R&D Investment and Financial Frictions*

Oscar M. Valencia†
Toulouse School of Economics -LERNA

January 1, 2014

Abstract

R&D intensity for small firms is high and persistent over time. At the same time, small firms are often financially constrained. In this paper, we propose a theoretical model that explains the coexistence of these two stylized facts. We show that self-financed R&D investment can distort effort allocated to different projects in a firm. In a dynamic environment, we show that it is optimal for the firm to invest in R&D projects even given borrowing constraints. We find that beyond a certain threshold, there is effort substitution between R&D and production. When transfers from the investor to the entrepreneur are large enough, R&D intensity decreases with respect to financial resources. We show that conditional on survival, firms that are more innovative and financially constrained grow faster and exhibit higher volatility

Key Words: Moral Hazard, Endogenous Borrowing Constraints, Technological Change.
JEL Codes: 041,031,D86

1 Introduction

Apple, Dell and Google are examples of successful startups in mid-1970s and 1990s and currently, their revenues are comparable with the GDP of countries likes Ecuador, Croatia and Latvia[1]. They started as small firms with high intangible investment and were successful in

*I wish to gratefully acknowledge for the constant support of Francois Salanié and Andre Grimaud. I appreciate the helpful comments of Klauss Walde and the participants of Macro Workshop at TSE.

†Contact information: 21 Allée de Brienne, 31000 Toulouse, Batiment F, Bureau 004. Email: oscarmvalencia[at]yahoo.com

[1]Source: Fortune Magazine http://money.cnn.com/magazines/fortune/fortune500/2011/full_list/301_400.html and IMF data taken for 2011 Macroeconomic Outlook

getting funding from outside investors. Those examples are not isolated cases, the following empirical facts for United States show that the firm's dynamics is intrinsically linked to the development of the capital markets.

1. *Small firms exhibit higher R&D intensity*

R&D intensity is measured as the ratio between R&D investment and sales. Based on the Compustat database, Caves (1998) shows that R&D intensity is constant over time for a set of firms that are publicly traded for the period 1973-1986. This suggests that R&D intensity is independent of firm size (see Klette and Kortum (2004) for a survey). More recent studies that used an updated version of the database (1999–2007) show downturn relationship between R&D intensity and firm size. (see Akcigit (2009) and Park (2011)). For example, Akcigit (2009) estimates that a 10 rise in firm size (measured by sales) is associated with a 2.65 decrease in R&D intensity for the period 1980–2005.

Park (2011) identifies a common pattern where small firms with high R&D intensity have significant growth through joint ventures. This study shows that in the early 1970s, small start-ups do not have the means to invest in R&D. However, in the 1980s exhibit rapid growth in joint ventures, start-ups with zero revenue and high R&D investment. Small firms find it easier to attract funding, technical support and networking which facilitated investment in R&D. It should be noted that this expansion of R&D activity appeared at a time when the financial system also expanded, and this not only made it easier to find funding but also increases options for diversifying the risk associated with R&D investment.

2. *The most innovative firms are often financially constrained.*

Why do firms with high R&D intensity suffer from a lack of finance? Hall (2002) argues that it is because the return on R&D investments is highly uncertain. Private information about the quality of projects creates a lemon problem between investors and entrepreneurs and the information gap drives a wedge between external and internal finance. Moreover, R&D activities are difficult to collateralise, which means that entrepreneurs may prefer to use internal resources to fund their R&D projects.

Other source of funding, such as joint ventures, are highly volatile. Gompers and Lerner (2006) provide empirical evidence that volatility in the joint venture industry is associated with different trends in technological innovation. For example, during the economic boom from 1998 to 2000, funding was 30 times higher than the levels of 1991. Investments were mainly made in Internet (39%) and telecomunication technologies (17%). Subsequent technological revolutions generated investment opportunities that created volatility in the stock markets. It was very difficult for small- and medium-sized firms to hedge against this risk which increased financial constraints.

At the aggregate level, there is evidence of a correlation between financial constraints and firm size distribution. Cabral and Mata (2003) study the distribution of firm size and the evolution of cohorts of Portuguese firms. They find that firm size distribution is skewed at the time of start-up, although its evolution over time follows a log-normal distribution. The authors find that firms that are able to overcome financial constraints became more efficient and determine the evolution of firm size distribution.

As we argue above, the development of the financial system is very important for fostering technological growth. For instance, Gorodnichenko and Schnitzer (2010) show that financial frictions in developing economies affect innovation and firm's export activities. Using the BEEPS World Bank Survey, they carried out a cross-sectional estimate of their average quantitative impact in developing economies where the financial market is poorly developed. They find that financial constraints are negatively related to the degree of innovation at firm level.

3. *R&D expenditure is stable over time and increases for small and young firms.*

Brown et al. (2009) analyse R&D investment smoothing patterns arising from high adjustment costs (e.g., wages for highly-skilled workers and training costs). They find that, over time, R&D smoothing can be a response to higher adjustment costs when the sources of financial investment are highly variable (i.e., highly volatile cash and equity flows). They argue that firms use cash reserves to smooth R&D. In particular, they find that young firms use cash holdings to reduce R&D volatility by about 75%. This occurred in the period 1998–2002 when there was a consecutive boom and bust cycle in United States' equity markets. They document an upward trend in both cash flow and R&D expenditure from 1970 to 2006. From 1998 to 2002 equity issues and cash flow declined sharply, but R&D investment remained relatively constant. This suggests that a cash reserve acts as a buffer-stock that prevents dramatic variation in the firm's R&D investment.

We propose a theoretical model that reconciles the empirical findings outlined above. A dynamic model is set up that includes technological shocks and moral hazard in the allocation of effort between standard production and R&D activities. A distinguishing feature of our approach is that the borrowing constraint is endogenous, as it is explained by technological risk. The model is based on an entrepreneur who is cash constrained and raises external funds from an outside investor. As the division of effort between standard production and R&D is non observable by the investor, the entrepreneur faces financial constraints.

The model resembles the relationship between innovative start-up and joint venture who provide financial resources to the entrepreneur once the first prototype is developed. They agree about the ownership of the project through shares and start the standardization phase. For example, in 1970s Apple Inc starts operation for assembly-line production for Apple II. The project was established thanks to the partnership between Jobs, Wozniak and the venture capitalist Mike Markkula. Similar story has Google where search engine projects were financially and technical supported in early stages by the co-founder of Sun-Microsystem (Andreas Bechtolsheim). These are few examples that allow us to motivate the relevance of the complementarity between entrepreneurial, innovation effort and financial market development to explain the firm's growth dynamics.

Our framework is based on an entrepreneur who has two types of projects: the first is standard production, the second is R&D. In each project, they exert effort that has a certain degree of substitutability. Final production is a combination of the stock of intermediate goods and the level of entrepreneurial effort, while the rate of growth of intermediate goods is determined by the R&D effort. Therefore, the effort allocated to standard production affects current cash flow, while R&D effort affects the value of the firm's equity.

There is an outside investor who provide resources and who is repaid by the entrepreneur. The repayment is a share of total production, such as company stock. Therefore, the investor's objective is to align incentives for effort provision by controlling the entrepreneur´s cash flow. Our benchmark economy is characterised by efficient allocations under full information. There is a central planner that maximises the aggregate surplus according to resource constraints and the law of motion for intermediate goods. The efficient contract is given by the effort allocated to each activity, independent of firm size. Hence, under full information, Gibrat's Law is satisfied. As we mentioned above, this result is standard in the endogenous growth model and firm dynamics' literature, but it is not supported by recent empirical literature.

Under asymmetric information, there is a conflict of interest between the investor and the entrepreneur as the level of effort allocated to each activity is non-observable. Production of the final good is an imperfect measure of the level of effort allocated to standard production, and the entrepreneur is privately informed about R&D effort. Hence, the information asymmetry affects the investor's surplus as it has an impact on the expected present value of the entrepreneur's repayment. The investor faces a tradeoff between maximising current cash flow or maximising the value of the entrepreneur's equity.

We find that the optimal contract leads to different allocations of financial resources, depending on whether it is optimal for the entrepreneur to invest in R&D or not. When it is optimal for the entrepreneur to invest in R&D, production effort increases and is concave with respect to finance provided by the investor. In turn, R&D effort decreases and is convex when there are more financial resources available. The investor uses the repayment and continuation value (future financial transfers) to reduce the misallocation of productive resources within the firm.

Several simulation exercises are implemented to study the sensitivity of the optimal contract. We evaluate the impact of falling productivity and increasing correlation between projects. In the first case, we find that a fall in firm's productivity is associated with a reduction in both standard production and R&D intensity. This leads to tighter borrowing constraints, increases in repayments made to the investor (in order to maintain incentives to provide effort to standard production), and reduced profit for the investor.

In the second case, increases in the degree of correlation between projects has a positive effect on R&D intensity which carries a spillover effect on standard production. The number of projects funded by the investor fell, because the entrepreneur has a greater incentive to invest in R&D. In this case, the optimal repayment to the investor decreased and the positive impact on standard production led to increases in the investor's profits.

Our work also contributes to the theory of endogenous growth models. In this paper, growth is driven by increasing the number of varieties, as in Romer´s model (1996). The rate at which the number of varieties increases is given by the R&D effort, which is non observable by the investor. We then introduce technological risk into the production of varieties to obtain a balanced growth path.

We analyse the impact of productivity and task substitutability shocks on the main statistical moments of the firm, in particular expected growth and aggregate variance. In both cases, the driving force is R&D intensity. In the first case, when financial constraints are tight, the entrepreneur has a greater incentive to allocate effort to R&D, which has a positive impact on expected growth. This effect is reinforced by the impact of the optimal contract on the allocation of effort. There are two effects on aggregate variance. First, high R&D intensity is related to high firm volatility and financial resources have a negative overall impact on variance. Consequently, small firms are positively correlated with high variance and binding financial constraints.

This paper is structured as follows: Section 2 looks at the related literature. The dynamic model and the optimal contract are presented in Section 3. The implications of the optimal contract on the firm's dynamics are examined in Section 4. In Section 5 we present the main conclusions.

2 Related Literature

This paper is related to main two strands of the literature. The first concerns R&D investment, firm dynamics and financial frictions. The second is recent literature on the relationship between dynamic contracts and borrowing constraints.

Financial frictions can affect R&D by creating barriers to entry. Aghion et al. (2007) provide an empirical study of how R&D frictions affect entry and post-entry growth of firms. They find that financial constraints matter for the entry of small firms. Therefore, a reduction in these barriers fosters growth because it allows small firms to take advantage of opportunities and enable reallocation of resources in favour of the most efficient firms. While the authors find that financial development also has a positive effect on post-entry growth of small firms, it has the reverse or null effect on large firms. In our model, financial frictions arise from asymmetric information problems. The investor does not observe how the entrepreneur allocates effort between production and R&D. Financial frictions come from the misallocation of internal resources between R&D and production.

However, this effect has an important cyclical component. Wälde and Woitek (2004) find that R&D investment for G7 countries tends to be pro-cyclical. Aghion et al. (2010) analyse the implications of volatility for short and long-term R&D investment. In periods of recession, the firm's earnings decline and so does its ability to finance R&D. Consequently, R&D investment has a greater effect on firms when they are financially constrained, which can amplify productivity and output. Moreover, there is evidence from OECD countries where R&D investments are more sensitive to liquidity shocks that there is a negative correlation between volatility and growth.

Most studies analyse the implications of financial frictions for total factor productivity (TFP) dynamics. Although these papers do not study R&D decisions directly, they are useful as a benchmark to help us understand how financial frictions impact TFP and, therefore, firm dynamics. It is well known that R&D is an important component of TFP dynamics. Cooley and Quadrini (2001) study the dependence of firm dynamics on its size and age. They integrate persistent productivity shocks and financial frictions to replicate the main features of firm dynamics. Financial frictions are modelled as a premium on equity with respect to reinvesting profits. They also study the cost of debt default (i.e., costly state verification) and find that higher levels of debt are associated with higher volatility in the firm's profits.

Clementi and Hopenhayn (2006) setup a model of firm dynamics with financial frictions. In this model, market incompleteness is presented as a problem of asymmetric information between borrowers and lenders in an intertemporal setting. They study repercussions of borrowing constraints on firm's growth and survival. Borrowing constraints are modelled as a commitment problem that limits investment opportunities. The authors predict that while the conditional probability of survival would increase with the value of the firm's equity, the failure rate would decrease with firm size. Borrowing constraints are endogenous, while productivity dynamics are exogenous. Consequently, negative productivity shocks make borrowing constraints binding and this increase the cost of capital and decrease the growth of the firm.

Along the same lines, Midrigan and Xu (2010) study the extent to which financial frictions account for misallocation and, therefore, TFP losses. They find that TFP losses in emerging economies are around 5–7%. This cost corresponds to the reallocation effect. In fact, as long as the firm accumulates internal funds, it is less constrained and becomes more efficient as it avoids large swings in productivity.

Moll (2010) shows how financial constraints are less binding when idiosyncratic productivity shocks are persistent. The study finds that the ability of entrepreneurs to accumulate internal resources depends largely on the persistence of productivity shocks over time. A firm that faces a sequence of positive productivity shocks accumulate more internal resources and relax borrowing constraints. TFP losses are associated with capital market imperfections, and they depend on the autocorrelation of idiosyncratic productivity shocks. They account for about 20% of productivity losses in developing countries.

The main characteristic of our model is that TFP is endogenous and persistent due to R&D investment; furthermore, the misallocation of effort between activities limits the ability to allocate internal resources to either standard production or R&D. As demonstrated by Buera and Shin (2013), reallocation is costly and can lead to, in the case of R&D investment, higher fixed costs. Their study developed a model to analyse the implications of financial frictions on productivity. It studied at a two-sector economy in which financial frictions distort capital accumulation and entrepreneurial talent. The main feature of this model is that establishments are differentiated in terms of fixed costs. Establishments in industries with large fixed costs are more dependent on external finance, and, therefore, more financially fragile. The authors argue that this mechanism explain differences in productivity between manufacturing and services in less-developed economies.

Our paper is also related to recent literature about the relationship between dynamic contracts and borrowing constraints. DeMarzo and Sannikov (2006) study the relationship between the investor and the entrepreneur in a moral hazard environment . In this model, the entrepreneur has incentive to divert part of their cash flow for their own benefit, and therefore, reduce the firm's mean cash flow. To align incentives, the investor must either control wages and reduce the funds allocated to the project or threaten to terminate the contract prematurely. Given this setup, the optimal contract is for the entrepreneur to have a share of equity in the project and acquire a credit line so that, in the case of failure, they are able to recover part of the funds invested in the project.

In our setup, there is also an incentive for the entrepreneur to reduce effort with respect to the full information case, and thus to reduce the cash flow offered to the investor. The main difference is that the diverted cash flow is allocated to activity that is productive for the firm. However, there is a trade-off because, although there are incentives to reduce current cash flow, there is also an intertemporal effect due to the fact that R&D can increase future cash flow through its impact on growth and future equity. In our model, the optimal contract gives the investor an appropriate share of equity in the project in such a way that they can take advantage of the firm's growth opportunities.

Another relevant article is Biais et al. (2007), which analyse large-scale risk prevention contracts. This study environment of moral hazard in which the individual exerts effort to prevent large losses. The main difference with respect to the earlier literature is that the principal could alter the size of the project. Thus, they have an additional tool for aligning incentives with the entrepreneur. The optimal contract shows that the investment depends on the entrepreneur's performance history. If, over time, the entrepreneur accumulated a history of bad performance, the optimal strategy for the principal is to downsize.

In our model, the investor uses the repayment and the continuation value as tools to align incentives between activities. The optimal contract shows that, although there is a region where it is optimal for the entrepreneur to make both types of effort, the effect on R&D decreases as the investor increases the continuation value. Therefore, the investor can alter the growth rate of the project using the continuation value.

DeMarzo et al. (2012) present a model in which financing constraints arise endogenously from moral hazard between the owner and the manager of the firm. The distinct characteristic of this framework is capital accumulation. The authors find a difference between marginal and average Q that is persistent over time. The main implication of this study is that investment is positively correlated with past profitability, past investment, and past managerial compensation. In our framework, the accumulation process is based on R&D investments that increase the variety of goods. The degree of financial slack depends largely on the complementarity of tasks. If activities complement each other, it is optimal for the investor to offer a repayment plan such that the entrepreneur is incentivised to exert effort in both activities. This will relax borrowing constraints and accelerate the growth of the firm.

3 The Model

We analyse R&D investment frictions in a growth environment. Hence, in this section we establish a dynamic model of technological uncertainty. In order to characterise dynamic contracts, we follow the approach proposed by Sannikov (2008) and apply it to the case of stochastic technological growth and multiple efforts. In the first subsection, we describe the environment of the model. In the second subsection we study the first-best benchmark. We then study optimal contracts under asymmetric information.

3.1 Technology, profits, and Information

Time is continuous. At each time period, there is an infinite lived entrepreneur that allocates effort to standard production and R&D task. R&D activities stochastically increase productivity. The entrepreneur contracts with an investor to obtain financial resources.

Technology

The entrepreneur produces a homogeneous good y_t . They use as inputs a stock n_t of existing intermediate goods and their own effort e_1 like entrepreneurial effort. Formally:

$$y_t = n_t \nu e_{1,t} + \varepsilon_t \tag{1}$$

where ε_t is stochastic disturbance such that $\varepsilon_t \sim N(0, \sigma^2)$ and ν is an exogenous productivity parameter that directly affects productivity in standard production task.

We assume complementarity between effort in standard production and the stock of intermediate goods. Through R&D activities, the entrepreneur can increase the stock of intermediate goods, which are accumulated by the following technology:

$$dn_t = n_t (\eta dq + dm) \tag{2}$$

where the parameter $\eta > 0$ measures the rate at which intermediate goods are accumulated over time. The accumulation of intermediate goods has two components: First, the firm can build up them according to the Poisson process $q(t)$. The entrepreneur chooses an R&D effort e_2 in order to increase n . Increments dq are determined by:

$$dq(t) = \begin{cases} 0 \text{ with probability} & 1 - e_2 dt \\ 1 \text{ with probability} & e_2 dt \end{cases}$$

The second component is the Poisson process dm, which characterises obsolescence. In particular, a firm that produces n_t units of goods faces a hazard rate μn of becoming a firm of size $n - 1$. The loss of goods is represented by:

$$dm(t) = \begin{cases} 0 \text{ with probability} & 1 - \mu dt \\ -1 \text{ with probability} & \mu dt \end{cases}$$

This specification implies that the mean technology growth rate is determined by R&D effort, $\mathbf{E}(\dot{n}/n) = \eta e_2 - \mu$.

Information Structure and Strategies

We characterise the information environment in which the entrepreneur and investor interact. Consider the set of technological shock stochastic processes,$\mathcal{Q}$; this set contains all possible technologies that are determined in turn by all the information generated by its path $\mathcal{Q} = \{q_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ defined in a space $(\Omega, \mathbf{F}, \mathbf{P})$. Here, Ω is the sample size, $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denotes the information set and $\mathbf{P}$ is the probability measure that follows a Poisson process with intensity e_2.

One strategy for the entrepreneur is to allocate effort to standard production and R&D $e_i \in E$ for $i = 1, 2$. The entrepreneur's effort is itself a stochastic process $e_i = \{e_{i,t} \in E; 0 \leq t \leq \infty\}$

and measurable with respect to $\mathcal{F}_t$. This means that, based upon the path of $\mathcal{Q}$, it is possible to determine effort e_i.

Strategies for the investor are defined in terms of a repayment function in which there is also a stochastic process $\psi = \{\psi_t \in \Psi; 0 \leq t \leq \infty\}$ where ψ_t is determined by the observed output $\psi_t\,(y_j; 0 \leq j \leq t)$.

Profits

The entrepreneur is risk-neutral and expected discounted profits are given by:

$$\pi^E = E\left[r\int_0^\tau \exp\left(-rs\right)\left[\left(\left(1-\psi_s\right)y_s - nc\left(e_{1,s}, e_{2,s}\right)\right)ds\right] + \exp\left(-r\tau\right)R\right] \tag{3}$$

where total production y_t is given by [1]. ψ is the repayment from the entrepreneur to the investor. We assume that the repayment is a share of total output y, such as shares. $c\left(e_{1,t}, e_{2,t}\right)$ represents the unit cost effort for both activities, where $c(e_1, e_2)$ is defined as:

$$c\left(e_1, e_2\right) = \frac{1}{2}\left(e_1^2 + e_2^2\right) + \gamma e_1 e_2 \tag{4}$$

with $\gamma < 1$. When $\gamma = 0$, activities are independent; when γ is high, activities are highly correlated. Once the contract is terminated, the entrepreneur receives a payoff $R \geq 0$ from an external party.

The investor is risk-neutral, and derives profit from the discounted repayment that they expect to receive from the entrepreneur. Note that the investor only observes aggregate output, which is an imperfect measure of the entrepreneur's level of effort.

The investor's profits are represented by:

$$\pi^I = E\left[r\int_0^\tau \exp\left(-rs\right)\left(\left(\psi_s y_s\right)ds\right) + \exp\left(-r\tau\right)L\right] \tag{5}$$

where L is the expected value liquidation value of the project's assets. Therefore the contract specifies the repayment to investor ψ and the termination stopping time $\tau \geq 0$. If the business is liquidated, the investor receive a scrap value L. Timing is explained as follows:

Timing

At time 0

- The investor propose a contract $\left[(y_{t'})_{0\leq t'\leq t}\mapsto\psi_t\right]_{0\leq t\leq\infty}$ to the entrepreneur.

- The entrepreneur either accepts or refuses the contract. If the contract is rejected, the game ends and there is no production.

- Entrepreneur establish a path of effort in production e_1 and R&D e_2

At each t

- The level of output is realized y .

- Entrepreneur makes a transfer to the investor according to the level of output $\psi(y)$.

3.2 The First-Best

In this subsection, we study the case in which the level of effort given to each activity is observable and verifiable by the investor. The social surplus is the expected discounted profits by the entrepreneur and investor. Therefore, the first-best allocation is the solution to the following problem:

$$V=\max_{e_1,e_2}E\left[r\int_0^\tau\exp(-rs)\left(y_s-nc\left(e_{1,s},e_{2,s}\right)\right)ds+\exp(-r\tau)(R+L)\right]$$

subject to the law of motion of accumulation of goods[2]. Using the Change Variable Formula for a Poisson process Walde (2008), the Bellman equation of this problem is :

$$rV(n)=\max_{e_1,e_2}r\left[y-nc\left(e_1,e_2\right)\right]+e_2\left[V((1+\eta)n)-V(n)\right]+\mu n\left[V(n-1)-V(n)\right]\tag{6}$$

The first term represents current profit, while the second and third terms measure how the social surplus changes when there is a "technological jump". We first assume that there are interior solutions, which gives first order conditions:

$$[e_1]\quad\nu=c_{e_{1,t}}\left(e_{1,t},e_{2t}\right)\tag{7}$$

$$[e_2]\quad\frac{V((1+\eta)n)-V(n)}{n}=c_{e_{2,t}}\left(e_{1,t},e_{2t}\right)\tag{8}$$

First-order condition [7] means that the marginal product of providing effort in production must be equal to the marginal cost, which is constant. The second first-order condition shows that, at the margin, one unit of R&D effort must be equal to the marginal gain for the entrepreneur of increasing the stock of goods per unit of input. This marginal gain is measured as the difference between intertemporal profits when the entrepreneur invests in R&D and the case where there is no innovation.

We now consider the case of corner solutions: Suppose first that $e_1 \neq 0, e_2 = 0$. In this case, the level of effort $e_1 = \bar{e}_1$ is constant for all $s < \tau$. Note that in the other case $e_1 = 0, e_2 \neq 0$, there is no incentive to produce intermediate goods since the total production is zero. The following proposition shows the socially efficient contract under full information.

Proposition 1 : *If $\{e_1^*, e_2^*\}$ is interior the solution to the program [6], then $V^{both} = f(e_1^*, e_2^*)\, n$ where $f(e_1^*, e_2^*) = \max_{e_1,e_2} \frac{r\left(\nu e_1^* - c\left(e_1^*, e_2^*\right)\right)}{r - \left(\eta e_2^* - \mu\right)}$ and the Gibrat law holds (i.e R&D intensity is independent of the firm size).*

This proposition uses the property of homogeneity of degree one in the profits of the entrepreneur and the investor in such way that the social surplus is linear with respect to the number of innovations. Based on this assumption and a frictionless environment, we arrive at Gibrat's Law, as in Klette and Kortum (2004).

From Proposition 1, we know that R&D intensity and effort are independent of firm size. However, we now look at the level of effort given to different activities. How does the allocation of effort between activities distort the intertemporal margins? Let us take as an example the cost function [4]; we can subtract [7] and [8] to get:

$$\frac{V((1+\eta)\, n) - V(n) - (1-\psi_t)}{n} = (1+\gamma)\,(e_2 - e_1)$$

Therefore, the efficient allocation of effort given to each activity depends on the difference between the marginal gain of the innovation and the share of output that is consumed by the entrepreneur. If R&D is profitable, then R&D effort efficiency is greater than the efficient level of production. This result is standard in a Schumpeterian endogenous growth model where there is over-investment in R&D.

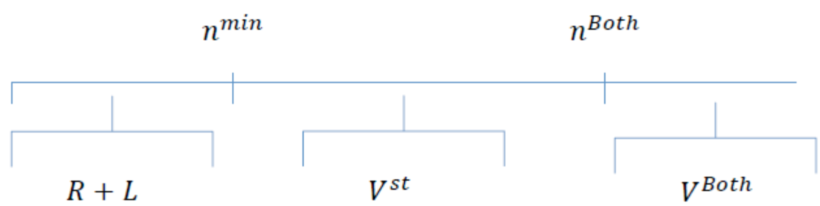
Consider the corner solutions where there is contract termination. This is the case when the entrepreneur only provides effort in standard task. That means $e_1 = \nu; \quad e_2 = 0$. It implies that value function is given by $V^{st} = n\frac{v^2}{2}$. The contract termination is given by the following condition:

$$V^T = \min_{n^{min}} \left\{ V^{st}, R + L \right\}$$

Where n^{min} is a threshold that determines the firm minimum scale level to operate in the market. Beyond that threshold the social return is $R + L$, the reservation value for the entrepreneur and the investor´s liquidation value. There is other firm size threshold that determines the incentives for the firm to provide effort in both activities n^c . The value of continuation is given by:

$$V^C = \max_{n^c} \left\{ V^{st}, V^{both} \right\}$$

Figure 1: Optimal Stopping



The Equilibrium

An equilibrium in this economy is a collection of stochastic processes $\{y_t, \psi_t, e_{1,t}, e_{2,t}\}_{t \geq 0}$ that are $\mathcal{F}_t$ - adapted and a r such that in each time period t:

- Given r, ψ the entrepreneur maximize his profits.
- The monopolistic investor chooses the repayment ψ such that they maximize their profits.

and limited liability condition:

- $\psi(y) \leq y$

In the next subsections we consider the case in which the entrepreneur holds private information about the level of effort. We characterise the optimal decisions of each agent and the optimal contract.

3.3 Moral Hazard

We now consider the case in which the level of effort given to each activity is non observable. Therefore, the problem that the investor solves is to choose the repayment such that it implements levels of effort that are incentive-compatible with the entrepreneur's decisions, in such a way that the entrepreneur's discounted profits are maximised. This can be described as follows:

$$\pi^I = \max_{\psi_t,\tau} E\left[r\int_0^\tau \exp\left(-rs\right)\left(\left(\psi_s y_s\right)ds\right) + \exp\left(-r\tau\right)L\right]$$

Subject to the incentive compatibility constraint:

$$\left(e^*_{1,s}, e^*_{2,s}\right) = \arg\max_{\hat{e}_{1,s},\hat{e}_{2,s}} E\left[r\int_0^\tau \exp\left(-rs\right)\left[\left(\left(1-\psi_s\right)y_s - nc\left(\hat{e}_{1,s},\hat{e}_{2,s}\right)\right)ds\right] + \exp\left(-r\tau\right)R\right]$$

and the promise-keeping condition in which the investor delivers to the entrepreneur a certain level of utility in order to incentivise them to participate in the contract.

$$E\left[r\int_0^\tau \exp\left(-rs\right)\left[\left(\left(1-\psi_s\right)y_s - nc\left(e_{1,s},e_{2,s}\right)\right)ds\right] + \exp\left(-r\tau\right)R\right] \geq \widehat{W} \tag{9}$$

To analyze the optimal contract we follow the approaches of Spear and Srivastava (1987), Abreu et al. (1986), and more recently, Sannikov (2008). The general idea is that the principal uses the continuation value as a contractual instrument to keep track of incentives for the agent. Therefore, the contract can be written according to the continuation value, W_t, which acts as a state variable for the principal.

Optimal contract allocations depend on historical information because the principal needs to track the whole history of technological shock in order to infer information about the agent's effort. Under the promised utility framework, represents the main statistics of the contract, such as the level of effort given to each activity, the output in each period, transfers from the investor to the entrepreneur, and variation with respect to the different realizations that the technology takes. The principal uses the continuation value W_{t+dt} as a control variable, takes W_t as given and turns W_t into a new endogenous state variable. In this way, the problem can be written in recursive form and standard dynamic programming techniques can be applied.

Following the Sannikov procedure, the first step is to study the continuation value as a diffusion process. This allows us to characterise the dynamics of the promised utility.

Second, we establish the conditions for incentive-compatible allocation of efforts. Third, using the previous input, we can rewrite the problem of the investor in recursive form and study the intertemporal conditions of the optimal contract.

3.3.1 The continuation value of the entrepreneur

The contract specifies a flow of repayments $\{\psi_t, 0 < t < \infty\}$, and a sequence of efforts, $\{e_{1,t}, e_{2,t}; 0 < t < \infty\}$. We define the continuation value of the entrepreneur's profits when $\hat{e}_1, \hat{e_2}$ are adopted :

$$W_t = E_{\hat{e}_{1,t},\hat{e}_{2,t}}\left[r\int_t^\tau \exp\left(-r\left(s-t\right)\right)\left[\left(\left(1-\psi_s\right)y_s - n_s c\left(e_{1,s}, e_{2,s}\right)\right)dt\right] + \exp\left(-r\left(\tau-t\right)\right)R \mid \mathcal{F}_s\right] \tag{10}$$

In order to characterise the dynamics of the promised utility, we use the standard techniques of the martingale representation theorem. This is an equivalence theorem that states that every martingale can be represented by an alternative process. In our context, we extend this theorem for a Poisson process (see Biais et al. (2010) and Bjork (2011)). We first show that the expected discounted profit of the entrepreneur is a martingale and then, that it can be represented by an alternative process. Based on this equivalence, we express the expected profit as a function of the promised profit, and therefore obtain an expression of the evolution of the continuation profit. The next two results show the application to our particular problem.

Lemma 1 : *π^E defined by [3] is a $\mathcal{F}_t$- adapted martingale.*

Using this result we can characterise the continuation value as a diffusion process:

Proposition 2 : *Let q and m be two independent Poisson process and let $p = q + m$ be also a Poisson process with intensity $e_2 - \mu$ that admits the following martingale representation:*

$$dN_s = dp - \left(e_{2,s} - \mu\right)ds \tag{11}$$

then there exists a measurable process $h = \{h_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ such that:

$$dW_t = r\left(W_t - \left(1-\psi_t\right)y_t + nc\left(e_{1,t}, e_{2,t}\right) - h_t\left(e_{2,t}-\mu\right)\right)dt + rh_t dp \tag{12}$$

The continuation value grows with the entrepreneur's discounted rate and decreases with their net current profits. The technological innovation affects the dynamics of the continuation utility in two ways. First, there is a negative effect that comes from the effort given to R&D, $h_t e_{2,t}$. Alternatively, there is a positive effect as more innovations increase the number of goods. This is captured by $\frac{1}{\eta}\left(dn_t/n_t\right) = dq$, where the stochastic component is given by the amount of goods that the firm accumulates.

Given the structure of the cost function, R&D effort affects not only the creation of goods in the economy, but also generates a 'crowding-out' effect with respect to production effort. Therefore, as R&D effort raises, the arrival rate of new goods increases, while the level of effort given to current production decreases. The factor h_t measures the responsiveness of the continuation value to technological uncertainty. As Biais et al. (2010) explain, this factor represents the "minimum penalty" that provides an incentive to the agent to exert R&D effort.

3.3.2 Incentive Compatibility

Based on changes in the continuation value, we can characterise the best response effort from the entrepreneur. That means that we can measure how the continuation value varies with each level of effort. Let us define the cost functions denoted by $c(e_1, e_2)$ for several cases. When the entrepreneur exerts high effort in both activities, the cost function is given by $c(e_1, e_2) = c_1$; when they only apply effort to production, the cost function is $c(e_1, \tilde{e}_2) = c_2$. Alternatively, it is $c(\tilde{e}_1, e_2) = c_3$ when the entrepreneur gives all their effort to R&D. In the case where the agent exerts no effort in either production or R&D, the cost is $c(0,0) = c_0 = 0$ and is normalised to zero.

Allocations that are incentive compatible imply two conditions: first, they require local incentive constraints that state that the entrepreneur receives more profits when they exert high effort compared to the case where they only exert high effort in one of the two activities:

$$E_{e_{1,t},e_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[\left(1-\psi_t\right)\nu e_1 - c_{1,t}\right]ds\right] \geq E_{\tilde{e}_{1,t},e_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[\left(1-\psi_t\right)\nu\tilde{e}_{1,t} - c_{2,t}\right]ds\right]$$

and

$$E_{e_{1,t},e_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[(1-\psi_t)\,\nu e_1 - c_{1,t}\right]ds\right] \geq E_{e_{1,t},\tilde{e}_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[(1-\psi_t)\,\nu \tilde{e}_1 - c_{3,t}\right]ds\right]$$

Secondly, these allocations require a global incentive constraint under which the entrepreneur's profits from providing high effort in both activities are higher than if no effort had been exerted in either of them:

$$E_{e_{1,t},e_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[(1-\psi_t)\,\nu e_1 - c_{1,t}\right]ds\right] \geq E_{\tilde{e}_{1,t},\tilde{e}_{2,t}}\left[r\int_0^\infty exp(-rs)n_t\left[(1-\psi_t)\,\nu \tilde{e}_1 - c_{0,t}\right]ds\right] = 0$$

The following proposition characterizes the incentive compatibility constraint:

Proposition 3: *The pair of effort (e_1, e_2) is incentive compatible if and only if :*

$$\frac{h}{r}e_2 - nc_1 \geq \frac{h}{r}\tilde{e}_2 - nc_2 \tag{13}$$

and

$$(1-\psi)\,\nu e_1 - c_1 \geq (1-\psi)\,\nu\tilde{e}_1 - c_3 \tag{14}$$

Proposition 3 shows that incentive compatibility includes not only current profit but also the sensitivity of R&D intensity to the continuation value of the entrepreneur's profit. Therefore, a strategy (e_1, e_2) is optimal if, and only if, in each time period the agent maximises current profits and the expected impact of R&D intensity on the continuation value.

Incentive constraint [13] shows that the expected gain for an entrepreneur following strategy e_2 is higher than for other strategies. This incentive constraint is affected by the sensitivity factor that measures the impact of the frequency of innovation on profits. The second incentive constraint measures the production margin. The net profit from exerting high production effort is greater than exerting low effort in every time period. Adding these two constraints together, we obtain the following expression:

$$\frac{h}{nr}\left(e_2 - \tilde{e}_2\right) + (1-\psi)\,\nu\left(e_1 - \tilde{e}_1\right) \geq c_3 - c_2 \tag{15}$$

The aggregate incentive constraint shows that the expected gain for the entrepreneur from production and R&D is higher that the opportunity cost of performing only one of the tasks. As shown by Sannikov (2008), the set of incentive compatibility constraints satisfies:

$$h = nrc_{e_2}\left(e_1, e_2\right)$$

and

$$\left(1-\psi\right)\nu = c_{e_1}\left(e_1, e_2\right) \tag{16}$$

3.3.3 Recursive Representation and Optimal Contract

In this subsection we describe the investor strategy outlined in Section 3.2 in recursive form. Based on previous results, we use the continuation value as a state variable for the investor. The characterization of the evolution of the state variable shows that it only depends on current variables. The investor's objective is to obtain the highest profit $\pi^I\left(W\right)$ while delivering a level of W to the agent. Consequently, the investor's problem can be expressed as current profits plus expected discounted profits $\frac{1}{dt}E_t d\pi^I\left(W\right)$. Using equation [12] and the Change Variable Formula for a Poisson process, we compute [12] as follows:

$$Ed\pi^I\left(W_t, n_t\right) = r\pi^I_w\left[W-\left(1-\psi\right)y + nc\left(e_1,e_2\right) - he_2\right]dt + e_2\left[\pi^I\left(\widetilde{W}\right) - \pi^I\left(W\right)\right]$$

Therefore, the Bellman equation for the investor is given by

$$r\pi^I\left(W,n\right) = \max_{e_1,e_2,\psi} r\left(\psi y\right) + r\pi^I_w\left[W-\left(1-\psi\right)y + nc\left(e_1,e_2\right) - he_2\right] + e_2\left[\pi^I\left(\widetilde{W}\right) - \pi^I\left(W\right)\right] \tag{17}$$

where $\widetilde{W} = W + rh$.

The model has the property of a constant return to scale, therefore:

$$\pi^I\left(W,n\right) = n\pi^I\left(\frac{W}{n},1\right) = n\pi^I\left(z\right)$$

where $z = \frac{W}{n}$. We can express the Bellman equation as:

$$r\pi^I\left(z\right) = \max_{e_1,e_2,\psi} r\left(\psi e_1\nu\right) + r\pi_z\left[z-\left(1-\psi\right)e_1\nu + c\left(e_1,e_2\right) - he_2\right] + e_2\left[\pi^I\left(\widetilde{z}\right) - \pi^I\left(z\right)\right] \tag{18}$$

with the following boundary conditions:

$$\pi_z(\bar{z}) = -1, \pi_{zz}(\bar{z}) = 0$$

where $\bar{z}$ upper bound of state z

The first term in the Bellman equation corresponds to the current reward (net repayment). The second term captures the expected discounted value of the profit, which in turn has been decomposed into drift in the entrepreneur's continuation value and the effect of the technological jump on the investor's profits. The drift in the entrepreneur's continuation value impacts the marginal value of the investor profits, which is affected by the difference between the delivered and current profits. If the entrepreneur's current profits are high enough, the impact of delivering one unit of continuation value to the entrepreneur decreases the investor's profits.

R&D effort affects the investor's profits in two ways. Firstly, it can negatively affect the investor's marginal profits since it reduces the provision of effort in standard production for the entrepreneur . The second, positive effect is that increases the investor's profits when there is a technological jump.

To solve the Bellman equation, we assume that in the event of a technological jump, the investor's profit follows:

$$\pi^I(\tilde{z}) = \pi^I(z) + \phi(z)\, dq \tag{19}$$

where $\phi(z)$ is a sensitivity parameter with $\phi_z(z) > 0, \phi_{zz}(z) < 0$ and which satisfies the Inada conditions $\lim_{z\to 0}\phi_z(z) = +\infty$ and $\lim_{z\to+\infty}\phi_z(z) = 0$.

Equation [19] shows that in the event of a technological shock, the investor's profits are increased by stochastic disturbances to the continuation value. $\phi(z)$ measures sensitivity to changes in the continuation value of investor's profits. Thus, when there is a technological shock, the investor faces decreasing marginal returns on profit.

Expected increments in the investor's profits are calculated as follows:

$$\mathbf{E}\left[\pi(\tilde{z}) - \pi(z)\right] = \phi(z)\, e_2 dt \tag{20}$$

Expected investor's profits in a short period of time are proportional to the level of R&D effort and the sensitivity factor. To characterise the optimal contract, we look at the case where allocations are incentive compatible, i.e., when the entrepreneur has incentives to exert high effort in both activities e_1^*, e_2^*. Hence, the optimal contract analyses how much of the entrepreneur's continuation profit the investor delivers to the entrepreneur given that the entrepreneur provides high effort and participates in the contract. The optimal contract implies that there is a trigger strategy where there are several thresholds of continuation utility in which the agent is incentivised to exert high effort in only one, or both activities.

Let $\pi^{prod}(z), \pi(z),$ *and* $\pi^{R\&D}(z)$be the level of investor profits, in the cases when the entrepreneur provides either only production effort, effort to both activities, or only R&D effort. The following two propositions characterise the optimal contract:

Proposition 4: *The optimal contract is a set of* $e_i, i = 1, 2,$ ψ *and continuation values* $z = \frac{W}{n}$, *such that these are solution to [17] and there is a threshold* $\hat{z}$ *that satisfies:*

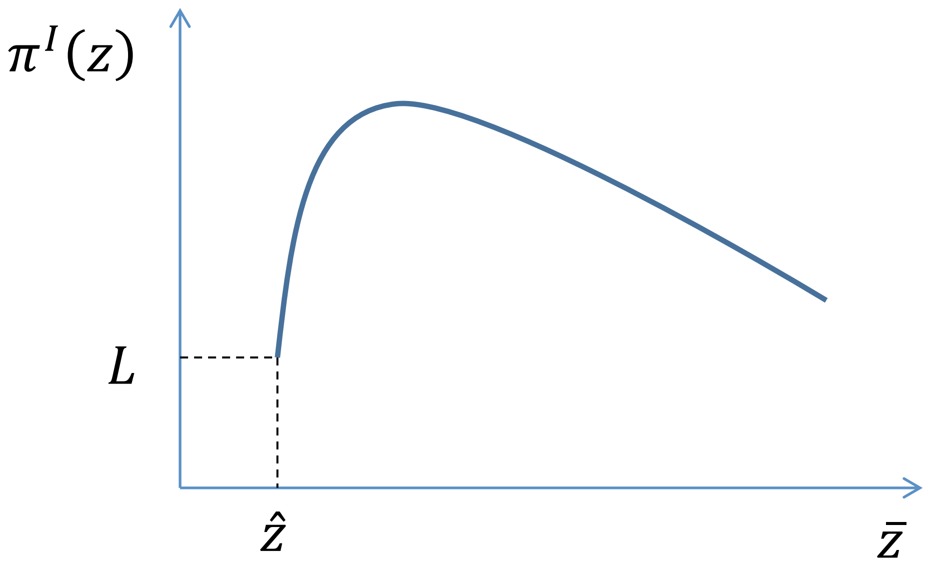
$if \quad z > \hat{z} \quad then \quad e_1^*$, *is increasing and concave with respect to* z *in* $[\hat{z}, \bar{z}]$, e_2^*, ψ^{sb} are decreasing and convex *with respect to* z *in* $[\hat{z}, \bar{z}]$.

In all other cases, investment only in production or in R&D is suboptimal $\pi^{prod}(z) < \pi(z)$ $\pi^{R\&D}(z) < \pi(z)$. *The thresholds are determined by the indifference points.*

Proposition 5 : Let be $\pi(z)$ *a continuous and differentiable function in* $(\hat{z}, \bar{z})$, $\pi(z)$ *is concave with respect to* z *in* $[\hat{z}, \bar{z}]$.

In the interval $[0, \hat{z})$, the investor obtains negative profits. Therefore $[0, \hat{z})$ is an inaction region that determines the free-entry condition of the problem. The relevant region for the entrepreneur is $(\hat{z}, \bar{z}]$ where effort is exerted in both activities. As stated in Proposition 5, the investor's profits are concave. As Figure 2 shows, initially, profit must cover the scrap value. The shape of the curve depends on the magnitude of the misallocation between R&D and production.

Figure 2: Investor´s Value Function

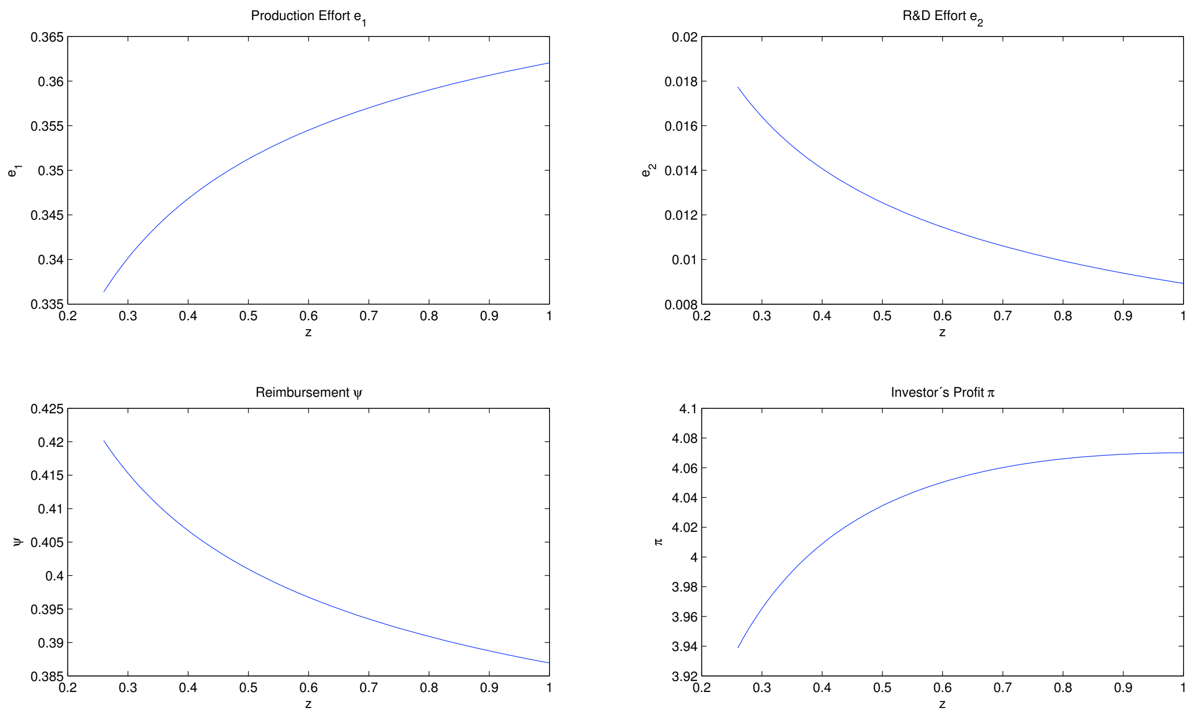


In Figure 3 we parametrised a standard optimal contract. The optimal contract requires that the entrepreneur provides a monotonically increasing level of effort in production but there are marginal decreasing returns as the continuation value increases. R&D effort has the opposite effect, i.e., low continuation values are associated with high R&D intensity. The degree of misallocation increases as the continuation value rises.

The optimal repayment imposed by the investor captures potential changes in effort levels of the entrepreneur. It implies that the investor creates incentives over time, assigning rents to the entrepreneur to generate high levels of effort. Furthermore, as repayments decrease, the entrepreneur invests more resources in R&D. As we can see in 3, beyond a certain value of $\hat{z}$, the entrepreneur has the resources to increase their innovative activity. However, the optimal contract shows that substitution effects between activities lead to a decrease in R&D intensity over time.

When the investor is interested in the effort expended in both activities, the continuation value will affect not only the entrepreneur's current profits, but also their future profits through R&D investment. The continuation value will affect the sensitivity factor that alters R&D effort and, therefore, the entrepreneur's profit. The sensitivity factor captures the opportunity cost for the entrepreneur of exerting R&D effort (or not doing so).

Figure 3: Typical Optimal Contract: $r = 0,20$;$\gamma = 0.5$; $\nu = 0.6$



4 Firm Dynamics

In this section we analyse the impact of financial frictions on firm dynamics. We study the potential impact of misallocation on the firm's growth rate and variance. We use the framework developed in Klette and Kortum (2004).

We study the evolution of the firm through the dynamics of the stock of goods n. The investor's financing decisions affect the firm's growth rate through the impact on R&D intensity. For now, we consider that an entrepreneur takes as given the financial resources provided by the investor. The firm changes the stock of goods over a time interval $\triangle t$ in accordance with the following continuous time Markov chain:

1. $\Pr\left(z(t+\triangle t) - z(t) = z + k \mid z(t)\right) = z\eta e_2\left(z\right)\triangle t + o\left(\triangle t\right)$
2. $\Pr\left(z(t+\triangle t) - z(t) = z - k \mid z(t)\right) = z\mu\triangle t + o\left(\triangle t\right)$
3. $\Pr\left(z(t+\triangle t) - z(t) > z + k \mid z(t)\right) = o\left(\triangle t\right)$

4. $\Pr\left(z(t+\triangle t)-z(t)=0 \mid z(t)=i\right)=1-z\left(\eta e_2-\mu\right)\triangle t+o\left(\triangle t\right)$

To characterize the life cycle of the firm over time, we calculated the forward-looking Kolmogorov differential equation. This equation describes the probability distribution of firm size over time:

$$p_z\left(t+\triangle t \mid z_0\right)=\begin{array}{c}\left(z-k\right)p_{z-k}\left(t \mid z_0\right)e_2\triangle t+\left(z+k\right)p_{z+k}\left(t \mid z_0\right)\mu\triangle t \\ zp_z\left(t \mid z_0\right)\left(1-\left(\eta e_2-\mu\right)\triangle t\right)+o\left(\triangle t\right)\end{array}+$$

Taking the limit when $\triangle t\to 0$ we obtain the following expression:

$$\dot{p}_z\left(t \mid z_0\right)=\begin{array}{c}\left(z-k\right)p_{z-1}\left(t \mid z_0\right)e_2+\left(z+k\right)p_{z+k}\left(t \mid z_0\right)\mu \\ zp_z\left(t \mid z_0\right)\left(1-\left(\eta e_2-\mu\right)\right)\end{array}+ \tag{21}$$

A firm of size $z-k$ grows to size z with probability $\left(z-k\right)e_2$. There is a probability $n\mu$ that a firm with size $z+k$ downsizes from z to $z-k$. Similarly, the probability that a firm does not innovate and remains the same size is given by $z\left(1-\left(e_2-\mu\right)\right)$. The following lemma presents the main statistical moments for the firm:

Lemma 2 (Klette and Kortum (2004)): *Given z, the expected rate of firm's growth and variance are given by:*

$$g\left(z\right)=\max\left\{0,\exp\left[\left(e_2\left(z\right)-\mu\right)t\right]-1\right\} \tag{22}$$

$$var\left(z\right)=\frac{e_2\left(z\right)+\mu}{e_2\left(z\right)-\mu}\exp\left[\left(e_2\left(z\right)-\mu\right)t\right]\left[\exp\left(e_2\left(z\right)-\mu\right)t-1\right] \tag{23}$$

We can now relate the dynamics of the firm to access to credit using the properties of the optimal contract. We measure the firm's financial constraints as $\mid \hat{z}-\bar{z} \mid$.

The question that arises is how substitution between standard production and R&D affects the firm's growth rate? To answer this, we study the dynamics of the growth rate over time. We start by assuming that it follows an exponential distribution. This assumption is reasonable in the sense that in the long run, firms tends to disappear. According to the optimal contract, beyond $\hat{z}$, the firm does not grow because the level of R&D intensity is zero. $\overline{z}$ defines the maximum resources provided by the investor and is taken exogenous. In the following proposition we study the firm's growth and variance at different stages of financing.

Proposition 6: *Conditional on survival, a firm is more financially constrained, grows on average faster and exhibits higher variance.*

The proposition shows that the threshold $\hat{z}$ changes with variations in the investor's continuation value z. Changes in this threshold determine the entrepreneur's incentives. Increasing the threshold generates incentives for the entrepreneur to devote more resources to R&D than standard production. This has a positive effect on the average growth rate because higher R&D has a positive effect when financing is scarce.

On the other hand, changes in z have two impacts on variance:$\frac{\partial var(z)}{\partial z} = \frac{\partial var(z)}{\partial e_2(z)}\frac{\partial e_2(z)}{\partial z}$. The first is that R&D intensity is positively related to individual variance. As it is shown in proposition 6, the R&D intensity is positive related to the individual variance. Second, based on the optimal contract, when there are more financial resources, the final effect is amplified. This effect is maintained at the level of the firm when is added over the range of all financial resources.

4.1 Numerical exercises and Comparative statics

In this subsection we study the sensitivity of the optimal contract to negative productivity shocks ν and changes in the substitutability parameter γ. In particular, we look at how idiosyncratic shocks affect the firm's borrowing constraints. Parameters are based on Compustat data at the level of firm for the period 1980–2007. Table 1 shows the main parameters used in the simulations which are set according with the following the first order conditions in steady-state to calibrate an average economy (see table 2).

$$z^{ss} = r\nu \left(e_1^{ss} + \frac{1}{1-r} \right)$$

$$\gamma^{ss} = \frac{e_1^{ss}}{e_2^{ss}}(1+\nu) - \frac{\nu}{e_2^{ss}}\left(\frac{r}{1-r}\right)$$

$$\psi^{ss} = \frac{1}{1-r} - e_1^{ss}$$

Table 1: Benchmark Parameters

γ	ν	r	ψ^{ss}
0.5	0.03	0.04	39%

Table 2 presents average values for a representative firm for the following: production effort (e_1), R&D intensity (e_2), repayments (ψ), growth $E\left(g^A\right)$ and variance var^A. The 'Data' row of Table 2 specifies the benchmark values used in the exercise. Production effort is set to be the contribution of capital to total output for the period of the sample. R&D intensity, growth and variance are averaged values for the period 1980–2007.

Table 2: Average Moments of the Model

	e_1^{ss}	e_2^{ss}	$E\left(g^A\right)$	var^A
Data	34%	3.1%	1.83%	1.45%

Next, we study the impact of the optimal contract when there is a negative idiosyncratic shock on productivity (ν) and when there is a positive shock on the substitutability parameter (γ) .

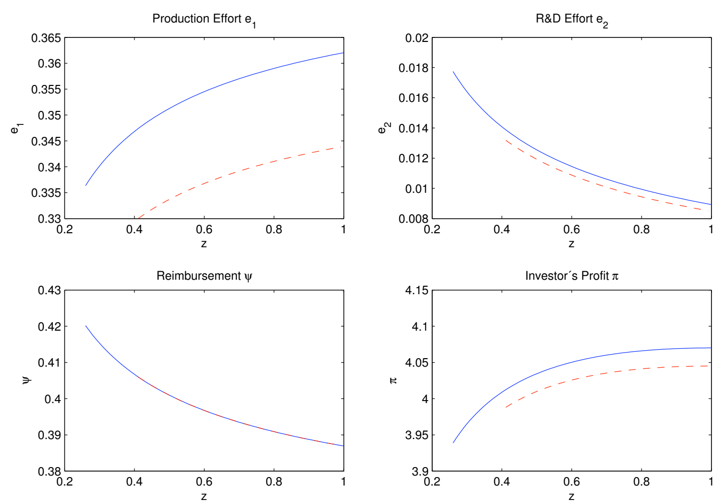
4.1.1 Productivity Shocks

Let us consider a firm that receives a negative productivity shock ν. The following proposition characterises the impact of comparative statics on financial constraints.

Proposition 7: *A negative productivity shock exacerbates misallocation and credit constraints within the firm .*

A negative productivity shock leads to a reduction in the optimal levels of effort given to both production and R&D. As we can see from Proposition 7, idiosyncratic changes in productivity proportionally affect policy functions e_1, e_2. The effect on reimbursement is cancelled out since it is the remainder of the effort allocated to production and R&D. The optimal reimbursement follows a similar shape to the non-shock case but in the range $(\hat{z}^{shock}, \bar{z}]$. Consequently, the investor‘s optimal level of profit is reduced and as $\hat{z}$ increases, the financial resources available to the entrepreneur are reduced . In Figure 4 the dashed blue line represents the benchmark and dotted red line shows the shock. The impact of a 5% fall in the firm's productivity leads to an average reduction by 5.06% in production effort; furthermore there is a clear effect on R&D intensity, which is reduced by an average of 4.35%. The reimbursement rate decreases slightly by around 1%. While the profit curve is shifted to the right and is reduced by 0.63%.

Figure 4: Optimal Contract and Negative Productivity Shock $r = 0,20;\gamma = 0.5;\ \nu = 0.6;$ $\nu^{shock} = 0.57$



4.1.2 Substitution effect between standard production task and R&D.

The impact of shocks on the substitutability parameter γ is shown in Figure 5 and Figure 6. When activities are less independent (i.e., rises by 5 %), they are more correlated (see Figure 5). R&D intensity increases by 4.68% and generates a spillover effect on production, which increases to 4.54%. As R&D intensity rises, the borrowing constraint is binding and fewer resources are devoted to financing entrepreneur's projects. The optimal repayment decreases as γ increases and the best strategy for the investor to reduce repayments to increase the correlation between R&D and production. This shock provides more information about the provision of effort through y. Information asymmetry is reduced and entrepreneur's output increases. The optimal contract is a powerful incentive to put effort into standard production and leads to an increase in the investor's profits.

Figure 5: Effects of a Shock of Substitutability Between Activities $r = 0,20$;$\gamma = 0.65$; $\nu = 0.6$; $\gamma^{shock} = 0.6825$

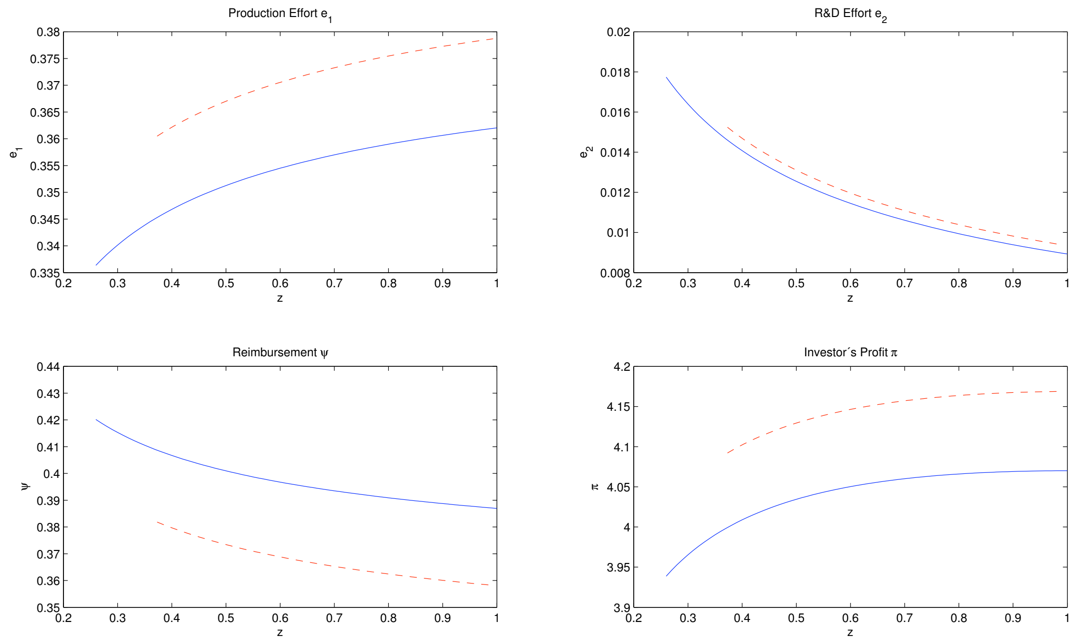
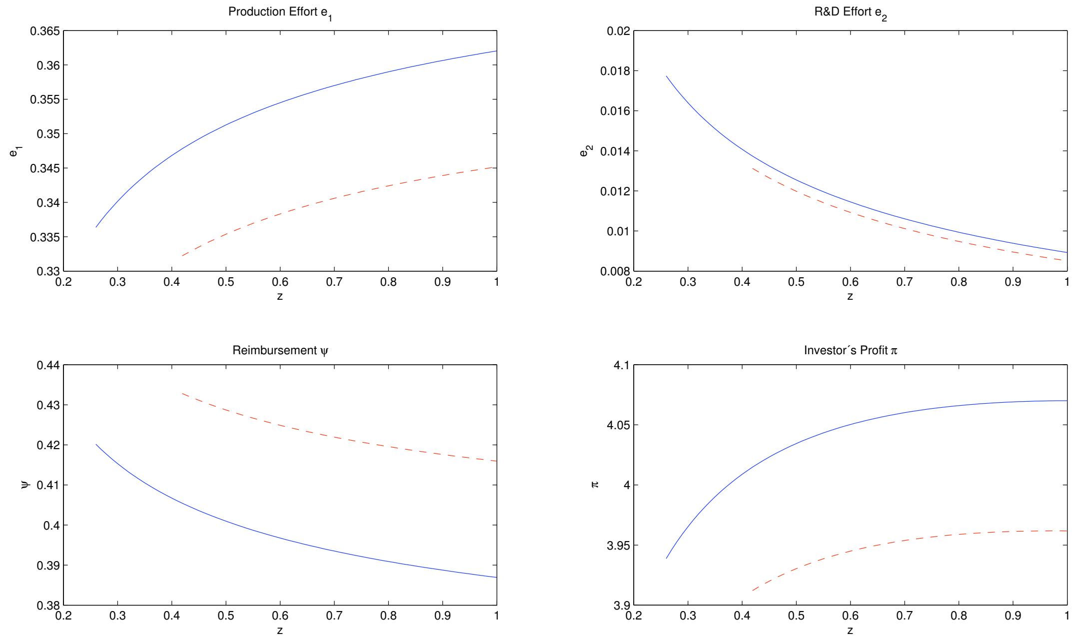


Figure 6 shows the impact when there is high substitutability ($\gamma = 0.6175$) between standard production and R&D. In this case the effect is reversed; lower R&D intensity means less production effort and final output drops. In Figure 6, effort given to standard production falls to 4.6 % while R&D intensity is reduced to 4.54%. In this case, the optimal repayment must incentivise production. In fact, in our example, the repayment increases until it reaches 7.20% and profits are reduced by 2.6%. In this case, the investor left to finance 21.3% of total projects, while in the case of high substitutability they finance 83% of projects.

Figure 6: Effects of a Shock of Substitutability Between Activities $r = 0,20$;$\gamma = 0.65$; $\nu = 0.4$; $\gamma^{shock} = 0.6175$.



4.1.3 Combined Shock: Negative productivity shock ν and less substitutability between tasks γ.

Here, we evaluate the impact when there is a contraction in productivity (a reduction in of 5%), but at the same time projects are highly correlated (increases γ by 5%). In this case we find that R&D generate a spillover effect that outweighed the negative effect of the contraction. Nevertheless, the entrepreneur's repayments are reduced because of lower production in the standard task and the increase in R&D.

The overall impact is that effort declines by around 0.73%. R&D also falls by 0.28% while the repayments to the investor required to retain their interest in the project are reduced to around 7.04%. The investor's profit increases as the positive impact on total output is higher than the fall in the reimbursement rate.

Figure 7: Combined Shock: Negative productivity shock and Less Substitutability between tasks

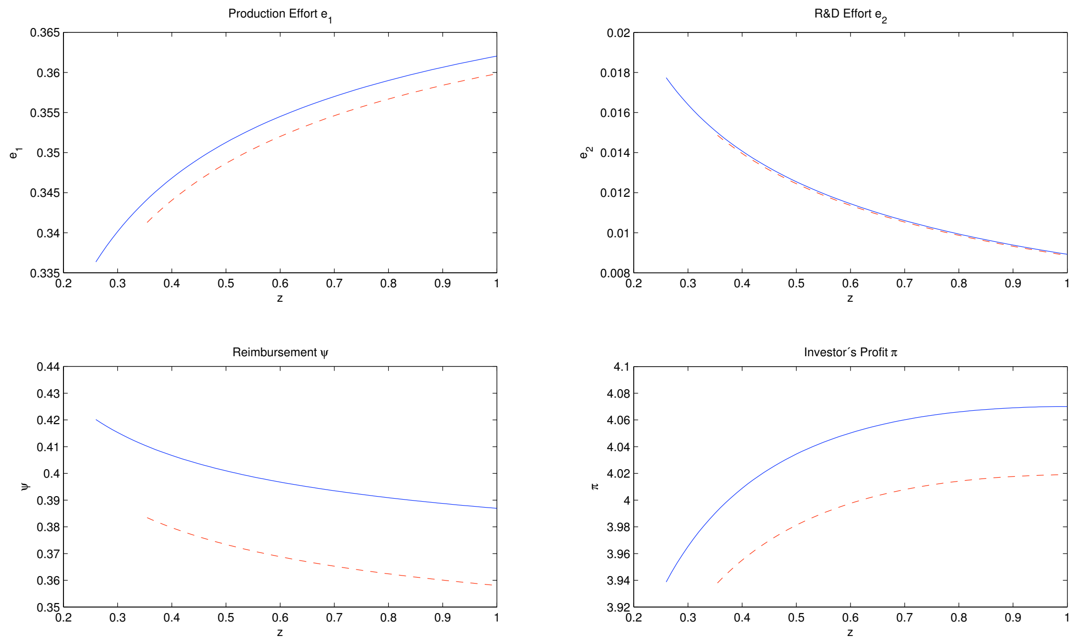


Table 3 summarises the impact on the optimal contract of the simulations presented above. The impact of the productivity shock is greater than changes in the substitutability parameter γ. A similar effect is seen in R&D intensity. The ratio e_1/e_2 is a proxy that measures the reallocation of effort between standard production and R&D. The impact of changes in ν and γ are of a similar magnitude. Reimbursements and profits are highly sensitive to changes in γ, while changes in productivity shock ν have little effect.

With respect to the firm's statistical moments, γ has a persistent impact on growth rate and variance. The impact on variance is greater for the combined shock. The last row of 3 represents a proxy measure of borrowing constraints. This proxy is the percentage of projects that are funded by the investor (FP). When there is high correlation between standard production and R&D, borrowing constraints are relaxed.

Table 3: Average response (av) to changes in ν and γ

	$\triangle\%\nu = -5\%$	$\triangle\%\gamma = 5\%$	$\triangle\%\gamma = -5\%$	$\triangle\%\nu = -5\%, \triangle\%\gamma = 5\%$
$\triangle\%e_1^{av}$	-5.06%	4.54%	-4.59%	-0.73%
$\triangle\%e_2^{av}$	-4.35%	4.68%	-4.54%	-0.28%
$\frac{e_2^{av}}{e_1^{av}}$	0.75%	0.05%	0.05%	-0.13%
$\triangle\%\psi^{av}$	0.07%	-7.15%	7.20%	-7.04%
$\triangle\%\pi^{av}$	-0.63%	2.39%	-2.62%	1.33%
$\triangle\%E\left(g\right)$	-4.79%	5.16%	-5.02%	-0.32%
$\triangle\%var$	-4.03%	4.33%	-4.20%	17.1%
$\%\,Financed\,Projects$	66.7%	82.7%	78.7%	74%

5 Concluding Remarks

We set up a model of R&D intensity in the presence of borrowing constraints. Borrowing constraints are due to the entrepreneur's misallocation of effort to different activities. R&D investment is not pledgeable for the investor, given the low value of collateral. Nevertheless, it is beneficial for firm expansion. When the entrepreneur increases the level of R&D investment, the borrowing constraint becomes binding and, therefore, some projects are not financed. We showed that at the intertemporal level, R&D resources are allocated in such way that there are equal implicit returns between production and R&D activities.

In this model, the optimal dynamic contract between an investor and an innovative entrepreneur implies that there are incentives for both production and R&D effort. Specifically, we find that R&D intensity decreases as borrowing constraints are relaxed. This finding is consistent with the empirical literature and the magnitude of the impact depends on the degree of correlation between the firm's activities.

References

Abreu, D., Pearce, D. G., and Stacchetti, E. (1986). Toward a theory of discounted repeated games with imperfect monitoring. (791).

Aghion, P., Angeletos, G.-M., Banerjee, A., and Manova, K. (2010). Volatility and growth: Credit constraints and the composition of investment. *Journal of Monetary Economics*, 57(3):246–265.

Aghion, P., Fally, T., and Scarpetta, S. (2007). Credit constraints as a barrier to the entry and post-entry growth of firms. *Economic Policy*, 22:731–779.

Akcigit, U. (2009). Firm size, innovation dynamics and growth. (1267).

Biais, B., Mariotti, T., Plantin, G., and Rochet, J.-C. (2007). Dynamic security design: Convergence to continuous time and asset pricing implications. *The Review of Economic Studies*, 74(2):345–390.

Biais, B., Mariotti, T., Rochet, J.-C., and Villeneuve, S. (2010). Large risks, limited liability, and dynamic moral hazard. *Econometrica*, 78(1):73–118.

Bjork, T. (2011). An introduction to point process from a martingale point of view.

Brown, J. R., Fazzari, S. M., and Petersen, B. C. (2009). Financing innovation and growth: Cash flow, external equity, and the 1990s r&d boom. *The Journal of Finance*, 64(1):151–185.

Buera, F. J. and Shin, Y. (2013). Financial frictions and the persistence of history: A quantitative exploration. *Journal of Political Economy*, 121(2):221 – 272.

Cabral, L. M. and Mata, J. (2003). On the evolution of the firm size distribution: Facts and theory. *American Economic Review*, pages 1075–1090.

Caves, R. E. (1998). Industrial organization and new findings on the turnover and mobility of firms. *Journal of Economic Literature*, 36:1947–1982.

Clementi, G. L. and Hopenhayn, H. A. (2006). A theory of financing constraints and firm dynamics. *The Quarterly Journal of Economics*, 121(1):229–265.

Cooley, T. F. and Quadrini, V. (2001). Financial markets and firm dynamics. *American Economic Review*, pages 1286–1310.

DeMarzo, P. M., Fishman, M. J., He, Z., and Wang, N. (2012). Dynamic agency and the q theory of investment. *The Journal of Finance*, 67(6):2295–2340.

DeMarzo, P. M. and Sannikov, Y. (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724.

Gompers, P. and Lerner, J. (2006). *The Venture Capital Cycle, 2nd Edition*, volume 1 of *MIT Press Books*. The MIT Press.

Gorodnichenko, Y. and Schnitzer, M. (2010). Financial constraints and innovation: Why poor countries don't catch up. NBER Working Papers 15792, National Bureau of Economic Research, Inc.

Hall, B. H. (2002). The financing of research and development. *Oxford Review of Economic Policy*, 18(1):35–51.

Klette, T. J. and Kortum, S. (2004). Innovating firms and aggregate innovation. *Journal of Political Economy*, 112(5):986–1018.

Midrigan, V. and Xu, D. Y. (2010). Finance and misallocation: Evidence from plant-level data. Technical report, National Bureau of Economic Research.

Moll, B. (2010). Productivity losses from financial frictions: Can self-financing undo capital misallocation?

Park, S. (2011). R&d intensity and firm size revisited. Job Market Paper UCLA.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *The Review of Economic Studies*, 75(3):957–984.

Spear, S. E. and Srivastava, S. (1987). On repeated moral hazard with discounting. *The Review of Economic Studies*, 54(4):599–617.

Walde, K. (2008). *Applied Intertemporal Optimization*. Number econ1 in Books. Business School - Economics, University of Glasgow.

Wälde, K. and Woitek, U. (2004). R&d expenditure in g7 countries and the implications for endogenous fluctuations and growth. *Economics Letters*, 82(1):91–97.

6 Appendix: Proofs

For convenience we interchange notation for partial derivatives as $\frac{\partial f(x,y)}{\partial x}$ and $f(x,y)$ with respect to x is $f_x(x,y)$.

In this appendix for convenience we will constantly interchange notation when we refer to partial derivatives, as $\frac{\partial f(x,y)}{\partial x}$ $f(x,y)$ with respect to x and also the same for $f_x(x,y)$.

Proof Lemma 1

We need to show that for a given $s < t$ $E\left(\pi_t^E \mid \mathcal{F}_s\right) = \pi_s^E$. Setting $s > t$ and computing the expected value for π_t^E process we have:

$$\phi_t = E_t\left[r\int_0^\tau e^{-rt}y\left(t\right)dt\right]$$

$$\phi_s = E_s\left[r\int_0^\tau e^{-rs}y\left(s\right)dt\right]$$

Descomposing in two terms:

$$\phi_t = r\int_0^t e^{-rx}y\left(x\right)dt + e^{-rt}rE_t\left[r\int_t^\tau e^{-r(x-t)}y\left(t\right)dt\right]$$

$$\phi_s = r\int_0^s e^{-rx}y\left(x\right)dt + e^{-rs}rE_s\left[r\int_s^\tau e^{-r(x-s)}y\left(t\right)dt\right]$$

Computing $E_t\left(\phi_s\right)$

$$E_t\left(\phi_s\right) = E_t\left(r\int_0^s e^{-rx}y\left(x\right)dt\right) + rE_t\left[r\int_s^\tau e^{-rx}y\left(t\right)dt\right]$$

Solving for $E_t\left(\phi_s - \phi_t\right)$

$$E_t\left(\phi_s - \phi_t\right) = E_t\left(r\int_0^s e^{-rx}y\left(x\right)dt\right) + rE_t\left[r\int_s^\tau e^{-rx}y\left(t\right)dt\right] - r\int_0^t e^{-rx}y\left(x\right)dt - rE_t\left[r\int_t^\tau e^{-rx}y\left(t\right)dt\right]$$

Computing the integral difference:

$$E_t\left(\phi_s - \phi_t\right) = E_t\left(r\int_t^s e^{-rx}y\left(x\right)dt\right) + rE_t\left[r\int_s^\tau e^{-rx}y\left(t\right)dt\right] - rE_t\left[r\int_t^\tau e^{-rx}y\left(t\right)dt\right]$$

$$E_t\left(\phi_s - \phi_t\right) = E_t\left(r\int_t^s e^{-rx}y\left(x\right)dt\right) + rE_t\left[r\int_s^\tau e^{-rx}y\left(t\right)dt\right] - rE_t\left[r\int_t^s e^{-rx}y\left(t\right)dt\right] - rE_t\left[r\int_s^\tau e^{-rx}y\left(t\right.\right.$$

Which gives: $E_t\left(\phi_s - \phi_t\right) = 0$ Q.E.D.

Proof Proposition 1

We guess that the solution of the value function [6] is linear for a number of innovations n. Considering the case of interior solutions, and replacing the guess $V = f(e_1^*, e_2^*)\, n$, the Bellman equation is given by:

$$rf(e_1^*, e_2^*) = r\left[\nu e_{1,t}^* - c(e_1^*, e_2^*)\right] + f(e_1^*, e_2^*)\left[e_2\eta - \mu\right]$$

solving for $f(e_1^*, e_2^*)$

$$f(e_1^*, e_2^*) = \frac{r\left[\nu e_{1,t}^* - c(e_1^*, e_2^*)\right]}{\left[r - (e_2^*\eta - \mu)\right]}$$

Using the first order conditions, we obtain the following system of equations:

$$[e_1] \quad \nu = c_{e_{1,t}}(e_{1,t}, e_{2t}) \tag{24}$$

$$[e_2] \quad \eta = c_{e_{2,t}}(e_{1,t}, e_{2t}) \tag{25}$$

Where policy functions are independent of firm size. Q.E.D.

Proof Proposition 2

This proof follows Sannikov (2008). As Lemma 1 proves, π^E is a martingale; from the Martingale Representation theorem for a Poisson process (see Bjork (2011) page 38). there exists a predictable process h such that:

$$U_t^E = U_0^E + \int_0^t \exp(-rs)\, h_s dN_s \tag{26}$$

where dN_s is defined in[11]. Differentiating[3] and[26] with respect to time we obtain:

$$\frac{dU_t^E}{dt} = \exp(-rt)\, h_t \left(\frac{dp}{dt} - e_{2,t} + \mu\right)$$

and for [26]:

$$\frac{dU_t^E}{dt} = r\exp(-rt)\left[(1-\psi_t)\, y_t - nc(e_{1,t}, e_{2,t})\right] - \exp(-rt)\, W_t + \exp(-rt)\frac{dW_t}{dt}$$

Equalizing both expressions and solving for dW_t we obtain

$$dW_t = r\left(W_t - (1-\psi_t)\, y_t + nc\left(e_{1,t}, e_{2,t}\right) - h(W_t)\left(e_{2,t} - \mu\right)\right) dt + rh(W_t)dp.$$

Q.E.D.

Proof Proposition 3

We start by analysing potential deviations in effort at the intertemporal level. Let $\tilde{U}$ the lifetime profits of the entrepreneur that follows a strategy $\tilde{e}_i$ for $i = 1, 2$ up to time t, and who then switches to strategy e_i for $i = 1, 2$ from t to ∞. We focus on the deviation in R&D effort. Let us consider the case when the agent exerts effort from $[0, t]$ by $(e_1, \tilde{e_2})$ for each t. Then, from $[t, +\infty]$ they switch to strategy (e_1, e_2). The value function of this potential deviation is given by $\tilde{V}^{R\&D}$ which is equivalent to:

$$\tilde{V}^{R\&D} = r\int_0^t exp(-rs)n_t\left[\left((1-\psi_t)\,\nu e_1 - c\left(e_1, \tilde{e}_2\right)\right) ds\right] + exp\left(-rt\right) W_t \tag{27}$$

Since V_t is a martingale, we can express $exp\left(-rt\right) W_t$ as:

$$exp\left(-rt\right) W_t = V_t - \int_0^t \exp\left(-rs\right)\left[\left((1-\psi_s)\,\nu e_{1s} - c\left(e_{1,s}, e_{2,s}\right)\right) ds\right] \tag{28}$$

Inserting equation [27] and simplifying yields:

$$\tilde{V}^{R\&D} = r\int_0^t exp(-rs)n_t\left[c\left(e_1, e_2\right) - c\left(e_1, \tilde{e}_2\right)\right] ds + V_t \tag{29}$$

Inserting equation [**??**] in [29] :

$$\tilde{V}^{R\&D} = r\int_0^t exp(-rs)n_t\left[c\left(e_1, e_2\right) - c\left(e_1, \tilde{e}_2\right)\right] ds + V_0 + \int_0^t \exp\left(-rs\right) h_s dN_s \tag{30}$$

We can express the Poisson process for the possible deviation from R&D level $\tilde{e}_2$:

$$d\tilde{N}_s = dp - \left(\tilde{e}_{2,s} - \mu\right) ds \tag{31}$$

Using equation [11] then $dN_s = d\tilde{N}_s + \tilde{e}_{2,s}ds - e_{2,s}ds$ and replacing in equation [30] we get:

$$\tilde{V}^{R\&D} = r\int_0^t exp(-rs)\left[n_s\left(c\left(e_1,e_2\right)-c\left(e_1,\tilde{e}_2\right)\right)+\frac{h_s}{r}\left(\tilde{e}_2-e_2\right)\right]ds+V_0+\int_0^t \exp\left(-rs\right)h_s dN_s \tag{32}$$

Therefore, if the pairs of effort (e_1, e_2) are incentive compatible, then the drift of the process [32] is negative. In fact $\tilde{V}^{R\&D}$ under the process $\{e_1, \tilde{e}_2\}$ satisfies:

$$E_{e_1,\tilde{e_2}}\left(\tilde{V}^{R\&D}\right) \leq V^{R\&D}$$

This means that following strategy $(e_1, \tilde{e}_2)$ on average provides less utility than following strategy (e_1, e_2). In the case where the incentive compatibility condition [13] does not hold, the drift of [32] is positive. Then:

$$E_{e_1,\tilde{e_2}}\left(\tilde{V}^{R\&D}\right) > V^{R\&D}$$

which implies strategy $e_2 \succ \tilde{e}_2$.

Now consider the case of change in production effort, from e_1 to $\tilde{e_1}$ denoted by the value function $\hat{V}$. Like the previous case, we consider the utility value of an arbitrary strategy $\tilde{e_1}$ as follows:

$$\hat{V} = r\int_0^t exp(-rs)n_t\left[\left(\left(1-\psi_t\right)\nu\tilde{e}_1 - c\left(\tilde{e}_1,e_2\right)\right)ds\right] + exp\left(-rt\right)W_t \tag{33}$$

Using equation [28] and [32], $\hat{V}^{R\&D}$ can be expressed as:

$$\hat{V} = r\int_0^t exp(-rs)n_t\left[\left(1-\psi_t\right)\nu\left(\tilde{e}_1-e_1\right)+c\left(e_1,e_2\right)-c\left(\tilde{e}_1,e_2\right)\right]ds+V_0+\int_0^t \exp\left(-rs\right)h_s dN_s \tag{34}$$

If the incentive compatibility condition [14] is satisfied, the drift of equation [34] becomes negative, therefore:

$$E_{\tilde{e}_1,e_2}\left(\hat{V}\right) \leq \hat{V}$$

In the case when [14] does not hold, the drift of [34] is positive; then:

$$E_{\tilde{e}_1,e_2}\left(\hat{V}\right) > \hat{V}$$

Then $e_1 \succ \tilde{e}_1$. In addition, if the pair of effort satisfies both incentive compatibility conditions, [13] and [14] satisfied the global incentive constraints because the drift in both cases is positive. Q.E.D.

Proof Proposition 4

We start with the case when there is no R&D effort. Then, the Bellman equation is as follows:

$$r\pi\left(z\right)=\max_{e_1,\psi} r\left(\psi e_1\nu\right)+r\pi_z\left[z-\left(1-\psi\right)e_1\nu+c\left(e_1\right)\right]$$

Then, the first order conditions are:

$$[e_1]:\quad \nu\psi+\pi_z\left(-\left(1-\psi\right)\nu+e_1\right)=0 \tag{35}$$

$$[\psi]:\quad \pi_z=-1 \tag{36}$$

As the contract is incentive compatible for each implementation of e_1, then $e_1^{prod}=\left(1-\psi\right)\nu$, inserting the first order condition [35] we obtain $\psi=0$; therefore, the investor's profits are negative when the agent only exerts effort in production. Consequently, the optimal level of production effort is constant and equal to the productivity parameter ν, $e_1^{prod}=\nu$, which is not profitable for the investor.

When the entrepreneur exerts effort in both activities, we assume that the expected technological increments are given by $\mathrm{E}(\pi\left(\bar{z}\right)-\pi\left(z\right))=\phi\left(z\right)e_2dt$, hence, inserting the incentive constraint for e_2 we have the following Bellman equation:

$$r\pi\left(z\right)=\max_{e_1,e_2,\psi} r\left(\psi e_1\nu\right)+r\pi_z\left[z-\left(1-\psi\right)e_1\nu+\frac{e_1^2}{2}+\frac{e_2^2}{2}+\gamma e_1e_2-r\left(e_2-\gamma e_1\right)e_2\right]+\phi\left(z\right)e_2^2$$

The first-order conditions are:

$$[e_1]:\quad \nu\psi+\pi_z\left(\gamma e_2\left(1+r\right)\right)=0 \tag{37}$$

$$[e_2]:\quad r\pi_z\left(e_2+\gamma e_1-2re_2+r\gamma e_1\right)+2e_2\phi\left(z\right)=0 \tag{38}$$

$$[\psi]:\quad \pi_z=-1 \tag{39}$$

Solving the equation system, we obtain the policy function for e_2,e_1,ψ:

$$e_1^{sb} = r\nu\left[1 - \frac{1}{1 + \frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}}\right] \tag{40}$$

$$e_2^{sb} = \frac{r\nu}{\left[\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma} + \gamma\left(1 - r\left(1+r\right)\right)\right]} \tag{41}$$

$$\psi^{sb} = 1 - r\nu\left[1 + \frac{r\left(1+r\right)}{r\left(1+r\right) + \phi\left(z\right)}\right]\left(\frac{\phi\left(z\right)}{r\gamma\left(1+r\right)} - \frac{2}{\gamma}\right) \tag{42}$$

where $e_1^{sb}, e_2^{sb}, \psi^{sb}$ are the levels of effort in each activity and the repayment to the investor under moral hazard. Next we compute the derivatives with respect to z to evaluate if the policy functions are concave or convex:

$$\frac{\partial e_1^{sb}}{\partial z} = \frac{\phi_z\left(z\right)\nu}{\left(1 + \frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}\right)^2\gamma\left(1+r\right)} > 0$$

and

$$\frac{\partial^2 e_1^{sb}}{\partial z} = \frac{\phi_{zz}\left(z\right)\nu}{\left(1 + \frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}\right)^2\gamma\left(1+r\right)} - \frac{2\left(\phi_z\left(z\right)\right)^2\nu\left(1 + \frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}\right)}{r\left(\left(1 + \frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}\right)^2\gamma\left(1+r\right)\right)^2} < 0$$

Respectively for e_2^{sb}

$$\frac{\partial e_2^{sb}}{\partial z} = \frac{-\phi_z\left(z\right)\nu}{\left[\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma} + \gamma\left(1 - r\left(1+r\right)\right)\right]^2\gamma\left(1+r\right)} < 0$$

and

$$\frac{\partial^2 e_2^{sb}}{\partial z} = \frac{-\phi_{zz}\left(z\right)\nu}{\left[\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma} + \gamma\left(1 - r\left(1+r\right)\right)\right]^2\gamma\left(1+r\right)} + \frac{2\left(\phi_z\left(z\right)\right)^2\left(\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma} + \gamma\left(1 - r\left(1+r\right)\right)\right)}{r\left(\left(\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma} + \gamma\left(1 - r\left(1+r\right)\right)\right)^2\gamma\left(1+r\right)\right)^2} > 0$$

and for the repayment we have:

$$\frac{\partial \psi_1^{sb}}{\partial z} = -\left[\frac{\partial e_1^{sb}}{\partial z} + \frac{\gamma}{\nu}\frac{\partial e_2^{sb}}{\partial z}\right]$$

As we know the sign of the derivatives for each level of effort we now need to compare the magnitudes of the derivatives to find the total sign. Therefore:

$$\frac{\partial \psi_1^{sb}}{\partial z} = -\left[\left(\frac{\partial \phi(z)}{\partial z}\right)\frac{1}{\gamma(1+r)}\left(\frac{\nu}{\left(1+\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}\right)^2} - \frac{\gamma}{\left(\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}+\gamma(1-r(1+r))\right)^2}\right)\right]$$

We can see that for reasonable values of the parameters r and ν so that $\frac{\nu}{\left(1+\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}\right)^2} > \frac{\gamma}{\left(\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}+\gamma(1-r(1+r))\right)^2}$ then $\frac{\partial \psi_1^{sb}}{\partial z} < 0$ and the second derivative is:

$$\frac{\partial^2 \psi^{sb}}{\partial z} = \begin{array}{l} \frac{\phi_z(z)\nu}{\gamma(1+r)}\left[\frac{1}{\left(1+\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}\right)^2} - \frac{1}{\left(\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}+\gamma(1-r(1+r))\right)^2}\right]+ \\ \frac{2(\phi_z(z))^2}{r\gamma(1+r)}\left[\frac{1}{\left(\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}+\gamma(1-r(1+r))\right)^3} - \frac{\nu}{r\left(1+\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}\right)^3}\right] \end{array} > 0$$

The result yields Q.E.D.

Proof Proposition 5

Suppose that $\pi(z)$is a differentiable continuous function. Consider $z^* < z$ such that $z^* \in [\hat{z}, \bar{z}]$; from the optimal contract we know that π_z is non-increasing, meaning that there is $z_1 \in [z^*, \bar{z}]$ such that $\pi_{z_1}(z_1) \leq \pi_z(z)$. Integrating both sides of the inequality, we have:

$$\int_{z^*}^{z} (\pi_{z_1}(z_1))\, dz_1 \leq \int_{z^*}^{z} (\pi_z(z))\, dz_1$$

Using the fundamental theorem of calculus we have:

$$\pi(z) - \pi(z^*) \leq \pi(z^*)(z - z^*) \tag{43}$$

Re-arranging the terms gives the definition of concavity around $z^*, \pi(z) \leq \pi(z^*)(z - z^*) + \pi(z^*)$. Q.E.D.

Proof Proposition 6

Let us consider the aggregate growth rate as $\tilde{g} = \int_{\hat{z}}^{\bar{z}} g(z)\, dz$ which is exponentially distributed with rate $e_2 = \int_{\hat{z}}^{\bar{z}} e_2(z)\, dz$. The first step is compute the survival function. For any $t > 0$ we have that:

$$P\left(M>t;g,z\right)=\int_{0}^{t}\left(\exp\left(e_{2}\left(z\right)-\mu\right)x\right)\left(e_{2}\left(z\right)-\mu\right)dx$$

Which is equivalent to:

$$P\left(M>t;g,z\right)=1-\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)$$

The survival function is defined as:

$$S\left(g,z\right)=1-P\left(M>t;g,z\right)=\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)$$

The survival function collapses the distribution of the firm's growth over time. Therefore the expected growth of a firm that survives is given by:

$$g^{s}\left(z\right)=\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\right)\left(\exp\left(e_{2}\left(z\right)-\mu\right)t-1\right)$$

Next we aggregate the survival function over the range of financial resources that measure the aggregate growth of a firm that survive:

$\widetilde{g^{s}}=\int_{\hat{z}}^{\bar{z}}\left(\exp\left(e_{2}\left(z\right)-\mu\right)t\left(\exp\left(e_{2}\left(z\right)-\mu\right)t-1\right)\right)dz$. We compute $\frac{\partial\tilde{g^{s}}}{\partial z}$ using the Leibniz formula:

$$\frac{\partial\tilde{g^{s}}}{\partial z}=\int_{\hat{z}}^{\bar{z}}\frac{\partial g^{s}\left(z\right)}{\partial z}ds-g^{s}\left(\hat{z}\right)\frac{\partial\hat{z}}{\partial z}$$

Then we can compute $\frac{\partial g^{s}(z)}{\partial z}$ as: $\frac{\partial g^{s}(z)}{\partial z}=\frac{\partial g^{s}(z)}{\partial e_{2}}\frac{\partial e_{2}}{\partial z}$. The marginal response of the R&D effort on the rate of growth of a firm that survives is given by:

$$\frac{\partial g^{s}\left(z\right)}{\partial e_{2}}=\left(1+g\right)\left(t\left(1+2g\right)\right)>0$$

then from Proposition 4, we know in the optimal contract $\frac{\partial e_{2}(z)}{\partial z}<0$ for $z\in[\hat{z},\bar{z}]$ then $\frac{\partial g^{s}(z)}{\partial z}<0$. Our measure of borrowing constraint implies that $\frac{\partial\hat{z}}{\partial z}>0$ therefore $\frac{\partial\tilde{g^{s}}}{\partial z}<0$.

In order to analyze the variance, we the same argument: $\frac{\partial\tilde{S}}{\partial\hat{z}}=\frac{\partial var(z)}{\partial z}\frac{\partial z}{\partial\hat{z}}$. We study the aggregate variance for all realizations of z $v\tilde{a}r\left(z\right)=\int_{\hat{z}}^{\bar{z}}var\left(z\right)dz$, and compute:

$$\frac{\partial v\tilde{a}r}{\partial z}=\int_{\hat{z}}^{\bar{z}}\left[\frac{\partial var\left(z\right)}{\partial z}\right]dz-var\left(\hat{z}\right)\frac{\partial\hat{z}}{\partial z}$$

In the same way, we use the chain rule to find the derivative $\frac{\partial var(z)}{\partial z} = \frac{\partial var(z)}{\partial e_2(z)} \frac{\partial e_2(z)}{\partial z}$. Therefore we use lemma 2 to find $\frac{\partial var(z)}{\partial e_2(z)}$:

$$\frac{\partial var\,(z)}{\partial e_2\,(z)} = \frac{1+g}{e_2-\mu}\left(t\,(e_2+\mu) - 2\mu g\,(e_2-\mu)\right) > 0$$

We can see that $t\,(e_2+\mu) - 2\mu g\,(e_2-\mu) > 0$, which implies $\frac{\partial var(z)}{\partial e_2(z)} > 0$. In addition, we know for proposition 4 $\frac{\partial e_2(z)}{\partial z} < 0$ then $\frac{\partial var(z)}{\partial z} < 0$. In the case of borrowing constraint, we have that $\frac{\partial \hat{z}}{\partial z} > 0$ and if the $var\,(\hat{z}) > 0$ then the result yields Q.E.D.

Proof Proposition 7

In order to study the impact of a negative shock on productivity, we reformulate the policy function of the optimal contract as follows:

$$e_1^{sb} = r\nu\Psi_1 \tag{44}$$

where $\Psi_1 = \left[1 - \frac{1}{1+\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}}\right] > 0$

$$e_2^{sb} = r\nu\Psi_2 \tag{45}$$

and $\Psi_2 = \frac{1}{\left[\frac{\phi(z)}{r\gamma(1+r)}-\frac{2}{\gamma}+\gamma(1-r(1+r))\right]} > 0$

$$\psi^{sb} = 1 - r\nu\Psi_3 \tag{46}$$

where $\Psi_3 = \left[1 + \frac{r(1+r)}{r(1+r)+\phi(z)}\right]\left(\frac{\phi(z)}{r\gamma(1+r)} - \frac{2}{\gamma}\right) > 0$

Therefore, in the productivity parameter, all policy functions are linear, which implies that a fall in productivity is associated with a reduction in $e_1^{shock}, e_2^{shock}\psi^{shock}$. Inserting the investor's profit function:

$$\pi^{shock}\,(z) = \left(\psi^{shock} e_1^{shock}\nu\right) - \left[z - \frac{\left(e_1^{shock}\right)^2}{2} + \frac{\left(e_2^{shock}\right)^2}{2}\left(\frac{1}{2} - r\right) + r\gamma e_1^{shock} e_2^{shock}\right] + \frac{\phi\,(z)}{r}\left(e_2^{shock}\right)^2 \tag{47}$$

And computing the derivative with respect to ν we have:

$$\frac{\partial \pi^{shock}(z)}{\partial \nu} = e_1^{shock}\left(r\Psi_1 + \psi - r^2\nu\right) + e_2^{shock}\left[\Psi_2\left(2\phi(z) - r\left(\gamma - r - \frac{1}{2}\right)\right)\right] + \gamma\Psi_1\left(r + \psi^{shock}\gamma\nu\right) > 0 \tag{48}$$

Therefore a negative productivity shock is associated with a fall in investor profits and a rise in $\hat{z}$, which implies an exacerbation of borrowing constraints, Q.E.D.