{2}C(\tau)^2\sigma_m^2
\end{aligned}$$

for the coefficients $A(\tau)$, $B(\tau)$ and $C(\tau)$, with terminal conditions $A(0) = B(0) = C(0) = 0$. As the system is time-homogenous, for each time horizon τ, these coefficients need to be computed only once. Thus, at each time t, $E_t^P[\exp(\zeta \int_t^{t+\tau} v_s\, ds)] = \exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$.

The expectation in the first two differential equations is

$$E^P[e^{B(\tau)J_t^v}] = \int_0^\infty e^{B(\tau)J^v} \frac{e^{-J^v/\mu_v^P}}{\mu_v^P}\, dJ^v = \frac{1}{\mu_v^P}\int_0^\infty e^{-J^v\left(\frac{1}{\mu_v^P} - B(\tau)\right)}\, dJ^v = \frac{1}{1 - B(\tau)\mu_v^P}$$

and the integral above converges when $\left(\frac{1}{\mu_v^P} - B(\tau)\right) > 0$, which is indeed the case according to our estimates. Then

$$E^P[e^{B(\tau)J_t^v} - 1] = \frac{B(\tau)\mu_v^P}{1 - B(\tau)\mu_v^P}$$

is substituted in the first two differential equations, and the system is solved numerically.

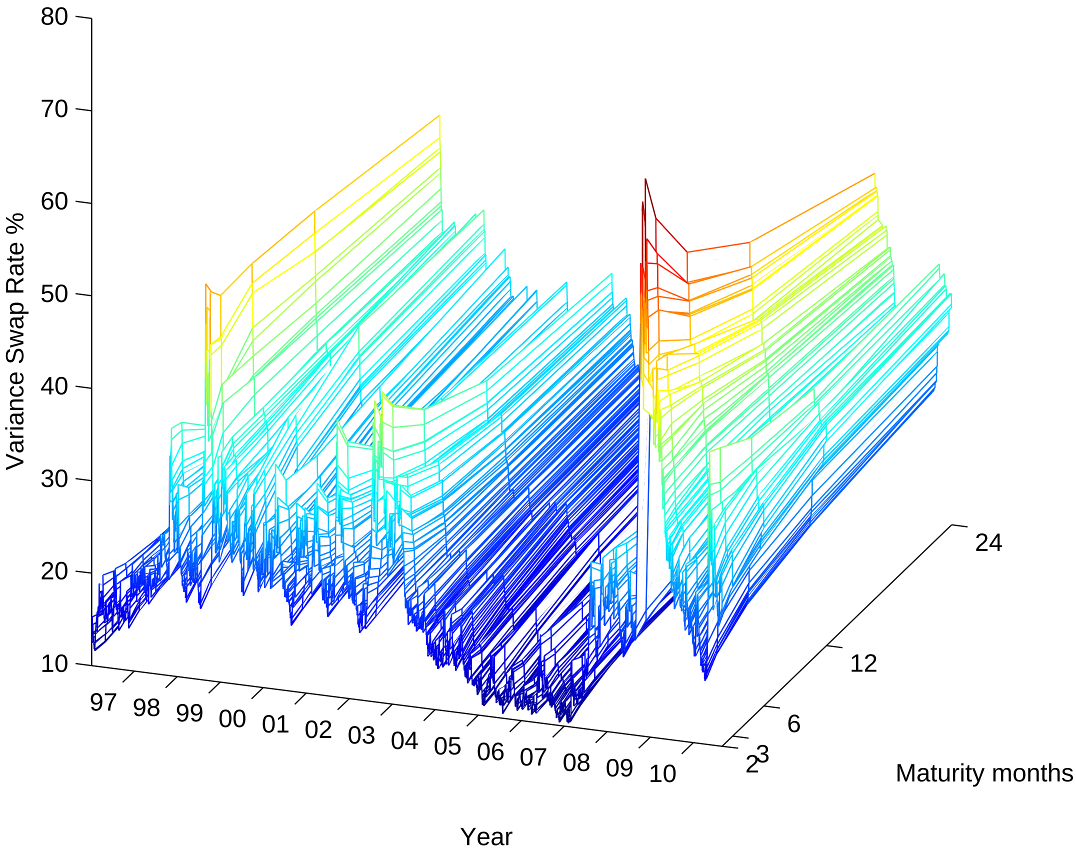


Figure 1. Term structure of variance swap rates. Values are in volatility percentage units, i.e., $\mathrm{VS}_{t,t+\tau}^{1/2} \times 100$, with 2-, 3-, 6-, 12-, and 24-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each time to maturity.

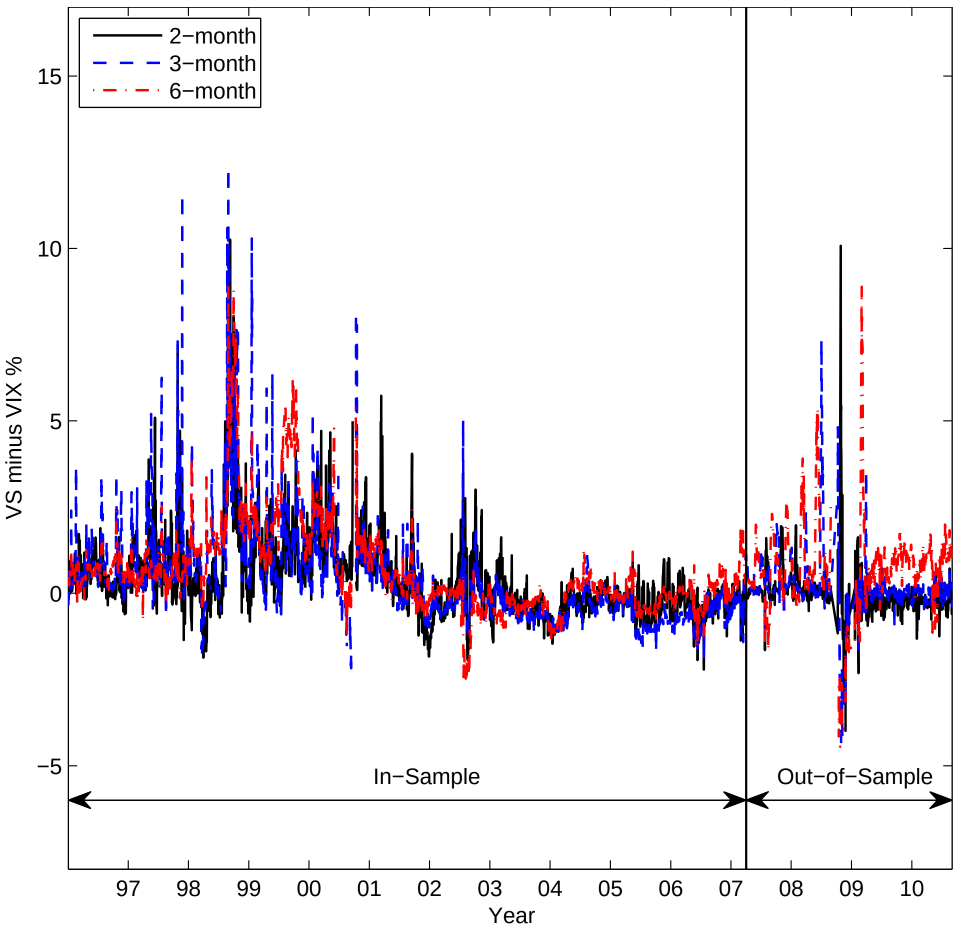


Figure 2. Term structure of model-free jump component in variance swap rates. VS rates minus calculated VIX-type indices for 2-, 3-, and 6-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each maturity. The difference is in volatility percentage units, i.e., $(\mathrm{VS}_{t,t+\tau}^{1/2} - \mathrm{VIX}_{t,t+\tau}^{1/2}) \times 100$.

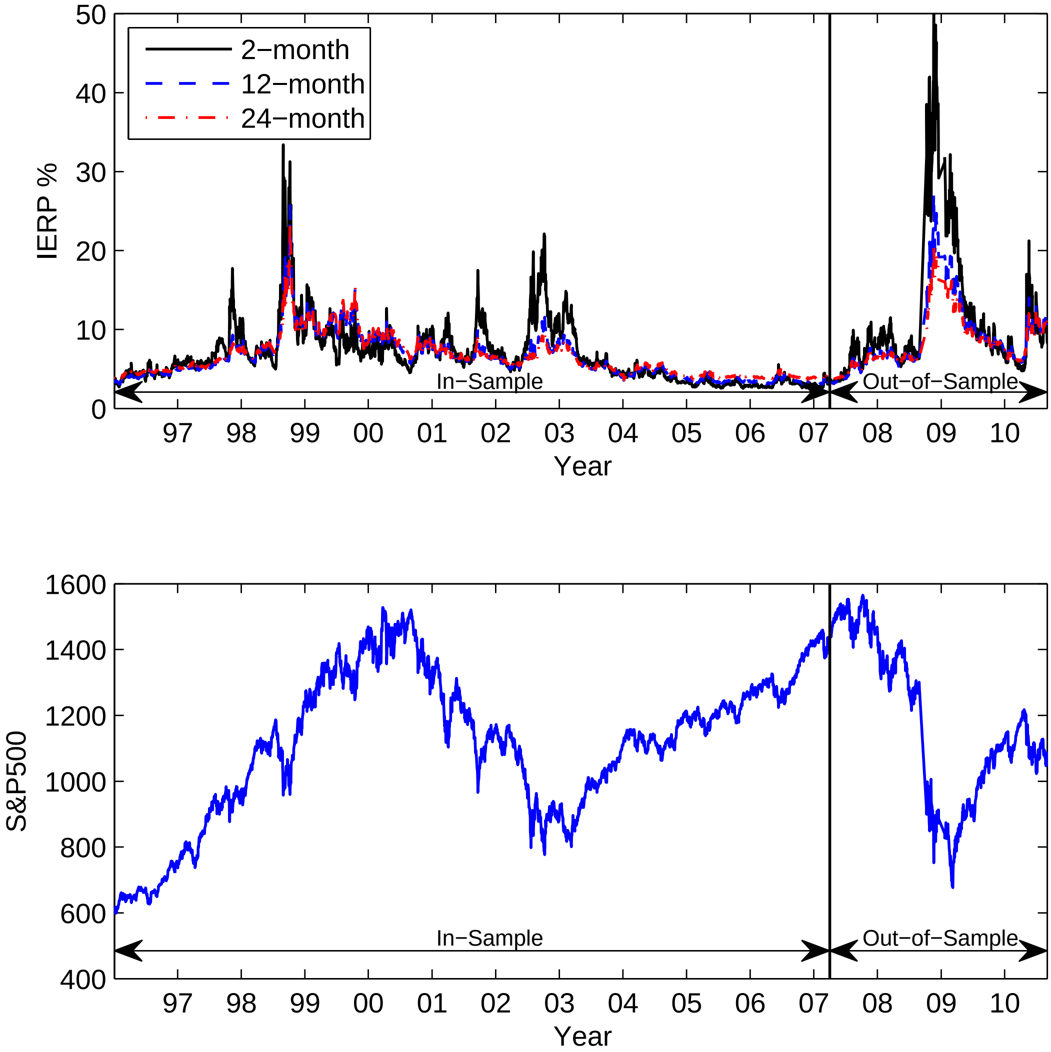


Figure 3. Term structure of integrated equity risk premia and S&P500 index. Upper graph: annualized integrated equity risk premia, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau) \times 100$. Lower graph: S&P500 index, S_t. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

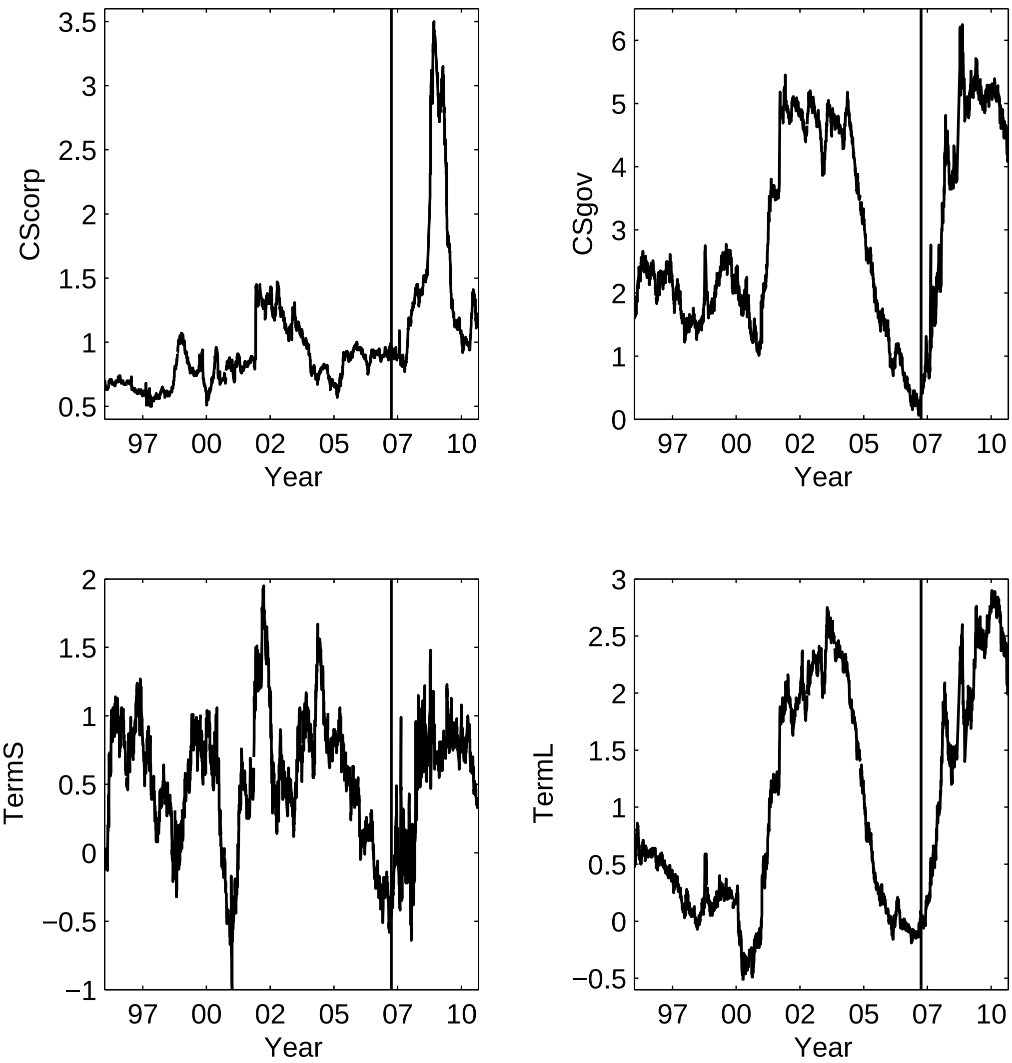
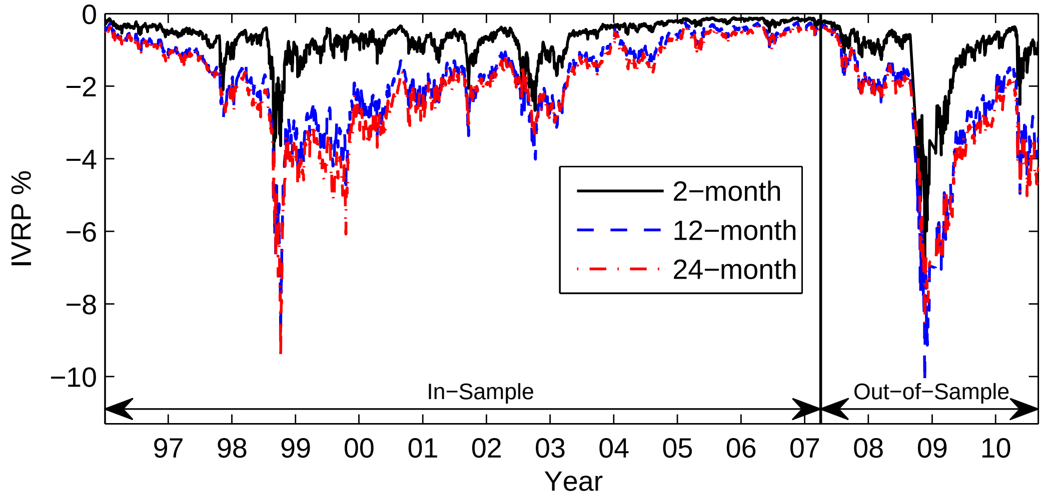


Figure 4. Time series plots of macro variables: CScorp the difference between Moody's BAA and AAA corporate bond yields, CSgov the difference between Moody's AAA corporate bond yield and 3-month Treasury securities, TermS the difference between the yields on 2-year and 3-month Treasury securities, TermL the difference between the yields on 10-year and 2-year Treasury securities. All variables are daily. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.



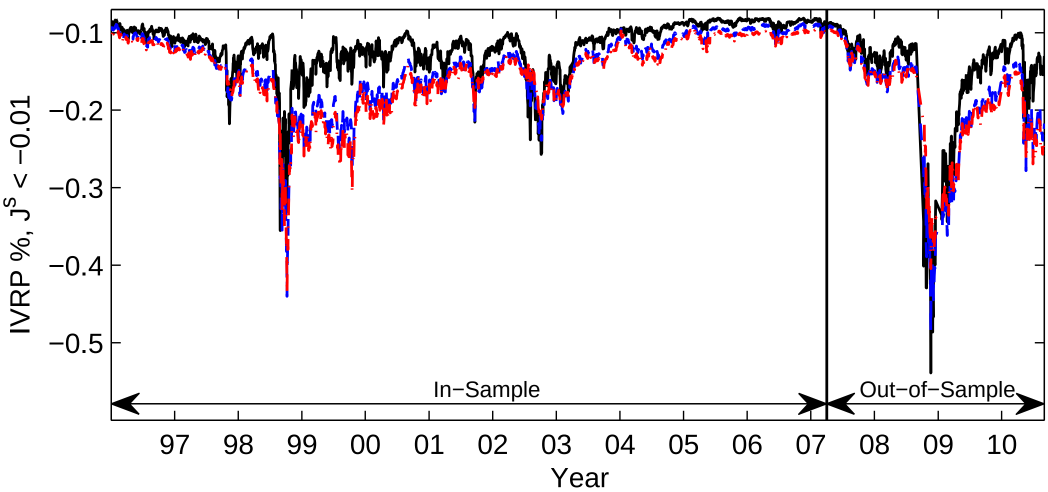


Figure 5. Term structure of integrated variance risk premia. Upper graph: integrated variance risk premia, i.e., $(E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}]) \times 100$. Lower graph: integrated variance risk premia due to price jump below $k = -0.01$, i.e., $(E_t^P[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}]) \times 100$. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

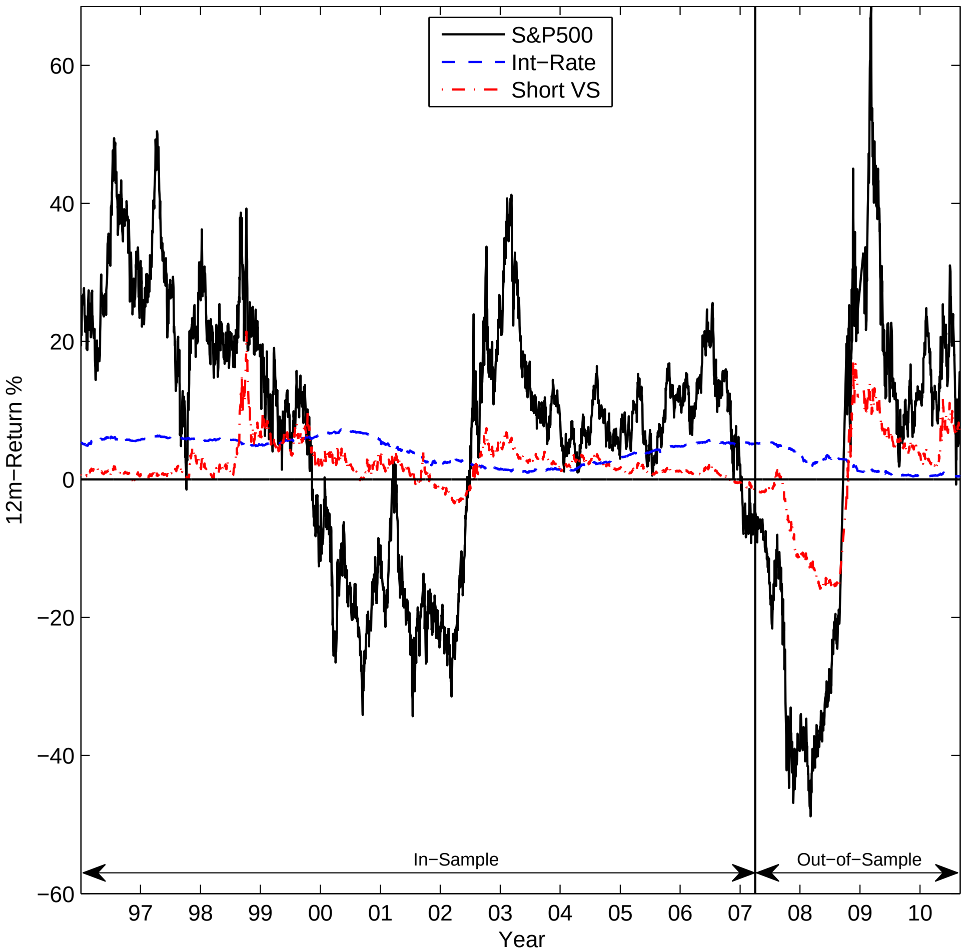


Figure 6. Returns of short positions in variance swap and long positions in the S&P500 index. Short VS (dash-dot line) denotes ex-post annual returns of the short-and-hold trading strategy based on 12-month VS, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$ for each day t in our sample, where τ is one year. S&P500 (solid line) denotes ex-post annual returns of the long-and-hold position on the S&P500 index, i.e., $S_{t+\tau}/S_t - 1$ for each day t in our sample, where τ is one year. Int-Rate (dash line) denotes the one-year interest rate for each day in our sample. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

Maturity	Mean	Std	Skew	Kurt	AC1	Q_{22}	ADF
Panel A: Variance Swap Rates							
2	22.14	8.18	1.53	7.08	0.982	62,908.97	−3.79
3	22.32	7.81	1.32	6.05	0.988	66,449.22	−3.52
6	22.87	7.40	1.10	4.97	0.992	69,499.72	−3.30
12	23.44	6.88	0.80	3.77	0.994	71,644.69	−2.82
24	23.93	6.48	0.57	2.92	0.995	72,878.68	−2.47
Panel B: Calculated VIX-type Indices, CBOE method							
2	21.74	7.63	1.53	7.22	0.985	63,551.35	−3.83
3	21.95	7.32	1.43	6.83	0.987	64,644.41	−3.72
6	22.08	6.85	1.16	5.58	0.991	65,736.38	−3.62
Panel C: Calculated VIX-type Indices, Carr–Wu method							
2	22.34	7.82	1.53	7.19	0.985	63,699.78	−3.83
3	22.34	7.46	1.42	6.78	0.989	65,703.56	−3.83
6	22.30	7.00	1.20	5.75	0.999	66,490.72	−3.70
Panel D: Realized Variances							
2	18.09	8.62	2.13	10.70	0.997	68,750.50	−4.96
3	18.21	8.47	2.13	10.43	0.998	73,156.42	−4.73
6	18.58	8.37	2.04	9.07	0.999	76,928.13	−3.44
12	19.07	7.88	1.54	5.97	0.999	78,412.59	−2.51
24	19.91	6.97	0.68	3.02	0.999	76,028.01	−1.97

Table 1. Summary statistics. Panel A: Summary statistics of the variance swap rates on the S&P500 index. Time to maturities are in months. The sample period is from January 4, 1996 to September 2, 2010, for a total of 3,624 observations for each time to maturity. The table reports mean, standard deviation (Std), skewness (Skew), kurtosis (Kurt), first order autocorrelation (AC1) the Ljung–Box portmanteau test for up to 22nd order autocorrelation (Q_{22}), the test 10% critical value is 30.81; the augmented Dickey–Fuller test for unit root involving 22 augmentation lags, a constant term and time trend (ADF), the test 10% critical value is −3.16. Panels B and C: summary statistics of the 2-, 3-, and 6-month VIX-type indices calculated using SPX options and applying the revised CBOE VIX and Carr–Wu methodologies, respectively. Panel D: summary statistics of ex-post S&P500 realized variances for various time to maturities. All variables are in volatility percentage units.

	SV1F		SV2F		SV2F-PJ		SV2F-PJ-VJ	
	Estim.	S.E.	Estim.	S.E.	Estim.	S.E.	Estim.	S.E.
κ_v^P	0.797	0.008	5.060	0.005	4.803	0.353	5.340	0.406
σ_v	0.272	0.002	0.525	0.003	0.419	0.009	0.394	0.006
κ_m^P			0.221	0.011	0.234	0.086	0.491	0.039
σ_m			0.154	0.002	0.141	0.002	0.167	0.001
θ_m^P	0.047	0.001	0.054	0.001	0.043	0.016	0.038	0.009
ρ	-0.674	0.008	-0.743	0.006	-0.713	0.010	-0.688	0.008
γ_1	1.303	2.537	0.742	2.591	-2.545	4.206	-5.054	5.495
γ_2	-1.322	1.173	-1.838	1.374	-2.244	0.851	-5.633	2.016
γ_3			-0.548	1.012	-0.673	0.610	-0.954	1.294
λ_0					3.669	0.621	2.096	0.467
λ_1					44.770	17.227	21.225	18.584
μ_j^P					0.010	0.008	-0.004	0.001
μ_j^Q					-0.001	0.009	-0.012	0.001
σ_j					0.038	0.003	0.043	0.000
μ_v^P							0.001	0.000
μ_v^Q							0.002	0.000
σ_{e_1}	0.006	0.000	0.004	0.000	0.004	0.000	0.004	0.000
σ_{e_2}	0.006	0.000	0.002	0.000	0.002	0.000	0.002	0.000
σ_{e_3}	0.011	0.000	0.003	0.000	0.003	0.000	0.007	0.000
σ_{e_4}	0.014	0.000						
ρ_e	0.288	0.016	-0.093	0.005	-0.088	0.006	-0.053	0.000
Log-likelihood	60,008.4		73,274.5		74,381.8		74,490.5	

Table 2. Model estimates. Estimation results for the Model (6)–(7) (labeled SV2F-PJ-VJ) and three nested models (labeled SV1F, SV2F and SV2F-PJ, respectively). For each model, estimate (Estim.) and standard errors (S.E.) are reported. The likelihood-based estimation procedure is described in Section 3. Variance swap rates with 2-, 3-, 6-, 12-, 24-month to maturity and S&P500 returns range from January 4, 1996 to April 2, 2007. Variance swap rates with 3- and 12-month (3-month) to maturity are assumed to be observed without errors (for the SV1F model). Variance swap rates with 2-, 6-, 24-month (and, for the SV1F model, 12-month) to maturity are assumed to be observed with errors whose standard deviations are σ_{e_1}, σ_{e_2}, σ_{e_3} (and σ_{e_4}), respectively, and correlation ρ_e. Interest rate $r = 4\%$ and dividend yield $\delta = 1.5\%$.

	In-Sample				Out-of-Sample			
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
	Heston		SJSV		Heston		SJSV	
$\widehat{\mathrm{VS}}_{2\mathrm{m}} - \mathrm{VS}_{2\mathrm{m}}$	-0.081	0.851	-0.151	0.739	0.259	1.420	0.194	1.036
$\widehat{\mathrm{VS}}_{6\mathrm{m}} - \mathrm{VS}_{6\mathrm{m}}$	0.002	1.119	0.058	0.397	-0.258	1.403	-0.209	0.470
$\widehat{\mathrm{VS}}_{24\mathrm{m}} - \mathrm{VS}_{24\mathrm{m}}$	1.001	2.950	0.144	0.555	0.469	3.074	-0.137	0.559

Table 3. Variance swap pricing errors. The pricing error is defined as the model-based VS rate minus observed VS rate, in volatility percentage units, i.e., $(E_t^Q[\mathrm{QV}_{t,t+\tau}]^{1/2} - \mathrm{VS}_{t,t+\tau}^{1/2}) \times 100$. The table reports mean and root mean square error of pricing errors for VS rate with 2-, 6-, and 24-month to maturity, under the Heston model and Model (6)–(7). In-sample period, used to estimate the models, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

	In-Sample		Out-of-Sample	
	Mean	Std	Mean	Std
DRP	4.66	3.99	9.30	9.49
JRP	2.38	0.57	3.05	1.36
VRP	−8.56	7.33	−17.08	17.42
LRMRP	−0.77	0.56	−1.20	0.68

Table 4. Spot risk premia. Diffusive risk premium $\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t$; Jump risk premium $\mathrm{JRP}_t = (E^P[e^J]-E^Q[e^J])(\lambda_0+\lambda_1 v_t)$; Variance risk premium $\mathrm{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\mathrm{LRMRP}_t = \gamma_3\sigma_m m_t$. Risk premia are based on Model (6)–(7). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010. Entries are in percentage.

	In-Sample		Out-of-Sample		In-Sample		Out-of-Sample	
Maturity	Mean	Std	Mean	Std	Mean	Std	Mean	Std
	Equity				Variance			
2	6.68	3.71	11.00	8.29	−0.63	0.47	−1.18	1.06
6	6.37	3.08	9.63	5.81	−1.23	0.87	−2.21	1.72
12	6.28	2.79	8.82	4.43	−1.59	1.11	−2.71	1.83
24	6.45	2.52	8.28	3.40	−1.79	1.22	−2.89	1.74
	Price Jump Contribution				$J^s < -1\%$ Contribution			
2	2.40	0.49	2.97	1.11	−0.12	0.03	−0.16	0.07
6	2.44	0.44	2.89	0.84	−0.13	0.04	−0.18	0.08
12	2.52	0.43	2.88	0.68	−0.14	0.05	−0.19	0.07
24	2.71	0.46	2.97	0.59	−0.15	0.05	−0.19	0.07

Table 5. Term structure of integrated equity risk premia and integrated variance risk premia. Left panels: integrated equity risk premium, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau)\times 100$, and equity risk premium due to the price jump component. Right panels: integrated variance risk premium, i.e., $(E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}])\times 100$, and variance risk premium due to price jump J^s below -1%. Risk premia are based on Model (6)–(7). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

Mat.	Interc.	S&P	VIX	CScorp	CSgov	TermS	TermL	R^2
			Panel A: Integrated Equity Risk Premium					
2	−5.27	−20.22	0.59	1.99	−1.20	1.04	0.96	93.1
	(−10.33)	(−3.25)	(14.79)	(3.11)	(−3.45)	(2.72)	(2.25)	
3	−4.56	−16.06	0.53	1.71	−0.67	0.70	0.29	92.5
	(−10.33)	(−2.79)	(13.75)	(2.91)	(−1.84)	(1.79)	(0.63)	
6	−3.10	−8.21	0.41	1.15	0.33	0.07	−0.95	88.2
	(−7.35)	(−1.63)	(10.28)	(2.06)	(0.62)	(0.13)	(−1.35)	
12	−1.57	−1.96	0.31	0.65	1.07	−0.39	−1.91	79.2
	(−2.88)	(−0.44)	(6.94)	(1.05)	(1.44)	(−0.56)	(−1.86)	
24	0.01	1.20	0.23	0.26	1.39	−0.54	−2.37	70.9
	(0.02)	(0.31)	(5.31)	(0.42)	(1.73)	(−0.73)	(−2.12)	
			Panel B: Integrated Variance Risk Premium					
2	0.88	2.72	−0.08	−0.26	0.18	−0.14	−0.16	93.3
	(13.36)	(3.50)	(−15.44)	(−3.21)	(4.20)	(−3.07)	(−3.11)	
3	1.13	3.13	−0.10	−0.31	0.17	−0.15	−0.13	93.3
	(14.48)	(3.16)	(−14.86)	(−3.10)	(2.92)	(−2.35)	(−1.76)	
6	1.52	2.89	−0.12	−0.36	−0.02	−0.05	0.14	90.6
	(14.12)	(2.06)	(−11.80)	(−2.44)	(−0.12)	(−0.42)	(0.87)	
12	1.65	1.06	−0.13	−0.29	−0.41	0.17	0.67	81.4
	(7.84)	(0.61)	(−7.39)	(−1.24)	(−1.41)	(0.64)	(1.68)	
24	1.46	−0.88	−0.11	−0.19	−0.73	0.37	1.09	70.3
	(4.71)	(−0.48)	(−4.98)	(−0.60	(−1.71)	(0.95)	(1.83)	

Table 6. Regression analysis for integrated risk premiums. Panel A: regression analysis of the annualized integrated equity risk premium, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau) \times 100$, based on Model (6)–(7). For each maturity (Mat.), the integrated equity risk premium is regressed on a constant (Interc.), S&P500 returns, VIX index, CScorp the difference between Moody's BAA and AAA corporate bond yields, CSgov the difference between Moody's AAA corporate bond yield and 3-month Treasury securities, TermS the difference between the yields on 2-year and 3-month Treasury securities, TermL the difference between the yields on 10-year and 2-year Treasury securities. All variables are daily. Maturity is in months. The sample period ranges from January 4, 1996 to September 2, 2010. For each maturity, the first row reports point estimates, the second row reports (in parenthesis) t-statistics based on robust standard errors computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). R^2 is the adjusted R^2 in percentage. Panel B: corresponding regression analysis for the integrated variance risk premia, i.e., $(E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}]) \times 100$.

	In-Sample									
	Short Variance Swap					Long S&P500				
Horizon	2	3	6	12	24	2	3	6	12	24
Threshold										
Always	0.59	0.61	0.68	0.85	0.67	0.13	0.16	0.22	0.27	0.18
0	0.59	0.61	0.68	0.85	0.68	0.13	0.16	0.22	0.27	0.18
1/4	0.67	0.72	0.71	0.93	1.22	0.07	0.14	0.15	0.22	0.10
1/2	0.94	1.39	1.30	1.02	1.37	0.59	0.85	0.55	−0.03	−0.18
1	1.47	3.16	2.05	2.21	2.64	1.04	2.92	1.32	0.61	−0.33
	Out-of-Sample									
	Short Variance Swap					Long S&P500				
Always	0.23	0.17	0.08	0.03	0.07	−0.02	−0.10	−0.10	−0.18	−0.06
0	0.23	0.17	0.08	0.03	0.08	−0.02	−0.10	−0.10	−0.18	−0.05
1/4	0.67	0.21	0.10	0.04	0.26	0.36	−0.03	−0.10	−0.14	0.12
1/2	0.57	1.13	0.36	0.06	0.43	0.34	0.46	0.19	−0.09	0.29
1	0.32	0.76	1.84	2.47	2.98	0.09	0.11	1.20	1.33	1.51

Table 7. Sharpe ratios of short positions in variance swaps and long positions in the S&P500 index. For each day t in the sample, the expected profit from a short position in a VS contract is computed, i.e., $\mathrm{VS}_{t,t+\tau} - E_t^P[\mathrm{QV}_{t,t+\tau}]$. If the expected profit is n times larger than its standard deviation, then the VS contract is shorted. Otherwise no position is taken at day t. The column "Threshold" reports the number of standard deviations n. "Always" means the VS contract is always shorted. At time $t+\tau$, the actual profit is computed, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$, where $\mathrm{RV}_{t,t+\tau}$ is the ex-post realized variance. The notional amount in the VS contract is such that for each unit increase of the variance payoff, the contract pays out \$1. The investment strategy in the S&P500 is as follows. If at day t the VS contract with maturity $t+\tau$ is shorted, \$1 is invested in the S&P500 at day t. The position is held until $t+\tau$ and then liquidated. Sharpe ratios are computed using all the returns from each investment strategy. Interest rates are obtained by linearly interpolating the daily term structure of zero-coupon Treasury bond yields. VS contracts with 2-, 3-, 6-, 12- and 24-month to maturities are considered. The row "Horizon" reports the time to maturity. In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

	In-Sample			Out-of-Sample		
	S&P500	Int-Rate	Short VS	S&P500	Int-Rate	Short VS
	2-month Returns			2-month Returns		
S&P500	1.00	0.09	0.57	1.00	−0.31	0.63
Int-Rate	0.09	1.00	−0.01	−0.31	1.00	−0.25
Short VS	0.57	−0.01	1.00	0.63	−0.25	1.00
	12-month Returns			12-month Returns		
S&P500	1.00	0.05	0.30	1.00	−0.64	0.91
Int-Rate	0.05	1.00	−0.01	−0.64	1.00	−0.54
Short VS	0.30	−0.01	1.00	0.91	−0.54	1.00

Table 8. Correlations between returns of short positions in variance swaps, long positions in the S&P500 index and interest rates. Short VS denotes actual, ex-post returns of the short-and-hold VS position, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$ for each day t in our sample, where τ is 2- and 12-month. S&P500 denotes actual, ex-post returns of the long-and-hold S&P500 position, i.e., $S_{t+\tau}/S_t - 1$ for each day t in our sample, where τ is 2- and 12-month. Int-Rate denotes the annualized interest rate for 2- and 12-month time horizons observed at a daily frequency. In-sample period ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

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