Efficient Firm Dynamics in a Frictional Labor Market *

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Abstract

We develop and analyze a labor search model in which heterogeneous firms operate under decreasing returns and compete for labor by publicly posting long-term contracts. Firms achieve faster growth by offering higher lifetime wages that attract more workers which allows to fill vacancies with higher probability, consistent with empirical regularities. The model also captures several other observations about firm size, job flows, and pay. In contrast to existing bargaining models, efficiency obtains on all margins of job creation and destruction, both with idiosyncratic and aggregate shocks. The planner solution allows a tractable characterization which is useful for computational applications.

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1 Introduction

Search models of the labor market following the Diamond-Mortensen-Pissarides (DMP) framework have traditionally treated the production side very simplistically: Either a firm has only one job, or large firms operate with constant marginal product, both of which are usually equivalent.[1] Central insights from this literature are that even if matching is random and wages are the outcome of a pair-wise bargaining process without commitment, all firm-worker pairs take jointly efficient decisions, and overall efficiency obtains if the bargaining power is at the Hosios condition (see Hosios (1990)). This creates a natural benchmark, and many applications directly analyze the planner's problem (e.g., Merz (1995), Andolfatto (1996) and Shimer (2005b)).[2] These models have proven useful in many applied labor settings, and they dominate much of the labor policy discussion. Despite their success, they are silent about all aspects that relate to employer size, even though firm size and firm dynamics are empirically important for wages, job flows and aggregate employment. For example, larger firms are more productive, pay more, and have lower job flow rates (e.g., Davis, Haltiwanger, and Schuh (1996))); younger firms have higher exit rates and pay higher wages (e.g., Haltiwanger, Jarmin, and Miranda (2010)); and Moscarini and Postel-Vinay (2009) find that small and large firms contribute to the business cycle in different ways.

To capture the implications of firm heterogeneity in size, age and productivity, a large body of recent work has introduced multi-worker firms with decreasing returns to scale into standard labor search models. This allows researchers to address many new questions relating firm size to labor market outcomes.[3] But the three maintained assumptions, random matching together with bilateral bargaining and a lack of commitment about future wages, generate some questionable implications. Due to random search, the rate at which a firm fills its vacancies is independent of the firm's growth rate, while in the data faster-growing firms have much higher job-filling rates (see Davis, Faberman, and Haltiwanger (2010)). Moreover, due to the absence of commitment, a worker in a growing firm sees his wages decline with tenure, even if individual and aggregate conditions otherwise stay the same. This happens because additional employees reduce the marginal product, and renegotiations lead to lower wages for existing workers, yielding a within-firm externality. Finally, the normative implications are very different from the standard DMP model: the within-firm externality leads firms to hire excessively in order to depress wages of existing workers (see, e.g., Stole and Zwiebel (1996) and Smith (1999)); and even with wage commitments the randomness of the search process generates an across-firm externality that

[1] For surveys see, e.g., Mortensen and Pissarides (1999) and Rogerson, Shimer, and Wright (2005).

[2] Others leave the bargaining power a free parameter to be assessed as part of the calibration strategy (e.g., Hall (2005), Hagedorn and Manovskii (2008)).

[3] A subset of this work considers, for example, unemployment and efficiency (Bertola and Caballero (1994), Smith (1999), Acemoglu and Hawkins (2010), Mortensen (2009)), labor and product market regulation (Koeniger and Prat (2007), Ebell and Haefke (2009)), business cycles (e.g., Elsby and Michaels (2010), Fujita and Nakajima (2009)), and international trade and its labor market implications (Helpman and Itskhoki (2010)). We review other approaches with on-the-job search and constant returns in production below.

impedes efficiency (Hawkins (2010)).[4]

This work proposes and characterizes an alternative framework to think about firm dynamics in a frictional labor market. Firms can commit to long-term wage contracts, and can publicly post these contracts in order to attract unemployed workers (*competitive search*). Commitment allows firms to offer wage policies that are independent of the hiring of other workers, which remedies within-firm externalities. Posting introduces a competitive element into the labor market. Since workers can choose which contract to search for, those firms that want to hire faster raise the attractiveness of their offer in order to induce more workers to apply for the job. Meaningful firm dynamics arise when firms adjust their employment slowly over time rather than jumping immediately to their desired size. In the model the slow adjustment is due to prohibitively high costs of large-scale employment adjustments which can be micro-founded by the idea that recruitment takes up time of the existing workers (e.g., Shimer (2010)). Our contribution is to highlight three key aspects that might render this alternative approach a real contender for the study of firm dynamics in frictional labor markets: (1) Its ability to account for new empirical regularities such as the connection between firm growth and job-filling rates while retaining other desirable stylized relationships; (2) its efficiency properties which provide a natural benchmark; and (3) its tractability even in the presence of aggregate and idiosyncratic shocks. We discuss each in turn.

One main contribution concerns the positive implications of our approach. We show analytically in a steady-state version of the model that faster firm growth comes along with higher job-filling rates. Firms that want to grow can offer higher wages to attract more workers per job. We find this a very natural channel to obtain this relationship, even though the actual proof is non-trivial.[5] We also show analytically that the qualitative properties exhibit several of the empirical connections between firm size, growth and pay.[6] In this respect, our work extends the insights of the competitive theory of firm dynamics of Hopenhayn and Rogerson (1993) to an environment with frictional unemployment. Moreover, we demonstrate the applicability of this theory in a calibrated example and show that the model replicates several relevant cross–sectional relationships. For example, it captures the distribution of employment growth rates and the negative relationship between firm size and job creation and destruction rates. We also

[4]Hawkins (2010) shows that the standard Hosios condition fails since firms of different sizes (and hence different marginal productivities) exert different search externalities that cannot be balanced efficiently by a single bargaining power parameter.

[5]The formal proof requires the firms' value functions to be supermodular in firm size and productivity, which does not follow from standard arguments. Clearly, there are other possible explanations for the relation between firm growth and job-filling rates. Employers can "increase advertising or search intensity per vacancy, screen applicants more quickly, relax hiring standards, improve working conditions, and offer more attractive compensation to prospective employees" (Davis, Faberman, and Haltiwanger (2010), p. 23). Ours formalizes the last point about compensation, but any measure that is costly for the firm but attracts more workers would be roughly captured by this model.

[6]In regards to pay, we note that commitment allows firms to spread their payments in any desired way. In fact, most of our results concern the lifetime wages that they promise to their workers, but we discuss some particular implementations as well. For simplicity we initially use fixed-wage contracts which we relax later on.

find that wages are positively correlated with firm size and firm growth.

In terms of normative implications, the decentralized economy creates and destroys jobs efficiently both on the *extensive* margin of firm entry/exit and on the *intensive* margin of firm expansion/contraction, both in the presence of idiosyncratic and aggregate shocks. This gives a natural benchmark against which to judge the actual performance of the labor market. The ability to commit to future wages eliminates the within-firm externality, as it decouples a worker's earnings from the employment of other workers. The competitive element of wage posting eliminates the across-firm externality, because different firms can offer different contracts and fill vacancies at different rates which correspond to a modified Hosios condition. The result is technically demanding because infinitely-lived firms change their productivity over time as they move along their non-linear production function. At a fundamental level efficiency is due to *public posting* of contracts and *commitment to payments upon hiring*. While these elements are at the heart of many efficiency arguments in the competitive search literature, the subtle nature of search markets does not always render them sufficient to induce constrained efficiency, especially when choices along different margins interact.[7] We are not aware of a formal efficiency result for large firms operating under decreasing returns.[8]

Our third contribution concerns the tractability of the model, especially in the presence of aggregate shocks. The main difficulty in a model with heterogeneous firms is that policy functions depend on the distribution of firm size and productivity. Under competitive search and with positive entry of new firms, however, a firm's optimal job creation and job destruction choices turn out to be independent of the current firm distribution. They depend only on the aggregate productivity state, which substantially reduces computational complexity, making it feasible to solve the model without the need to resort to approximation techniques, such as those of Krusell and Smith (1998), that have been applied in the heterogeneous-firm search models of Elsby and Michaels (2010) and Fujita and Nakajima (2009) to analyze aggregate labor market dynamics. The reason for tractability is that firms compete publicly for workers, which links all decisions of existing firms to those of new entrants via an *aggregate arbitrage condition*. The logic is different from standard free-entry arguments,[9] but builds on Menzio and Moen (2010) whose approach is nevertheless substantially simpler because they consider short-lived firms whose histories are

[7]Posting and full specification in payments upon hiring are present for example in Galenianos and Kircher (2010), but efficiency fails because of an intensive margin (search intensity) on the workers' side which is not internalized. Guerrieri (2008) introduces an intensive margin through moral hazard and finds efficiency in steady state but not out of steady state. The subtlety of the arguments suggests that an efficiency proof is necessary both in and out of steady state.

[8]Hawkins (2006) suggests such an outcome on the basis of a static model, but his results are complicated by the stochastic nature of the hiring process and they do not extend to dynamic settings with shocks. Menzio and Moen (2010) do not obtain efficiency because they focus on lack of commitment, and Garibaldi and Moen (2010) abstract from decreasing returns.

[9]The idea of "block-recursivity" that underlies this approach goes back to Shi (2009) and Menzio and Shi (2009, 2010) and refers to the ability to compute the block of individual decisions as functions of the aggregate state only, while the block of aggregate distributions of workers or firms can be computed subsequently. Their proofs rely crucially on the fact that each wage is characterized by a free-entry condition, and they therefore consider single worker-firm pairs. With large firms and convex recruitment costs, free entry per market does not

truncated after two periods. Importantly, although firm policies react to the aggregate state, convex recruitment costs prevent firms from jumping to their optimal size immediately. As a result, aggregate variables such as the workers' job-finding rate crucially depend on the mix between new and old firms in the economy and exhibit a sluggish response to aggregate shocks, which has been documented in the data (e.g. Fujita and Ramey (2007)). This contrasts with standard labor-search models, where the job-finding rate is a jump-variable which is perfectly correlated with aggregate productivity.[10] While the common lack of amplification at moderate values of leisure applies, our model gives rise to a downward-sloping Beveridge curve in spite of endogenous job separations.

After a brief review of further related literature, we first present in Section 3 a simplified setup without aggregate or idiosyncratic productivity shocks. This allows for a teachable representation, it establishes the most important insights for dynamics of employment, job-filling rates and wage offers, and it demonstrates the efficiency of the decentralized allocation. In Section 4, we lay out the notationally more complex analysis that takes account of aggregate and idiosyncratic shocks, we prove the equivalence between the efficient and the decentralized allocations, and we characterize them using our aggregate arbitrage condition. Beyond the focus on these theoretical contributions, we consider a calibrated numerical example in Section 5 to illustrate the workings of the model and to point at its potential to capture the connections between firm dynamics and the labor market. We conclude in Section 6 with a discussion how risk aversion and worker heterogeneity might be introduced into this framework.

2 Further Related Literature

Competitive search dispenses with bargaining weights and generates a surplus split through explicit competition for labor. Much of this work considers large economies (see, e.g., Moen (1997), Acemoglu and Shimer (1999a), Shi (2001), Shimer (2005a), Eeckhout and Kircher (2010)) whose micro-foundations combine Bertrand-style contract posting with coordination frictions: sometimes multiple workers apply for the same job and only one of them can be hired (e.g. Peters (1991), Burdett, Shi, and Wright (2001), Galenianos and Kircher (2010)). These micro-foundations can be extended to a multi-worker firm setting if one assumes that excess applicants for one position cannot fill another position at the same firm. This arises if different vacancies relate to different qualifications: for example, vacancies for an electrician cannot be filled by applicants for the position of a mechanic or a carpenter, even though each position yields roughly the same contributions in terms of marginal product (therefore labor input is modeled

apply, and a different argument is necessary.

[10] Persistence in our model relies on the fact that the aggregate composition of firms evolves slowly since individual firms grow slowly over time. In standard models such as Pissarides (2000) and Shimer (2005b), vacancies jump with aggregate shocks, which generates tractability but no sluggishness in job-finding rates, whereas models with convex recruitment costs such as Acemoglu and Hawkins (2010) generate persistence while random search impedes tractability.

homogeneously). An alternative interpretation is that workers are literally identical and excess capacity can be substituted from one job to another, which means that posting additional jobs exhibits increasing returns; see Burdett, Shi, and Wright (2001), Hawkins (2006) and Lester (2010) for variations along these lines. In accordance with most work on large firms, we adopt the first interpretation and abstract from possible increasing returns in hiring.

Hawkins (2006) is the first to consider multi-worker firms with decreasing returns in a competitive search setup. He assumes that firms employ *finitely* many workers after receiving a stochastic number of applications. Since the number of applicants is stochastic, he shows that posting a wage alone is not sufficient to induce efficiency. Rather, the posted contract has to condition on the realized number of applicants. These contingencies make the model quite complicated, and results on efficiency and firm dynamics out of steady state are missing.

All worker flows in our model are transitions between unemployment and employment. Work following the lines of Burdett and Mortensen (1998), Postel-Vinay and Robin (2002), and Moscarini and Postel-Vinay (2010) focuses additionally on the worker flows between firms in random-search environments. These models feature firms of different sizes, and higher wages attract more workers because it is easier to steal workers away from other firms. Although these models have the potential to explain similar empirical regularities as our model, most contributions in this literature have abstracted from firm dynamics since firms are neither hit by idiosyncratic shocks nor by business-cycle shocks. Moscarini and Postel-Vinay (2010) do allow for aggregate shocks, but their requirement of rank-preserving hiring prevents the study of firm entry and firm-specific shocks. To our knowledge, the only model that explicitly focuses on firm dynamics is Lentz and Mortensen (2010), which combines decreasing returns with on-the-job search, but again it has no idiosyncratic or aggregate shocks.

In the competitive-search literature, job-to-job movements have been considered by Shi (2009), Menzio and Shi (2009, 2010), Garibaldi and Moen (2010) and recently Schaal (2010). Except for the last contribution, firm size in these models is not restricted by the operated technology, circumventing considerations induced by the difference between average and marginal product. Closely related to our contribution is the work by Garibaldi and Moen (2010) who also consider a competitive search model with heterogeneous firms, deriving a number of new insights for on-the-job search. As they assume constant-returns in production, the only determinant of firm size arises from convex vacancy creation costs, which are assumed to be independent of firm size or productivity. By implication, the current size of the firm ceases to be a state variable, and therefore firm growth and wages depend only on the productivity type but are independent of firm size. That is, their model is silent about the role of firm size and age for job creation; it also abstracts from idiosyncratic and aggregate productivity shocks.[11] The work by Schaal (2010) has a different focus on uncertainty shocks. In terms of modeling it differs by assuming linear vacancy costs, which have the implication that firms immediately jump to their desired

[11] Productivity shocks might induce some firms to shed some of their workforce. We note that a linear frontier such as in Garibaldi and Moen (2010) would imply that a firm that fires some workers will fire all of them, unless there are strictly convex firing costs.

size. It also allows the use of the same market-by-market free entry considerations as in the work on block-recursivity by Shi (2009) and Menzio and Shi (2009, 2010). In particular, firm characteristics are not linked to the contracts it offers, implying for example that firm growth is not linked to job-filling rates or to the lifetime wages offered. Furthermore, the job-finding rate for unemployment workers is a jump variable, perfectly correlated with aggregate productivity. In our work, in contrast, firms face limitations to expand the workforce by posting more vacancies, so that they have to rely on posting higher wages if they want grow fast. They are no longer indifferent between contracts, and job–filling rates are linked to firm characteristics. Furthermore, many important aggregate variables, such as unemployment and job–finding rates, do depend on the firm distribution, which induces a sluggish response to business–cycle shocks.

In further relation to the random-search multi-worker-firm models mentioned in the introduction, it is worth pointing out that current applied work on business cycles mainly focuses on the intensive margin of hiring by considering a fixed number of firms (Elsby and Michaels (2010), Fujita and Nakajima (2009)). Similarly, Cooper, Haltiwanger, and Willis (2007) assess business cycle implications for a fixed number of firms, assuming zero bargaining power for workers. Our paper addresses additionally the entry and exit of firms, and this feature is in fact decisive to obtain a tractable solution.[12] The problem that bargaining might introduce unwarranted inefficiencies by assumption has also spurred other solutions than ours. For example, Veracierto (2008) and Samaniego (2008) consider general-equilibrium versions of the Hopenhayn and Rogerson (1993) model with frictionless labor markets and competitive wage setting. These approaches eliminate involuntary unemployment altogether. Veracierto (2009) introduces unemployment in an adaptation of the Lucas-Prescott island model that includes recruitment technologies. In contrast, competitive search allows the market to operate through decentralized wage setting, which attracts workers that optimally choose between search markets and are matched according to a standard matching function.

3 A Stationary Model

3.1 The Environment

The model is set in discrete time and is stationary; that is, there are neither idiosyncratic nor aggregate shocks. These will be introduced in Section 4.

Workers and Firms

There is a continuum of workers and firms, and workers are negligibly small relative to firms. That is, every active firm employs a continuum of workers. The mass of workers is normalized to one. Each worker is infinitely-lived, risk-neutral, and discounts future income with factor $\beta < 1$.

[12] See also the recent contribution by Hawkins (2011) who obtains tractability in a random-search model with linear vacancy costs and free entry.

A worker supplies one unit of labor per period and receives income $b \geq 0$ when unemployed. On the other side of the labor market is an endogenous mass of firms. Firms are also risk neutral and have the same discount factor β. Upon entry, the firm pays a set-up cost $K > 0$ and draws productivity x with probability $\pi_0(x)$ from the finite set $x \in X$. In this section, productivity stays constant during the life of the firm. In each period, a firm produces output $xF(L)$ with $L \geq 0$ workers, where F is a twice differentiable, strictly increasing and strictly concave function satisfying $F'(0) = \infty$ and $F'(\infty) = 0$. Firms die with exogenous probability $\delta > 0$ in which case all workers are laid off into unemployment. Furthermore, each employed worker quits the job with exogenous probability $s \geq 0$. Thus, workers' separation probability is exogenous at $\eta \equiv 1 - (1-\delta)(1-s)$.

Recruitment

Search for new hires is a costly activity. A firm with current workforce L that posts V vacancies incurs recruitment cost $C(V, L, x)$. An often used benchmark is the case where firms pay some monetary recruitment cost $C(V, L, x) = k(V)$ that is strictly increasing and strictly convex, but independent of the current size or productivity (for applications of this specific case, see e.g. Cooper, Haltiwanger, and Willis (2007), Koeniger and Prat (2007), Garibaldi and Moen (2010)). The convexity captures adjustment costs that prevent the firm from immediately growing large just by posting many vacancies.[13] A deeper micro-foundation for the convexity arises when recruitment costs are in terms of labor rather than goods, as proposed in Shimer (2010). This naturally leads to strictly convex costs even if the inputs into recruitment are linear. For example, if each vacancy costs $c \geq 0$ units of output but also requires $h > 0$ units of recruitment time from existing workers, then the total recruitment costs $C(V, L, x) = xF(L) - xF(L - hV) + cV$, comprising lost output and pecuniary costs, are strictly convex.[14] Taking no stance on either specification, we allow recruitment costs of the form $C(V, L, x) = xF(L) - xF(L - hV) + k(V)$ with at least one of the inequalities $h \geq 0$ and $k''(V) \geq 0$ strict. Our results can be extended to other cost functions as long as they obey the concavity and cross-partial restrictions that we outline in the proofs and that arise naturally for this specification.

Search and Matching

A recruiting firm announces a flat flow wage income w to be paid to its new hires for the duration of the employment relation. The assumption that the firm offers the same wage to all its new hires is no restriction. Indeed, it is straightforward to show that it is profit maximizing

[13] Of course, there can be other reasons why firms do not post very large numbers of vacancies to grow fast to large sizes, e.g. capital adjustment costs or a sluggish firm productivity process.

[14] Clearly no more workers can be engaged in hiring than are present at the firm already. To get the hiring process started, we therefore need to assume that a newborn firm is endowed with some initial workforce L_e (e.g., the entrepreneurs) who can undertake the initial hiring or production. In this case the production function is defined on the interval $L \in [-L_e, \infty)$; and recruitment activities of any firm are then constrained by its labor endowment: $hV \leq L + L_e$.

for the firm to post vacancies with identical wages at a given point in time.[15] Further, because of risk neutrality, only the net present value that a firm promises to the worker matters. Flat wages are one way of delivering these promises.[16]

There is no search on the job. Unemployed workers observe the vacancy postings and direct their search towards wages promising the highest expected lifetime income. Workers anticipate that different wage contracts come along with different job-finding probabilities. Specifically, every offered wage attracts a certain number of unemployed workers per vacancy, captured by an unemployment-vacancy ratio λ. This number is an equilibrium object that depends on how attractive the wage is relative to the other wages that are offered.[17] For any offered wage, unemployed workers and vacant jobs are matched according to a constant-returns matching technology, and the associated wage is paid every period that the worker is employed. Matching probabilities depend on the unemployment-vacancy ratio λ: a vacancy is matched with a worker with probability $m(\lambda)$ and a worker finds a job with probability $m(\lambda)/\lambda$. The function m is differentiable, strictly increasing, strictly concave, and it satisfies $m(0) = 0$ and $m(\lambda) \leq \min(1, \lambda)$ for all $\lambda \in [0, \infty)$. In particular, wages that attract more workers per vacancy induce higher matching rates for firms and lower matching rates for workers.

As discussed in the previous section, we follow most of the literature by assuming that each vacancy has its independent matching rate. Then the law of large numbers convention together with the assumption that workers are small relative to firms ensures that firms know with certainty that they hire $m(\lambda)V$ workers when they post V vacancies together with a wage offer that attracts λ workers per vacant job.[18]

Timing

Every period is divided into four stages. First, new firms are created and draw their productivity. Second, production and search activities take place. Third, vacancies and unemployed workers are matched, and a fraction s of workers leave their firm. And fourth, a share δ of firms dies. Newly hired workers may never work (and receive no wage income) in the unlucky event that their employer exits the market at the end of the period.

[15] An intuition for this result is provided in the discussion of equation (4) below.

[16] This is a theory of the present value of offered wages. The implementation through constant wages might be viewed as the limiting case of risk-neutral firms and risk-averse workers, as risk-aversion vanishes. But other payment patterns are conceivable. Section 4.4 is more explicit about this, and discusses how various notions of commitment can be relaxed.

[17] In the terminology of the competitive search literature, the labor market segments into different "submarkets" (w, λ), indexed by the offered wage w and the corresponding unemployment-vacancy ratio λ. Following most of the literature, workers are restricted to apply for only one wage per period. Galenianos and Kircher (2010) and Kircher (2009) allow for search for multiple wages, and even though results differ, there are large segments of wages in which the market essentially resembles the restricted one-wage-per-period search models.

[18] Note that we view each vacancy as a separate job requiring individual skills. See the discussion on increasing returns to hiring in Section 2.

3.2 Equilibrium

Given that there are no aggregate shocks, we characterize a stationary equilibrium where a constant number of firms enters the market in every period and where the workers' unemployment value is constant over time.

Workers' Search Problem

In a stationary environment, a worker who is looking for a certain wage in one period is willing to search for the same wage in every period. Consider a worker who is always searching for wage w which attracts an unemployment-vacancy ratio λ. The worker follows a simple sequential search process which has been analyzed extensively going back to McCall (1965). Standard arguments give rise to the following equation for the flow value of unemployment:[19]

$$(1-\beta)U = b + \beta \underbrace{\frac{m(\lambda)}{\lambda}\Big(w-(1-\beta)U\Big)\frac{1-\delta}{1-\beta(1-\eta)}}_{\equiv\rho} . \tag{1}$$

The flow value of unemployment equals the current period payoff from unemployment together with an option value from searching, denoted by ρ. The search value is the probability of finding a job to the next period multiplied with the worker's job surplus, which is the present discounted value of flow gains $w-(1-\beta)U$.

Workers have a choice which wage contract they want to search for. In equilibrium all offered contracts have to deliver the same search value ρ. If one contract would be more attractive than others, then more workers would apply and drive up the unemployment-vacancy ratio, making the offered contract less attractive. Similarly, if a contract is less attractive than others and still has $\lambda > 0$, more workers would apply elsewhere, reducing the unemployment-vacancy ratio and making the contract more attractive. Therefore, when workers choose between all combinations (w, λ), they must yield the same search value ρ if they attract applicants. Rearranging means that (w, λ) has to fulfill

$$w = b + \beta\rho + \frac{\lambda}{m(\lambda)}\frac{1-\beta(1-\eta)}{1-\delta}\rho \quad \text{whenever} \quad \lambda > 0 . \tag{2}$$

This condition says that a firm can only recruit workers when its wage offer matches the workers' unemployment value $(1-\beta)U = b + \beta\rho$ plus a premium which is needed to attract workers to vacancies with job-finding probability $m(\lambda)/\lambda$. This premium is increasing in λ. This is a crucial insight. If a firm wants to attract more workers per vacancy in order to fill its vacancy at a faster rate, it has to offer a higher wage. This means that wages are always monotonically related to the job-filling rate. The relationship between worker-job ratios and wage offers is standard in the competitive search literature (e.g., Moen (1997), Acemoglu and Shimer (1999b)).

[19] Bellman equations for employed and unemployed workers are $E = w + \beta[(1-\eta)E + \eta U]$ and $U = b + \beta[m(\lambda)\lambda^{-1}(1-\delta)E + (1 - m(\lambda)\lambda^{-1}(1-\delta))U]$. Equation (1) follows by substituting the first into the second equation.

Firms' Recruitment Policy

Let $J^x(L,W)$ be the profit value of a firm with productivity x, an employment stock of L workers and a commitment to a total wage bill of W. An entrant firm's profit value is then $J^x(0,0)$. The firm's recruitment choice involves deciding the number of posted vacancies V as well as the offered wage w. The firm correctly anticipates that higher wage offers attract more applicants per vacancy. That is, the firm takes the optimal search decisions of workers, captured by equation (2), as given. Its recursive profit maximization problem is expressed as[20]

$$\begin{aligned} J^x(L,W) \;&=\; \max_{(w,\lambda,V)} xF(L) - W - C(V,L,x) + \beta(1-\delta)J^x(\hat{L},\hat{W}) \;, \\ & \text{s.t. } \hat{L} = L(1-s) + m(\lambda)V \;,\; \hat{W} = W(1-s) + m(\lambda)Vw \;, \\ & \lambda \geq 0,\; V \geq 0 \;,\; \text{and condition (2)} \;. \end{aligned} \tag{3}$$

The first line reflects the value of output minus wage and hiring costs, plus the discounted value of continuation with an adjusted workforce and its associated wage commitment. The second line captures that employment next period consists of the retained workers and the new hires. For the wages, since separations are random they reduce the wage bill proportionally, and new commitments are added for the new hires.[21]

The solutions to problem (3) are characterized by one intra-temporal and one inter-temporal optimality condition, equations (4) and (5) below, that we derive in the Appendix (proof of Proposition 1). We also show that the firm's recruitment strategy is independent of past wage commitments and can be described by two policy functions: $\lambda^x(L)$ denotes the number of applicants that a firm of size L and productivity x wants to attract per vacancy. This directly determines the wage it has to post according to (2). Given a choice of λ, the firm decides on the number of number of posted vacancies. We denote this policy function by $V^x(L,\lambda)$.

The intra-temporal optimality condition describes the choice between the two recruitment tools of the firm within a period: the number of posted vacancies V on the one hand, and the worker-job ratio λ (and thus the posted wage w) on the other:

$$C_1(V,L,x) \geq \beta\rho\frac{m(\lambda) - \lambda m'(\lambda)}{m'(\lambda)} \;,\; V \geq 0 \;, \tag{4}$$

with complementary slackness. This condition can be derived by minimizing the sum of recruitment costs and wage costs conditional on the requirement of hiring a given amount $H = m(\lambda)V$ of workers this period. The left-hand side gives the marginal cost of posting one more vacancy. When a firm posts more vacancies, it can lower its wage offer and the associated hiring rate while keeping the total number of hires constant. The net present value of wage savings are represented by the right-hand side, an increasing function of λ. From the assumptions on C follows that the

[20] The wage bill enters as a state variable because of wage commitments, but it turns out to be irrelevant for the firm's policy choices, as we see below.

[21] Note again that separation and exit rates are exogenous. In steady state and in the absence of shocks, this is without loss of generality, since no firm would want to layoff workers or exit the market. Section 4 considers endogenous exit and layoff rates in the presence of aggregate and idiosyncratic shocks.

left side is (weakly) decreasing in L and strictly increasing in V. Hence the implicit solution to this equation yields the policy function $V = V^x(L, \lambda)$ which is increasing in λ and (weakly) increasing in L. Intuitively, with higher λ the probability to fill a vacancy increases, and hence the firm is willing to bear higher marginal recruitment costs by advertising more jobs. And with higher L, marginal recruitment costs fall and hence the firm is inclined to post more vacancies. Clearly, if costs are linear in vacancies and independent of the other variables, then the optimal job-filling rate according to (4) is independent of any firm or size characteristics. In that case the model loses the connection between the desire to expand and the attractiveness of the job offers that are posted, because any adjustment can be achieved through vacancy creation on which there are no bounds even for the smallest firms. Equation (4) also provides an intuition why it is never optimal for a firm to offer different wages at a given point in time: the firm compares the wage costs against the vacancy costs, and there is a unique value that balances this trade-off.

The inter-temporal optimality condition describes how the firm grows over time. For a firm which hires in the current and in the next period ($\lambda, \hat{\lambda} > 0$), we obtain the Euler equation

$$xF'(\hat{L}) - C_2(\hat{V}, \hat{L}, x) - b - \beta\rho = \frac{\rho}{1-\delta}\left[\frac{1}{m'(\lambda)} - \frac{\beta(1-\eta)}{m'(\hat{\lambda})}\right] . \tag{5}$$

Here $\hat{L}$, $\hat{V}$, and $\hat{\lambda}$ are next period's values of employment, vacancy postings and worker-job ratio. Since next period's employment and vacancy creation are fully determined by the worker-job ratios in this and the next period (through (4) and the first constraint in (3)), this condition links the worker-job ratio and the related hiring rate inter-temporally. The left-hand side of this equation captures the marginal benefit of a higher workforce in the next period. If this is high, then the firm rather hires more workers now than to wait and hire them next period. This is captured by the right-hand side, which is increasing in the current worker-job ratio but decreasing in next period's worker-job ratio. In particular, this means that a more productive firm wants to achieve fast growth by offering a more attractive contract, thus raising the worker-job ratio and the job-filling rate.

The following proposition and its corollaries provide comparative statics results for the job-filling rate and the growth rate of firms. The job-filling rate is linked to the earnings offer, so that these comparative statics carry over to the offered net present value of wages to new hires. These characterization results depend crucially on the supermodularity of the value function, which renders this proof non-trivial. While standard techniques (Amir (1996)) can be applied when the cost function is independent of firm size and productivity, this is no longer true in the more general setting.

Proposition 1: *For any value $\rho > 0$, the firm's value function $J^x(L, W)$ is strictly increasing and strictly concave in L, increasing in x, strictly supermodular in (x, L), and continuous and decreasing in ρ. The firm's policy functions are independent of wage commitments W. For a hiring firm the policy function $\lambda^x(L)$ is strictly increasing in x and strictly decreasing in L. Posted vacancies $V^x(L, \lambda)$ are increasing in L and strictly increasing in λ.*

Proof: Appendix.

Since these results hold for any search value ρ, they also apply when this value is determined in general equilibrium. These results directly imply

Corollary 1: *Conditional on size, more productive firms pay higher lifetime wages and have a higher job-filling rate. Conditional on productivity, younger (and smaller) firms pay higher lifetime wages and have a higher job-filling rate.*

In the Appendix, we also prove

Corollary 2: *If parameter h in the recruitment technology is sufficiently small, more productive firms have a higher growth rate, conditional on size; and larger firms have a lower growth rate, conditional on productivity.*

A useful illustration how firms grow over time can be provided in a special case. Consider a firm with productivity x that enters in some period τ. Its job creation policy is then described by a sequence $(L_t, \lambda_t, V_t)_{t\geq\tau}$ starting from $L_\tau = 0$. Posted vacancies $V_t = V^x(L_t, \lambda_t)$ are the implicit solution of equation (4). The employment stock accumulates according to

$$L_{t+1} = (1-s)L_t + m(\lambda_t)V^x(L_t, \lambda_t) \ . \tag{6}$$

In the example with recruitment cost $C(V, L, x) = xF(L) - xF(L - hV) + cV$, equations (4) and (5) can be further simplified to an intertemporal equation which is independent of L_t:

$$\beta\rho\Big[m(\lambda_{t+1}) - \lambda_{t+1}m'(\lambda_{t+1})\Big] - \Big[(b+\beta\rho)h + c\Big]m'(\lambda_{t+1}) = \frac{\rho h}{1-\delta}\Big[\frac{m'(\lambda_{t+1})}{m'(\lambda_t)} - \beta(1-\eta)\Big] \ . \tag{7}$$

In Lemma 3 of the Appendix, we show that this equation has a unique steady state $\lambda^* > 0$ if recruitment costs are low enough, and λ_t converges to λ^* from any initial value $\lambda_\tau > 0$. Figure 1 shows the phase diagram for the system (6) and (7). The curve where the employment stock is constant ($L_t = L_{t+1}$) is downward sloping since (4) implies that $V^x(L, \lambda)/L$ is increasing in L. If the condition

$$xF'(0) > b + \beta\rho + \frac{\rho[1-\beta(1-\eta)]}{(1-\delta)m'(\lambda^*)}$$

holds, there exists a unique stationary employment level $L^* > 0$. The corresponding dynamics imply further that there is a downward-sloping saddle path converging to the long-run employment level. Graphically, the firm's policy function $\lambda^x(L)$ traces this saddle path.

It follows from these considerations that the firm's recruitment policy is characterized by a path of declining wage offers and job-filling rates along the transition to the firm's long-run employment level. Concavity of the firm's production function implies that the firm wants to spread out its recruitment costs across several periods. This statement remains true for other forms of the recruitment technology. Only when recruitment costs are linear in vacancies, $C(V) = cV$, the firm would choose a constant λ^* (and hence post the same wage in all periods). In that case, it would immediately jump to its optimal size by recruiting L^* workers in the entry period and then keep the employment level constant. As soon as recruitment costs are strictly convex, such a policy is not optimal, and it may not be feasible due to the capacity constraint

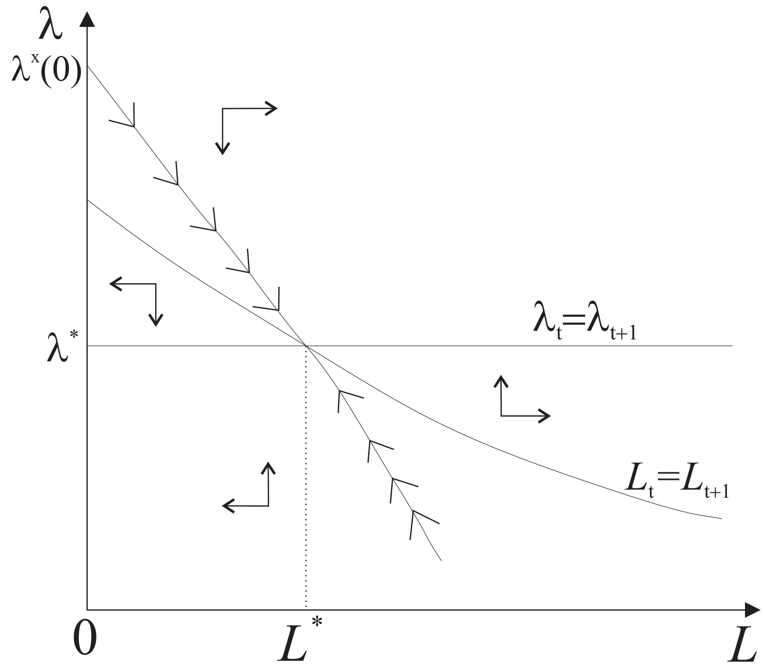

Figure 1: The firm's optimal recruitment policy follows the declining saddle path.

on labor input in recruitment. A further insight of this example is that the stationary firm size depends positively on x: a more productive firm grows larger and offers higher lifetime wages on its transition to the long-run employment level.

Firm Entry and General Equilibrium

No entrant makes a positive profit when the expected profit income of a new firm does not exceed the entry cost, that is,

$$\sum_{x \in X} \pi(x) J^x(0,0) \leq K \ , \tag{8}$$

with equality if firm entry is positive. This condition implicitly pins down the worker's job surplus ρ and therefore, via the firm's optimal recruitment policy, unemployment-vacancy ratios for all offered wages. In a stationary equilibrium, a constant mass of N_0 firms enters the market in every period, so that there are $N_a = N_0(1-\delta)^a$ firms of age a in any period. Let $(L_a^x, \lambda_a^x, V_a^x, w_a^x)_{a \geq 0}$ be the employment/recruitment path for a firm with productivity x. Then, a firm of age a with productivity x has L_a^x employed workers, and $\lambda_a^x V_a^x$ unemployed workers are searching for jobs which offer wage w_a^x. Therefore, the mass of entrant firms N_0 is uniquely pinned down from aggregate resource feasibility:

$$1 = \sum_{a \geq 0} N_0(1-\delta)^a \sum_{x \in X} \pi(x)[L_a^x + \lambda_a^x V_a^x] \ . \tag{9}$$

This equation says that the unit mass of workers is either employed or unemployed. We now define a stationary equilibrium.

Definition: A *stationary competitive search equilibrium* is a list

$$\Big(\rho, N_0, (L_a^x, \lambda_a^x, V_a^x, w_a^x)_{x\in X, a\geq 0}\Big)$$

such that

(a) Unemployed workers' job search strategies maximize utility. That is, the relationship between wages and worker-job ratios satisfies (2) for all (w_a^x, λ_a^x).

(b) Firms' recruitment policies are optimal. That is, given ρ and for all $x \in X$, $(L_a^x, \lambda_a^x, V_a^x)_{a\geq 0}$ describes the firm's growth path, obtained from the policy functions solving problem (3).

(c) There is free entry of firms, i.e. (8) and $N_0 \geq 0$ with complementary slackness.

(d) The number of entrant firms is consistent with aggregate resource feasibility, equation (9).

Since firms' behavior has already been characterized, it remains to explore equilibrium existence and uniqueness.

Proposition 2: *A stationary competitive search equilibrium exists and is unique. There is strictly positive firm entry provided that K is sufficiently small.*

Proof: Appendix.

It is worthwhile to briefly summarize some cross-sectional implications of our theory. Different firms in the equilibrium cross-section (x, L) have different recruitment and wage policies. Corollaries 1 and 2 point out that job-filling rates and firm growth rates depend positively on x and negatively on L. Hence, when recruitment time input h is small enough, firm growth rates correlate positively with job-filling rates. Such a relationship has been documented by Davis, Faberman, and Haltiwanger (2010) who find that firms that grow faster do so through a higher job-filling rate on top of expanding the number of vacancies. Furthermore, since job-filling rates relate directly to the earnings of new hires, the two corollaries also imply that faster-growing firms offer higher lifetime wages. Belzil (2000) documents such patterns after controlling for size and worker characteristics; he shows that wages, particularly those of new hires, are positively related to a firm's job creation. Lastly, a well established fact in labor economics is the positive relationship between size, productivity and pay (e.g., Brown and Medoff (1989), Oi and Idson (1999)). In our model, wages of new hires depend positively on x and negatively on L. Since more productive firms grow larger, a positive wage-size relation emerges in our model if the dispersion in productivity is large enough.[22] Moreover, such a relation also obtains if sufficiently

[22]We note that enough productivity dispersion is also required in models without commitment over future wages, and even more so because wages of *all* workers decline in a growing firm. In our model with wage commitment, more productive firms have grown faster in the past, and hence pay higher lifetime wages to their *existing* workers.

many firms are close to their long-run employment level (which happens if the exit rate δ is rather low): Among those firms with nearly constant employment, the larger (and more productive) firms need to hire more to keep their workforce constant. Therefore, among the cross-section of established firms the larger ones offer higher lifetime wages.[23] We emphasize again that our theory, with risk neutral workers and firms, only makes predictions about lifetime wage contracts, but it says nothing about individual wage-tenure profiles or about wage dispersion within firms.

3.3 Efficiency

The social planner decides at each point in time about firm entry, job creation and unemployment-vacancy ratios for all recruiting firms in the economy. The planner takes as given the numbers of firms that entered in some earlier period, as well as the employment stocks of all these firms. Formally, the planner's state vector is

$$\sigma = (N_a, L_a^x)_{a \geq 1, x \in X} \ ,$$

where N_a is the mass of firms of age $a \geq 1$, and L_a^x is employment of a firm with productivity x and age a. It is no restriction to impose that all firms of a given type (a, x) are equally large. The planner maximizes the present value of output net of opportunity costs of employment and net of the costs of firm and job creation. With $\hat{\sigma}$ to denote the state vector in the next period, the recursive formulation of the social planning problem is

$$\begin{aligned} S(\sigma) \quad = \quad & \max_{N_0, (V_a^x, \lambda_a^x)_{a \geq 0}} \left\{ \sum_{a \geq 0} N_a \sum_{x \in X} \pi(x) \Big[x F(L_a^x) - b L_a^x - C(V_a^x, L_a^x, x) \Big] \right\} - K N_0 + \beta S(\hat{\sigma}) \\ & \text{s.t. } L_0^x = 0, \ \hat{L}_{a+1}^x = (1-s) L_a^x + m(\lambda_a^x) V_a^x, \ a \geq 0, \ x \in X \ , \\ & \hat{N}_{a+1} = (1-\delta) N_a \ , \ a \geq 0 \ , \\ & \sum_{a \geq 0} N_a \sum_{x \in X} \pi(x) \Big(L_a^x + \lambda_a^x V_a^x \Big) \leq 1 \ . \end{aligned} \tag{10}$$

The last condition is the economy's resource constraint. It states that the mass of of all individuals that are attached to some firm of type (a, x), either as workers L_a^x or as unemployed workers queuing up for a job at this firm $\lambda_a^x V_a^x$, may not exceed one. We say that a solution to problem (10) is *socially optimal*.

Proposition 3: *The stationary competitive search equilibrium is socially optimal.*

Proof: Appendix.

The efficiency of equilibrium can be linked to a variant of the well-known Hosios (1990) condition. It says that efficient job creation requires that the share of the surplus that firms get

[23]Equations (4) and (5), evaluated at the stationary employment level, imply that V and λ are positively related (provided that $k'' > 0$). It follows that the worker-job ratio increases in the number of hires $m(\lambda)V$.

upon matching with a worker is equal to the elasticity of the job-finding rate with respect to the job-worker ratio $1/\lambda$ (which is one minus the elasticity of m). For a hiring firm ($V > 0, \lambda > 0$) we can exploit the first-order conditions for problem (3) (equations (58) and (59) in the Appendix) to obtain

$$C_1(V, L, x) = \Big[1 - \frac{\lambda m'(\lambda)}{m(\lambda)}\Big] m(\lambda)(1-\delta)\beta \Big\{ J_1^x(\hat{L}, \hat{W}) - \frac{b + \beta\rho}{1 - \beta(1-\eta)} \Big\} .$$

This equation compares the marginal cost of a vacancy on the left-hand side to the marginal benefit on the right-hand side. The term in squared brackets captures the surplus share accruing to the firm, and the last terms capture the expected discounted total surplus of a vacancy. With probability $m(\lambda)(1-\delta)$, the vacancy is filled and survives to the next period. The term in braces represent the marginal surplus of a filled job, which is the difference between the expected discounted marginal increase in net output, $J_1^x(\hat{L}, \hat{W})$, and the discounted value of the worker's opportunity cost. Overall, the equation states that firms create vacancies exactly to the point where their marginal benefit coincides with the value specified by the appropriate Hosios condition for large firms.[24]

4 Productivity Shocks and Firm Dynamics

We now extend the previous model to include both idiosyncratic (firm-specific) and aggregate productivity shocks. This extension allows us to explore not only two margins of job creation (firm entry and firm growth), but also the two margins of job destruction (firm exit and firm contraction). Output of a firm with L workers is $xzF(L)$ where $x \in X$ is idiosyncratic productivity and $z \in Z$ is aggregate productivity. Both x and z follow Markov processes on finite state spaces X and Z with respective transition probabilities $\pi(x_+|x)$ and $\psi(z_+|z)$. An entrant firm pays fixed cost K and draws an initial productivity level $x_0 \in X$ with probability $\pi_0(x_0)$. For a firm of age $a \geq 0$, let $x^a = (x_0, \ldots, x_a) \in X^{a+1}$ denote the history of idiosyncratic productivity, and let $z^t = (z_0, \ldots, z_t)$ be the history of aggregate shocks at time t. Write $\psi(z^t)$ and $\pi(x^a)$ for the unconditional probabilities of aggregate and idiosyncratic productivity histories.

We assume that an active firm incurs a fixed operating cost $f \geq 0$ per period. This parameter is required to obtain a non-trivial exit margin.[25] In this section we are as agnostic as possible about the recruitment cost function; we only assume that C is strictly increasing and convex in posted vacancies. Firms exit with exogenous probability $\delta_0 \geq 0$ which is a lower bound for the actual exit rates $\delta \geq \delta_0$. Similarly, workers quit a job with exogenous rate $s_0 \geq 0$ which provides a lower bound for the actual separation rates $s \geq s_0$.[26]

[24] See also the derivation of the Hosios condition in a bargaining context in Hawkins (2010).

[25] A non-trivial exit margin could also obtain in the presence of firing costs when $f = 0$.

[26] Although this model ignores many important worker flows, such as those between jobs and the flows in and out of the labor force, parameter s_0 also represents a measure of exogenous worker turnover, as in Fujita and Nakajima (2009).

The timing within each period is as follows.[27] *First*, aggregate and idiosyncratic productivities are revealed, new firms enter, and all firms decide about separations and exit. *Second*, firms decide about recruitment, and recruiting firms are matched with unemployed workers. An unemployed worker who has just left another job (due to firm exit, quit or layoff) can search for reemployment within the same period. And *third*, production takes place. In the following, we first describe the planning problem before we show its equivalence to a competitive-search equilibrium in Section 4.4.

4.1 The Planning Problem

The planner decides at each point in time about firm entry and exit, layoffs and job creation, as well as unemployment-vacancy ratios (i.e. matching rates) for all firm types. In a given aggregate history z^t, we denote by $N(x^a, z^t)$ the mass of firms of age a with idiosyncratic history x^a. Similarly, $L(x^a, z^t)$ is the employment stock of any of these firms. At every history node z^t and for every firm type x^a, the planner decides an exit probability $\delta(x^a, z^t) \geq \delta_0$, a separation rate $s(x^a, z^t) \geq s_0$, vacancy postings $V(x^a, z^t) \geq 0$, and a worker-job ratio $\lambda(x^a, z^t)$ for firms of type (x^a, z^t).[28] The numbers of firm types change between periods $t-1$ and t according to

$$N(x^a, z^t) = [1 - \delta(x^a, z^t)]\pi(x_a|x_{a-1})N(x^{a-1}, z^{t-1}) \ , \tag{11}$$

and the employment stock at any of these firms adjusts to

$$L(x^a, z^t) = [1 - s(x^a, z^t)]L(x^{a-1}, z^{t-1}) + m(\lambda(x^a, z^t))V(x^a, z^t) \ . \tag{12}$$

To simplify notation, we define $\overline{L}(x^a, z^t) \equiv [1 - s(x^a, z^t)]L(x^{a-1}, z^{t-1})$ as the employment stock after separations and before recruitment, which is an argument of the firm's recruitment cost function $C(V, \overline{L}, x)$.

At time $t = 0$, the planner takes as given the numbers of firms that entered the economy in some earlier period, as well as the employment stock of each of these firms. Hence, the state vector at date 0, prior to the realization of productivities, is summarized by the initial firm distribution $(N(x^{a-1}, .), L(x^{a-1}, .))_{a\geq 1, x^{a-1} \in X^a}$. In a given history z^t, the planner also decides the mass of new entrants $N_0(z^t) \geq 0$, so that

$$N(x_0, z^t) = [1 - \delta(x_0, z^t)]\pi_0(x_0)N_0(z^t) \text{ and } L(x_0, z^t) = m(\lambda(x_0, z^t))V(x_0, z^t) \ . \tag{13}$$

[27] The timing is slightly different from the previous section. With shocks arriving at the start of the period, it seems more sensible to allow firms to exit right after the shock, and also to allow workers to immediately find reemployment after separations have taken place. In the previous section, the different timing clearly distinguishes between the current recruitment costs and the future benefits in production. The characterization and efficiency results can be proven for either environment.

[28] To save on notation, we do not allow the planner to discriminate between workers with different firm tenure. Given that there is no learning-on-the-job, there is clearly no reason for the planner to do so. Nonetheless, the competitive search equilibrium considered in 4.4 allows firms to treat workers in different cohorts differently, which is necessary because firms offer contracts sequentially and are committed to these contracts. See the proof of Proposition 6 for further elaboration of this issue.

The planning problem is

$$\max_{\delta,s,V,\lambda,N_0} \sum_{t\geq 0,z^t} \beta^t \psi(z^t) \Bigg\{ -KN_0(z^t) + \tag{14}$$

$$\sum_{a\geq 0,x^a} N(x^a,z^t)\Big[x_a z_t F(L(x^a,z^t)) - bL(x^a,z^t) - f - C(V(x^a,z^t),\overline{L}(x^a,z^t),x_a)\Big]\Bigg\}$$

subject to the dynamic equations for N and L, namely (11), (12) and (13), and subject to the resource constraints, for all $z^t \in Z^{t+1}$,

$$\sum_{a\geq 0,x^a} N(x^a,z^t)\Big[\overline{L}(x^a,z^t) + \lambda(x^a,z^t)V(x^a,z^t)\Big] \leq 1 \ . \tag{15}$$

This constraint says that the labor force (employment plus unemployment) cannot exceed the given unit mass of workers. The first part of the sum, namely

$$\sum_{a\geq 0,x^a} N(x^a,z^t)\overline{L}(x^a,z^t) \ ,$$

are the workers that are employed in some firm after separations have taken place. The remaining part of the sum are unemployed workers queueing up for employment in one of the active firms with $V(.)$ vacancies and unemployment-vacancy ratio $\lambda(.)$. For instance, there are $N(x^a,z^t)$ active firms with productivity history x^a, each of which posts $V(x^a,z^t)$ vacancies attracting $\lambda(x^a,z^t)V(x^a,z^t)$ unemployed workers. We summarize a solution to the planning problem by a vector $(\mathbf{N},\mathbf{L},\mathbf{V},\boldsymbol{\lambda},\mathbf{s},\boldsymbol{\delta})$, with $\mathbf{N} = (N(x^a,z^t))_{a,t\geq 0}$ etc.

4.2 Characterization of the Planning Solutions

There is a convenient characterization of a planning solution which says that exit, layoff, and hiring decisions follow a recursive equation at the level of the individual firm. Let $\beta^t\psi(z^t)\mu(z^t) \geq 0$ be the multiplier on the resource constraint (15) in history node z^t. Intuitively, $\mu(z^t)$ is the social value of a worker in history z^t. Let $G_t(L,x,z^t)$ denote the social value of an existing firm with employment stock L, idiosyncratic productivity x and aggregate productivity history z^t. The sequence G_t obeys the recursive equations

$$G_t(L,x,z^t) = \max_{\delta,s,V,\lambda}(1-\delta)\Bigg\{xz_tF(\hat{L}) - b\hat{L} - \mu(z^t)[(1-s)L+\lambda V] - C(V,(1-s)L,x)$$

$$-f + \beta\sum_{\hat{x}\in X}\sum_{z_{t+1}\in Z}\pi(\hat{x}|x)\psi(z_{t+1}|z_t)G_{t+1}(\hat{L},\hat{x},z^{t+1}))\Bigg\} \tag{16}$$

$$\text{s.t.} \quad \hat{L} = (1-s)L + m(\lambda)V \ ,$$
$$\delta \in [\delta_0, 1], \ s \in [s_0, 1], \ \lambda \geq 0, \ V \geq 0 \ .$$

The interpretation of these equations is rather straightforward. The planner wants a firm with characteristics (L, x) to stay active in aggregate history z^t whenever the term in braces is non-negative, otherwise he sets $\delta = 1$. The term in braces gives the value of an active firm. In the current period, this value encompasses the firm's output net of the opportunity cost of employment, net of fixed costs and recruitment costs, and net of the social cost of workers tied to the firm in this period; these workers include those that are retained from the previous period, namely $(1-s)L$, and also λV unemployed workers who aim to find a job at the firm (of which $m(\lambda)V \leq \lambda V$ eventually find a job).

Proposition 4:

(a) *For given multipliers $\mu(z^t)$, there exist value functions $G_t : \mathbb{R}_+ \times X \times Z^{t+1} \to \mathbb{R}$, $t \geq 0$, satisfying the system of recursive equations (16).*

(b) *If $\mathbf{X} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \boldsymbol{\lambda}, \mathbf{s}, \boldsymbol{\delta})$ is a solution of the planning problem (14) with multipliers $\boldsymbol{\mu} = (\mu(z^t))$, then the corresponding firm policies also solve problem (16) and the complementary-slackness condition*

$$\sum_{x \in X} \pi_0(x) G_t(0, x, z^t) \leq K \ , \ N_0(z^t) \geq 0 \tag{17}$$

is satisfied for all z^t. Conversely, if $\mathbf{X}$ solves for every firm problem (16) with multipliers $\boldsymbol{\mu}$, and if condition (17) and the resource constraint (15) hold for all z^t, then $\mathbf{X}$ is a solution of the planning problem (14).

Whilst Proposition 4 is a useful characterization of planning solutions, it cannot be applied for computational purposes. The difficulty is that the multipliers $\mu(z^t)$ are non-stationary and depend on the initial firm distribution. However, a much more powerful characterization can be obtained under the provision that firm entry is positive in all states of the planning solution, so that the first inequality in (17) is binding. When this is the case, the firm-level value functions and the social value of a worker are independent of the firm distribution. This is our aggregate-arbitrage property, which we discuss in the introduction and which relates to the concept of block recursivity introduced by Shi (2009) and Menzio and Shi (2009, 2010).

To gain intuition for the independence from the distribution of existing firms, envision only a single period. The planner can assign unemployed workers either to existing firms or to new firms. If there are many existing firms, there are fewer workers left to be assigned to new firms. Nevertheless, the social value of any worker that is assigned to a new firm does not change: Each new firm has an optimal size, and if less workers are assigned to new firms, then proportionally less new firms will be created, leaving the marginal value of each worker unchanged. Therefore, as long as any new firms are created, efficient hiring by existing firms requires their marginal social benefit of hiring to be equal to the social benefit at the new firms. This logic extends to the case with many periods. A new firm today may also hire workers tomorrow, but its marginal social benefit of tomorrow's hires has to equal the marginal benefit of hiring at tomorrow's new firms, which depends on the next aggregate state but is again independent of the firm distribution. Thus, the social value of assigning a worker to any firm is tied to the social benefit created at new firms which depends on the aggregate state alone. While this ensures that job creation and destruction policies of individual firms are independent of the distribution of existing firms, the firm distribution does matter for the dynamics of aggregate labor market variables, such as the workers' job finding probability: If there are more existing firms, then more of the workers queue for their jobs and obtain a different probability of getting hired.

To see the independence of value functions from the firm distribution formally, suppose there are n aggregate states z_i, $i = 1, \ldots, n$, and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$ be a vector of social values in these states. Let $G^i(L, x, \mu)$ be the social value of a firm with employment stock L, idiosyncratic productivity x and aggregate productivity z_i, for $i = 1, \ldots, n$. $G = (G^i) : \mathbb{R}_+ \times X \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfy the Bellman equations

$$\begin{aligned} G^i(L, x, \mu) \quad = \quad & \max (1-\delta) \Big\{ x z_i F(\hat{L}) - b\hat{L} - f - \mu_i[(1-s)L + \lambda V] \\ & -C(V, L(1-s), x) + \beta \sum_{\hat{x} \in X} \sum_{z_j \in Z} \pi(\hat{x}|x) \psi(z_j|z_i) G^j(\hat{L}, \hat{x}, \mu) \Big\} , \end{aligned} \tag{18}$$

where maximization is subject to the same constraints as in problem (16). Positive entry in all aggregate states requires that the expected social value of a new firm is equal to the entry cost,

$$\sum_{x \in X} \pi_0(x) G^i(0, x, \mu) = K \ . \tag{19}$$

This characterization of planning solutions by $(G^i, \mu_i)_{i=1,\ldots,n}$ is particularly helpful for numerical applications. Despite considerable firm heterogeneity, the model can be solved by a recursive problem on a low-dimensional state space (18) and the (simultaneous) solution of a finite-dimensional fixed point problem (19). Importantly, the distribution of firms is irrelevant for this computation. After the corresponding policy functions have been calculated, the actual number of entrant firms $N_0(z^t)$ is obtained as a residual of the economy's resource constraint in a simulation of the model, and thus it *does depend* on the distribution of existing firms. It follows that the evolution of aggregate employment, output and job flows depend on the firm distribution as well. Because of the dependence of N_0 on the distribution of employment among existing firms, it cannot be guaranteed that the planning solution has positive entry in all state histories. Therefore, this property can only be checked ex-post in simulations of the model. Analytically, we prove that any solution of (18)–(19) which gives rise to positive entry in all state histories coincides with a solution to the planner's problem. We also find that a unique solution of these equations exists for small aggregate shocks.

Proposition 5:

(a) *Suppose that a solution of (18) and (19) exists which defines an allocation* $\mathbf{X} = (\mathbf{N}, \mathbf{L}, \mathbf{V}, \boldsymbol{\lambda}, \mathbf{s}, \boldsymbol{\delta})$ *satisfying* $N(z^t) > 0$ *for all* z^t*. Then* $\mathbf{X}$ *is a solution of the planning problem (14).*

(b) *If* K*,* f*, and* b *are sufficiently small and if* $z_1 = \ldots = z_n = z$*, equations (18) and (19) have a unique solution* $G(L, x, \mu)$ *with* $\mu_1 = \ldots = \mu_n$*. Moreover, if the transition matrix* $\psi(z_j|z_i)$ *is strictly diagonally dominant and if* $|z_i - z|$ *is sufficiently small for all* i*, equations (18) and (19) have a unique solution.*

4.3 Recruitment and Layoff Strategies

The reduction of the planning solution to problem (18) permits a straightforward characterization of the optimal layoff and hiring strategies. A firm with productivity x and employment stock L should dismiss workers (that is, $s > s_0$) in state $i = 1, \ldots, n$ iff

$$xz_i F'(L(1-s_0)) - b - \mu_i + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i) G_1^j(L(1-s_0), \hat{x}, \mu) < 0 \ . \tag{20}$$

This expression is the marginal social surplus of a worker at the employment stock $L(1-s_0)$ after worker turnover. If marginal worker surplus is negative, the firm lays off some workers until the marginal worker surplus is nil.

Conversely, for the firm to recruit workers, it must be that $\lambda > 0$ and $V > 0$. In that case, it follows from the first-order conditions for λ and V that[29]

$$C_1(V, L(1-s_0), x) = \mu_i \lambda \Big(\frac{m(\lambda)}{\lambda m'(\lambda)} - 1 \Big) \ . \tag{21}$$

As in the previous section, it follows from concavity of m and convexity of C that there is an increasing relation between the worker-job ratio and the number of posted vacancies at the firm. With higher λ, the probability to fill a vacancy increases, and hence the planner is willing to post more vacancies at higher marginal recruiting cost. Denote the solution to equation (21) by $V = V_i(\lambda, L, x)$, which is positive for $\lambda > \underline{\lambda}_i(L, x)$. The planner's optimal choice of λ for firm (L, x) in aggregate state i satisfies

$$xz_i F'(\hat{L}) - b + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i) G_1^j(\hat{L}, \hat{x}, \mu) = \frac{\mu_i}{m'(\lambda)} \ ,$$

with $\hat{L} = L(1-s_0) + m(\lambda)V_i(\lambda, L, x)$. Therefore, the firm recruits workers, if and only if

$$xz_i F'(L(1-s_0)) - b + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i) G_1^j(L(1-s_0), \hat{x}, \mu) > \frac{\mu_i}{m'(\underline{\lambda}_i(L, x))} \ . \tag{22}$$

[29] This equation is straightforward to derive and analogous to (4).

The two conditions (20) and (22) illustrate how the firm's strategy depends on its characteristics (L, x). Small and productive firms recruit workers and grow, whereas large and unproductive firms dismiss workers and shrink. Depending on the functional forms of $C(.)$ and $m(.)$, there can also be an open set of characteristics where firms do not adjust their workforce.[30] Similar patterns for employment adjustment are obtained in the models of Bentolila and Bertola (1990) and Elsby and Michaels (2010).

4.4 Decentralization

We now describe a competitive search equilibrium and demonstrate that competitive search gives rise to the same allocation as the planning solution characterized in Proposition 4. Firms offer workers a sequence of state-contingent wages, to be paid for the duration of the match. They also commit to cohort-specific and state-contingent separation probabilities. Contracts are contingent on the idiosyncratic productivity history of the firm at age k, x^k, and on the aggregate state history z^t at time t. Formally, a contract offered by a firm of age a at time T takes the form

$$\mathcal{C}_a = \Big(w_a(x^k, z^t), \varphi_a(x^k, z^t)\Big)_{k \geq a, t = T+k-a},$$

where $w_a(x^k, z^t)$ is the wage paid to the worker in firm history (x^k, z^t), conditional on the worker being still employed by the firm in that instant. $\varphi_a(x^k, z^t) \geq \delta_0 + (1-\delta_0)s_0$, for $k > a$, is the probability of a job separation prior to the production stage in history x^k. In the hiring period, a separation cannot occur, so $\varphi_a(x^a, z^T) = 0$ by definition.

The Workers' Search Problem

Let $U(z^t)$ be the utility value of an unemployed worker in history z^t, and let $W(\mathcal{C}_a, x^k, z^t)$ be the utility value of a worker hired by a firm of age a in contract $\mathcal{C}_a$ who is currently employed at that firm in history x^k, with $k \geq a$. The latter satisfies the recursive equation

$$\begin{aligned} W(\mathcal{C}_a, x^k, z^t) \;&=\; \varphi_a(x^k, z^t)U(z^t) + (1 - \varphi_a(x^k, z^t))\Big[w_a(x^k, z^t) \\ &\quad + \beta \sum_{x_{k+1}} \sum_{z_{t+1}} \pi(x_{k+1}|x_k)\psi(z_{t+1}|z_t) W(\mathcal{C}_a, x^{k+1}, z^{t+1})\Big] . \end{aligned} \tag{23}$$

An unemployed worker searches for contracts which promise the highest expected utility, considering that more attractive contracts are less likely to sign. The worker observes all contracts $\mathcal{C}_a$ and he knows that the probability to sign a contract is $m(\lambda)/\lambda$ when λ is the worker-job ratio at the offered contract. That is, potential contracts are parameterized by the tuple $(\lambda, \mathcal{C}_a)$. Unemployed workers apply for those contracts where expected surplus is maximized:

$$\rho(z^t) = \max_{(\lambda, \mathcal{C}_a)} \frac{m(\lambda)}{\lambda} \Big[W(\mathcal{C}_a, x^a, z^t) - b - \beta E_{z_t} U(z^{t+1})\Big] . \tag{24}$$

[30] Such inactivity states exist if marginal adjustment costs are strictly positive; this is either the case when $C_1(0, L, x) > 0$ or when $m'(0) < 1$. The latter condition says that matching frictions do not vanish asymptotically when $\lambda \to 0$.

Because an unemployed worker gets one chance to search in every period, his Bellman equation reads as

$$U(z^t) = b + \rho(z^t) + \beta E_{z_t} U(z^{t+1}) \ . \tag{25}$$

The Firms' Problem

A firm of age a in history (x^a, z^t) takes as given the employment stocks of workers hired in some earlier period, $(L_\tau)_{\tau=0}^{a-1}$, as well as the contracts signed with these workers, $(\mathcal{C}_\tau)_{\tau=0}^{a-1}$. The firm chooses an exit probability δ and cohort-specific layoff probabilities s_τ. For these probabilities to be consistent with separation probabilities specified in existing contracts, it must hold that $\delta \le \varphi_\tau(x^a, z^t)$ for all $\tau \le a-1$, and $s_\tau = 1 - (1 - \varphi_\tau(x^a, z^t))/(1-\delta)$ when $\delta < 1$, with arbitrary choice of s_τ when $\delta = 1$. The firm also decides new contracts $\mathcal{C}_a$ to be posted in V vacancies attracting worker-job ratio λ. It is no restriction to presuppose that the firm offers only one type of contract. When J_a is the value function of a firm of age a, the firm's problem is written as

$$\begin{aligned} J_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] \quad = \quad & \max_{(\delta,\lambda,V,\mathcal{C}_a)} (1-\delta)\Bigg\{ x_a z_t F\Big(\sum_{\tau=0}^{a} \hat{L}_\tau\Big) - W - C(V, \overline{L}, x_a) \\ & -f + \beta E_{x_a, z_t} J_{a+1}\Big[(\mathcal{C}_\tau)_{\tau=0}^{a}, (\hat{L}_\tau)_{\tau=0}^{a}, x^{a+1}, z^{t+1}\Big]\Bigg\} \end{aligned} \tag{26}$$

$$\text{s.t.} \quad \hat{L}_a = m(\lambda)V, \ , \lambda \ge 0, \ V \ge 0, \ \hat{L}_\tau = L_\tau \frac{1 - \varphi_\tau(x^a, z^t)}{1-\delta}, \ \tau \le a-1 \ , \tag{27}$$

$$\delta \in [\delta_0, \min_{0 \le \tau \le a-1} \varphi_\tau(x^a, z^t)], \ s_0(1-\delta) \le (1 - \varphi_\tau(x^a, z^t)) \ , \tag{28}$$

$$W = \sum_{\tau=0}^{a} w_\tau(x^a, z^t)\hat{L}_\tau \ , \ \overline{L} = \sum_{\tau=0}^{a-1} \hat{L}_\tau, \tag{29}$$

$$W(\mathcal{C}_a, x^a, z^t) = b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda \rho(z^t)}{m(\lambda)} \text{ when } \lambda > 0 \ . \tag{30}$$

The last condition is the workers' participation constraint; it specifies the minimum expected utility that contract $\mathcal{C}_a$ must promise in order to attract a worker queue of length λ per vacancy.

Definition: A competitive search equilibrium is a list

$$\Big[U(z^t), \rho(z^t), \mathcal{C}_a(x^a, z^t), \lambda(x^a, z^t), V(x^a, z^t), \delta(x^a, z^t), J_a(.), L_\tau(x^a, z^t), N(x^a, z^t), N_0(z^t)\Big],$$

for all $t \ge 0$, $a \ge 0$, $x^a \in X^{a+1}$, $z^t \in Z^{t+1}$, $0 \le \tau \le a$, and for a given initial firm distribution, such that

(a) Firms' exit, hiring and layoff strategies are optimal. That is, J_a is the value function and $\mathcal{C}_a(.)$, $\delta(.)$, $\lambda(.)$, and $V(.)$ are the policy functions for problem (26)-(30).

(b) Employment evolves according to

$$\begin{aligned} L_\tau(x^a, z^t) &= L_\tau(x^{a-1}, z^{t-1}) \frac{1-\varphi_\tau(x^a, z^t)}{1-\delta(x^a, z^t)} \ , \ 0 \le \tau \le a-1 \ , \\ L_a(x^a, z^t) &= m(\lambda(x^a, z^t)) V(x^a, z^t) \ , \ a \ge 0 \ . \end{aligned}$$

(c) Firm entry is optimal. That is, the complementary slackness condition

$$\sum_x \pi_0(x) J_0(x, z^t) \le K \ , \ N_0(z^t) \ge 0 \tag{31}$$

holds for all z^t, and the number of firms evolves according to (11) and (13).

(d) Workers' search strategies are optimal, i.e. (ρ, U) satisfy equations (24) and (25).

(e) Aggregate resource feasibility; for all z^t,

$$\sum_{a \ge 0, x^a} N(x^a, z^t) \Big[\lambda(x^a, z^t) V(x^a, z^t) + \sum_{\tau=0}^{a-1} L_\tau(x^a, z^t) \Big] = 1 \ . \tag{32}$$

Proposition 6: *A competitive search equilibrium is socially optimal.*

Proof: Appendix.

Discussion of Wages and Employment Commitment

It is not hard to see that a wage commitment is sufficient for a firm to implement its desired policy, even if it cannot commit to separation rates. Given risk neutrality, the firm can set the wages following any future history exactly equal to the reservation wage (i.e. the flow value of unemployment) which is the sum of unemployment income and the worker's shadow value, $b + \mu(z^t)$. It can achieve any initial transfer to attract workers through a hiring bonus. In this decentralization, the costs of an existing worker are always equal to his social value in the alternative: unemployment and search for another job. Since the flow surplus for any retained worker equals his shadow value, the firm's problem in this case coincides with the planner's problem (16), so that firing and exiting will be exactly up to the socially optimal level even though the firm does not commit to separation rates. Workers do not have any incentive to quit the job unilaterally, either, because they are exactly compensated for their social shadow value from searching.

Similarly, given employment commitment the wage-tenure profiles for individual workers are arbitrary because of risk-neutrality. As we show in the proof of Proposition 6, firms do not need to discriminate in separation rates between workers in different cohorts. Nonetheless, such equilibria are also possible; then workers with higher separation rates will be compensated through higher wage transfers, whereas workers with more stable jobs earn lower wages. Put differently, this model cannot say anything about individual wage-tenure profiles. It only pins down the surplus split between workers and firms.

In our numerical examples, we consider the benchmark case where wage profiles are not dispersed within the firm. That is, all workers within firm (L, x) in history z^t earn the same flow wage $w(L, x, z^t)$. In a competitive-search equilibrium, such a wage profile can be easily calculated using (23) and condition (30).

5 A Calibrated Example

We study the implications of this model by calibrating it to the U.S. labor market. The calibration proceeds in two steps. First, we choose model parameters to match selected long-run features of the U.S. labor market. Second, we study the model's response to aggregate productivity shocks. For illustration purposes we adopt as basic a specification as possible for the firm productivity process and for the vacancy cost function, even though in their general form they allow substantial additional degrees of freedom whose exploration might be useful in future applications.

We choose the period length to be one month and set $\beta = 0.996$ so that the annual interest rate is about 5 percent. We assume a CES matching function $m(\lambda) = (1 + k\lambda^{-r})^{-1/r}$ and set the two parameters k and r to target a monthly job-finding rate of 0.45 (Shimer (2005b)) and an elasticity of the job-finding rate with respect to the vacancy-unemployment ratio of 0.5 which belongs within the range of reasonable values reported in Petrongolo and Pissarides (2001). Since we also target the aggregate vacancy-unemployment ratio at $1/\lambda = 0.72$, we calculate the parameters k and r to attain the two targets at $\lambda = 1/0.72$.

The production technology is Cobb-Douglas with xL^α and a simple Markov process for idiosyncratic productivity. Particularly, we let idiosyncratic productivity attain one of ten equally distant values in the range $[x_{min}, 1]$, uniformly drawn upon entry. The transition process is such that idiosyncratic productivity changes from one month to the next with probability π and switches to a neighboring state with identical probabilities. Parameter α is set to 0.7 which gives rise to a labor share of roughly 2/3.[31] The parameters x_{min} and π are chosen to match two targets. Given that labor is a continuous variable in our model, we identify the labor input of one worker with the minimum firm size in the sample distribution, and we target the ratio between the mean firm size and the minimum firm size at 21.6, which is the average number of workers per firm in the Business and Employment Dynamics (BED) data set of the Bureau of Labor Statistics (BLS). Second, we target quarterly firm-level rates of job creation and job destruction of around 6.5% which is the average in BED data for the period 1990–2005, see Helfand, Sadeghi, and Talan (2007).

[31] Given that all capital is fixed at the level of a firm, this calculation of factor shares ignores capital income accruing from variable capital investment which would suggest a higher value of α. Our quantitative results do not change substantially when we choose somewhat larger values of α.

Table 1: Parameter choices.

Parameter	Value	Description	Target
β	0.996	Discount factor	Annual interest rate 5%
k	1.623	Matching fct. scale	job-finding rate
r	1.475	Matching fct. elasticity	Pissarides and Petrongolo (2001)
α	0.7	Prod. fct. elasticity	Labor share
c	0.1	Vacancy cost parameter	Hiring cost 14% of quarterly wage
x_{min}	0.31	Lowest productivity	Firm size (mean relative to min)
π	0.65	Transition probability	Job-creation rate
b	0.2	Flow value of leisure	Vacancy-unemployment ratio = 0.72
K	13.09	Entry cost	Job creation at opening firms
f	0.6	Flow operating cost	Job destruction at closing firms
δ_0	0.0011	Exogenous exit rate	Job destruction at closing large firms
s_0	0.02	Quit rate	Monthly quit rate

We deliberately choose a quadratic vacancy cost function $C(V) = cV^2$ and we set parameter c so that recruiting cost per hire are about 14 percent of quarterly wage income, following Hall and Milgrom (2008) and Elsby and Michaels (2010). We set the opportunity cost of employment (parameter b) to target a vacancy-unemployment ratio of 0.72 which is the number chosen by Pissarides (2009), based on the Job Openings and Labor Turnover Survey (JOLTS). We set the entry cost parameter K and the operating flow cost parameter f to target the extensive margins of job creation and job destruction. Based on BED data between 1990 and 2005, 16.6% of all quarterly job gains occur at opening firms and 17.2% of all job losses occur at closing firms (see Helfand, Sadeghi, and Talan (2007)).[32] With this choice of parameters, all firms with the lowest idiosyncratic productivity $x = x_{min}$ leave the market, whereas all others stay. By implication, only the smallest firms (those at the lowest four productivity levels) can make a transition to the lowest productivity state (and thus leave the market) from one quarter to the next. Nonetheless, in BED data 0.33 percent of jobs are lost at exiting firms whose employment is larger than mean employment (i.e. 20 workers or more). Hence we set the exogenous monthly exit rate at $\delta_0 = 0.0011$ to account for job destruction at exiting larger firms. The exogenous worker quit rate is set at $s_0 = 0.02$, which is roughly the monthly quit rate in JOLTS (Davis, Faberman, and Haltiwanger (2006)). Table 1 summarizes these parameter choices.

Due to the non-linearity of the model, we cannot match all targets exactly, but the fit is rather close (see Table 2). The job-finding rate is a bit lower than the target which is due to the fact that the matching function is concave and vacancy-unemployment ratios are dispersed across different firm types.

Figure 2 shows value and policy functions (separation and recruitment policies) for firms in

[32]The respective shares at the establishment level are somewhat larger (20.9% and 20.1%).

Table 2: Data moments and model statistics.

Variable	Data	Model
Job-creation rate (quarterly)	6.7%	6.9%
Job-destruction rate (quarterly)	6.3%	6.9%
Labor share	0.67	0.68
Workers per firm	21.6	21.3
Hiring cost (share of quart. wage)	0.14	0.14
Share of JC at openings (quarterly)	16.6%	17.5%
Share of JD at closings (quarterly)	17.2%	16.0%
Vacancy-unemployment ratio	0.72	0.72
job-finding rate (monthly)	45%	44%

Notes: The model statistics are calculated as a stationary firm distribution obtained from a simulation of 10000 entrant firms where each firm is subject to the idiosyncratic shock process and exits at productivity $x = x_{min}$. This gives a total of about $8 \cdot 10^5$ observations.

the nine active productivity states $x > x_{min}$. These policy functions confirm the insights from Section 3: conditional on size, more productive firms advertise more vacancies and fill any of them with a higher rate. And conditional on productivity, smaller firms recruit faster and create more jobs. The figure also shows that, for any productivity, there is a range of employment levels where firms neither shed workers nor recruit new workers, and hence shrink by the natural turnover $s_0 L$.

In a simulated stationary firm distribution, we find positive relations between firm growth and the two means of recruitment, vacancy postings and vacancy fill rates. The two relationships are shown in Figure 3(a) for the vacancy rate (i.e., vacancies as a share of employment) and in Figure 3(b) for the monthly job-filling rate (hires per vacancy). Qualitatively, these graphs correspond to the findings of Davis, Faberman, and Haltiwanger (2010) who also document a positive relationship between employer growth, the monthly vacancy rate and the job-filling rate in JOLTS data. In their study, vacancy postings seem to play a smaller role in accounting for differences in employment growth, whereas they are a more important factor in our simulation. As discussed earlier, convexity in the recruitment technology is the key factor for a positive relationship between firm growth and job-filling rates; with a linear recruitment technology, this relationship disappears.[33]

The model performs reasonably well in matching the dispersion of employment growth rates across firms. Using the Longitudinal Business Database (1992–2005), Davis, Haltiwanger, Jarmin, and Miranda (2006) obtain a cross-sectional dispersion (employment-weighted standard deviation) of annual employment growth rates for continuing firms of 0.37 (see Figure 8 in their

[33]When we choose a more general function $C(V) = cV^a$, we confirm that differences in job-filling rates vanish for values of a close to one. On the other hand, the spread does not become much larger when we choose values of a greater than two. Hence we decided to simply use a quadratic function.

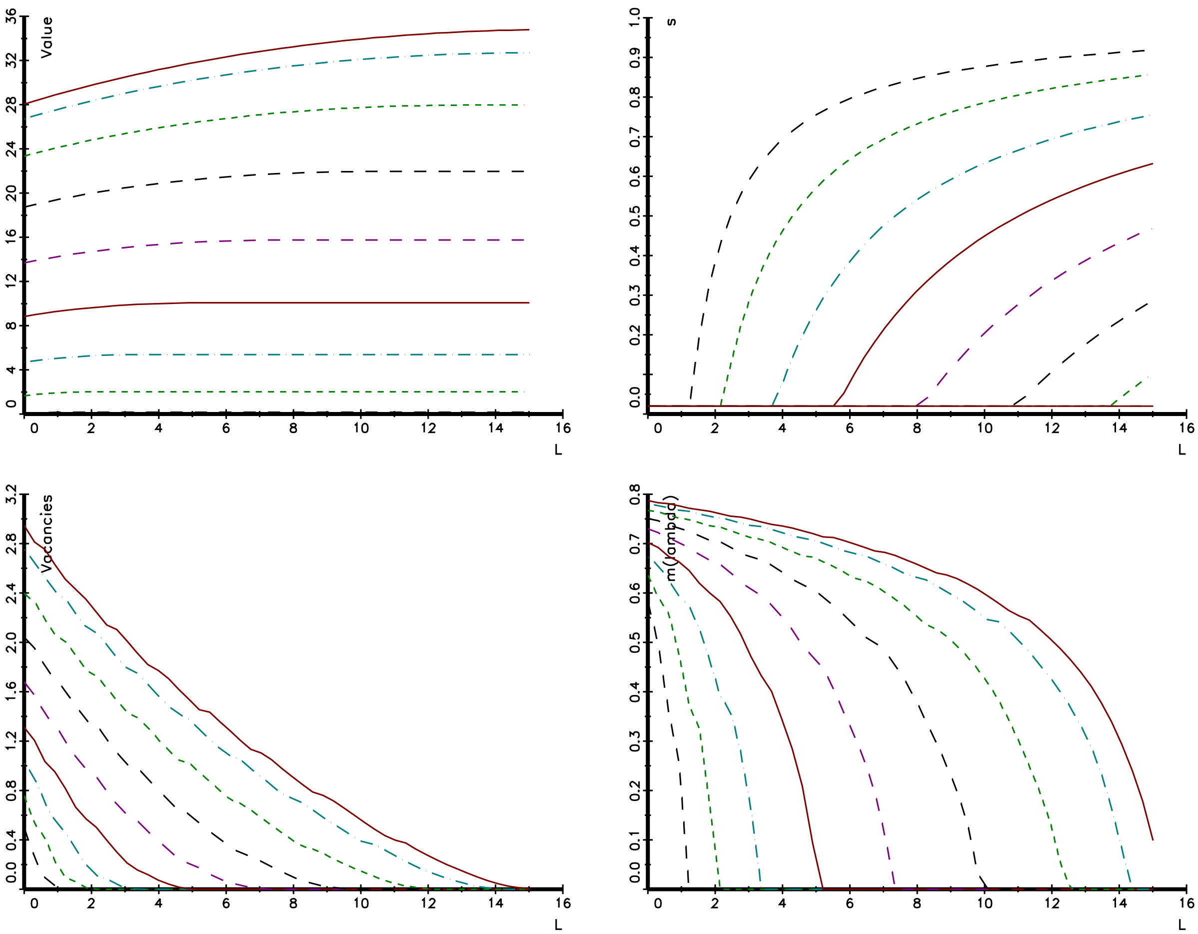

Figure 2: The firms' value functions (upper left), and the policy functions for separation rates s (upper right), for vacancies (lower left), and for job-filling rates $m(\lambda)$ (lower right).

paper). In our simulated stationary distribution, this dispersion measure is somewhat larger at 0.44. Table 3 also shows that the model does a good job in matching the distribution of employment growth rates.

Although our calibration of idiosyncratic productivity does not target the cross-sectional employment distribution,[34] we are still able to produce the negative relation between firm size (as measured by the rank in the size distribution) and rates of job creation and job destruction, and it also captures the negative relation between firm size and the rates of job creation and destruction at the extensive margin; see Figure 4.

We also calculate wage variation between firms, using the wage profiles where all workers within a firm earn the same. We find that wage dispersion across firms is rather small, with

[34] One possibility to account for the large variance and skewness of the employment distribution is to split $x = x_0 x_1$ into a permanent and a transitionary component where the former is set to match the employment distribution (see Elsby and Michaels (2010)). We leave such an extension for future work.

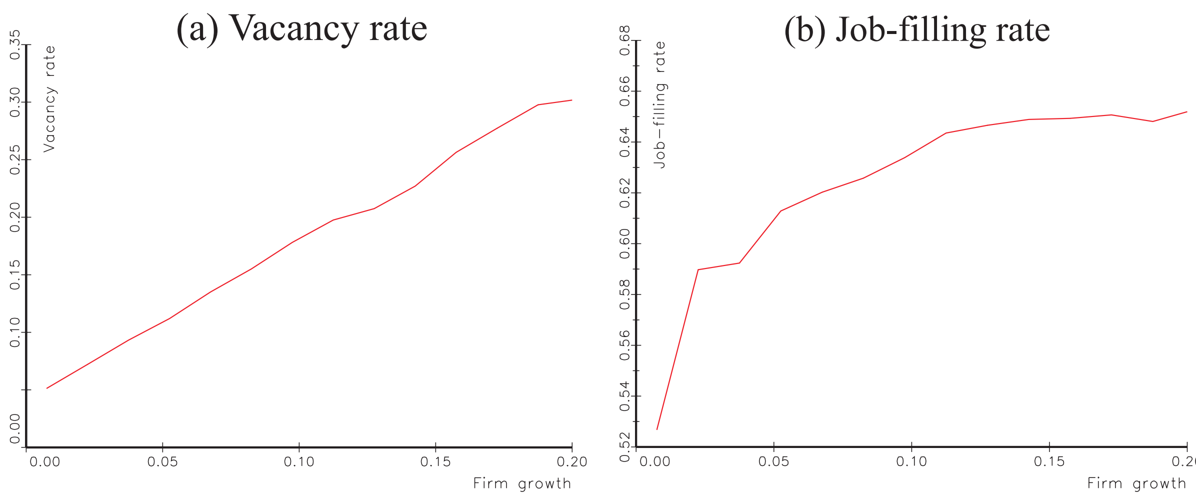

Figure 3: Vacancy rates (vacancies relative to employment) and job-filling rates (monthly hires per vacancy) across firm growth rates. **Notes:** The curves are calculated from a simulated firm distribution with $8 \cdot 10^5$ observations and 20 equally spaced intervals of the firm growth distribution, with firm growth defined as $2(L_t - L_{t-1})/(L_{t-1} + L_t)$.

Table 3: Distribution of employment growth

Growth rate interval	Data	Model
-2 (exit)	0.7	0.55
$(-2, -0.2]$	7.5	9.6
$(-0.2, -0.05]$	16.5	7.6
$(-0.05, -0.02]$	9.6	14.9
$(-0.02, 0.02)$	30.9	22.8
$[0.02, 0.05)$	9.9	14.0
$[0.05, 0.2)$	16.7	21.9
$[0.2, 2)$	7.5	8.0
2 (entry)	0.7	0.61

Notes: The table reports employment shares for intervals of quarterly employment growth rates. The empirical distribution is taken from Table 2 of Davis, Faberman, Haltiwanger, and Rucker (2008). The model statistics are calculated from a stationary firm distribution with $8 \cdot 10^5$ observations.

a standard deviation of log wages equal to 3.6%, in line with other work that abstracts from on-the-job search (see e.g. Hornstein, Krusell, and Violante (2009)). The model does link wage dispersion to firm size and firm growth. For instance, the wage difference between firms with log employment one standard deviation above average to those with log employment one standard deviation below average is 2.5% percent. Differences in firm growth account for more variation in wages: the differential between wages at firms that grow by more than 20 percent and those

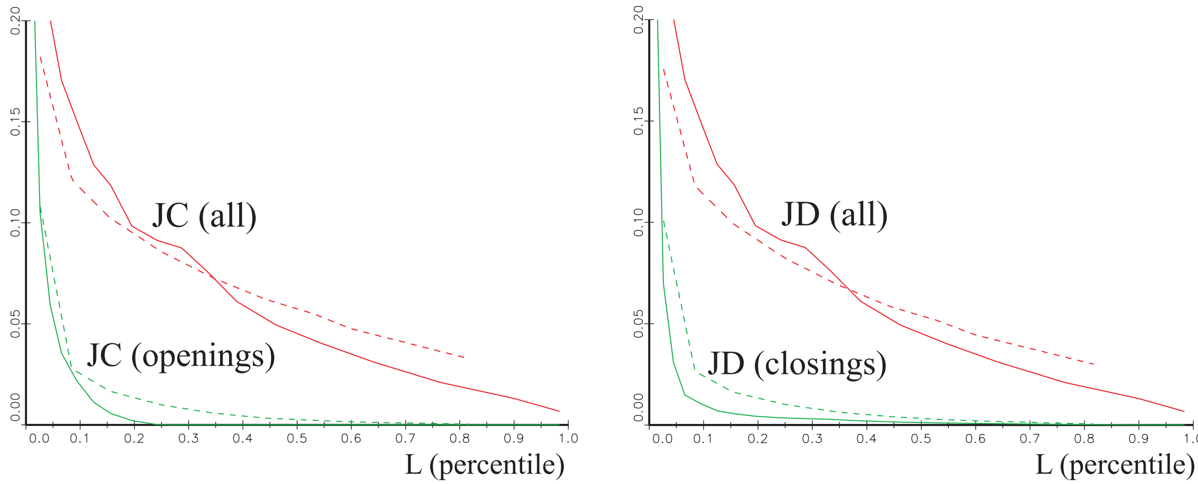

Figure 4: Quarterly job creation and job destruction rates (total and extensive margins) across firm sizes (percentiles of the employment distribution). **Notes:** The dashed curves are based on the nine reported firm–size classes in the BED. The solid curves are calculated from a simulated firm distribution with $8 \cdot 10^5$ observations and 20 equally spaced intervals of the employment distribution. When firms change size classes, job flows are attributed according to the dynamic–size allocation of the BLS (see Moscarini and Postel–Vinay (2009)).

that do not grow or shrink is 6.7 percent (almost two standard deviations).

To explore the impact of aggregate shocks, we first compute the model's impulse response to a permanent productivity increase. We find that our model accounts for a sluggish response of labor market aggregates. In a reduced-form vector autoregression, Fujita and Ramey (2007) show that the vacancy-unemployment ratio is much more persistent than labor productivity and that productivity shocks propagate gradually to market tightness and employment. Fujita and Ramey (2007) and Shimer (2005b) argue that standard search and matching models cannot replicate this pattern because market tightness is a jump variable which correlates perfectly with aggregate productivity. Our model with heterogenous firms and convex recruitment costs yields some propagation of unemployment and vacancies. Figure 5 shows the impulse response to a permanent one-percent increase in aggregate labor productivity. As higher productivity makes entry more attractive, wages must rise sufficiently to balance the gains from entry to its cost. We find that the reservation wage rises by 1.4% (and hence by more than productivity) which implies that optimal firm size falls, which induces a larger number of firms to enter. The impact response to the productivity and wage increase is that existing firms shed some of their workers so that unemployment rises in the first quarter.[35] Over time, however, more and more firms enter and vacancies rise gradually to a higher level, following a hump-shaped pattern. After the first three months, unemployment sluggishly declines to a permanently lower level. Figure 5(c)

[35] The impact response of unemployment would be dampened or even reversed if entry costs increase together with productivity.

also shows that the job-finding rate responds only gradually to the aggregate productivity shock. After the initial decline (due to the rise in unemployment), the job-finding rate takes about a year to adjust to the permanently higher level. We emphasize again that this sluggish response is entirely driven by the convex recruitment costs at the firm level. With linear vacancy costs, the job-finding rate would immediately jump to its new steady-state level.[36] This result differs from Elsby and Michaels (2010) who obtain a somewhat less sluggish response of the job-finding rate in a *random* search model with linear recruitment costs. It is however consistent with Acemoglu and Hawkins (2010) who obtain persistence of aggregate shocks due to a convexity in the recruitment technology. It is also consistent with the models of Schaal (2010) and Hawkins (2011) who assume linear vacancy costs and find that the job-finding rate is a jump variable.

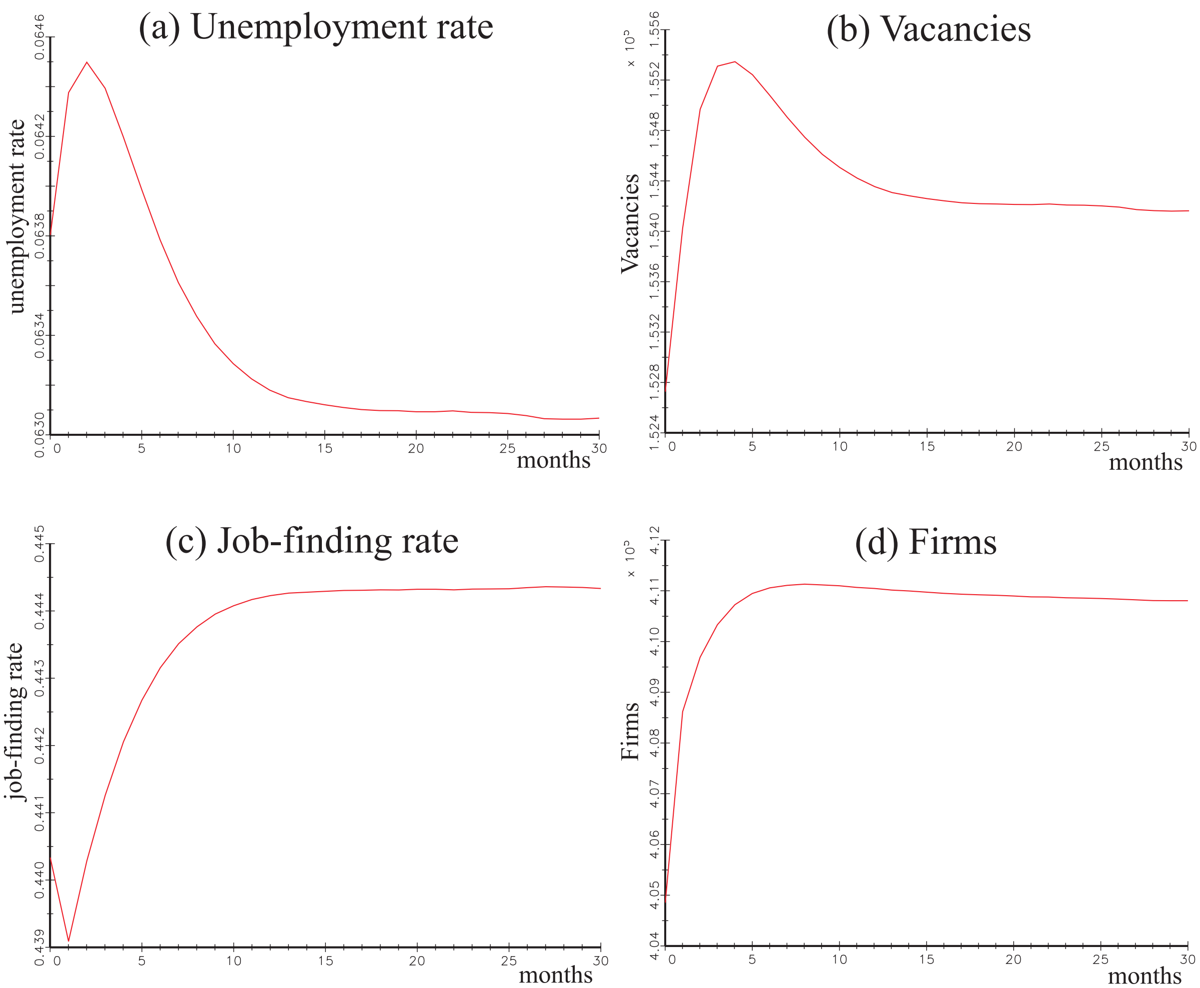

Figure 5: Impulse response to a permanent 1% increase in aggregate labor productivity. The graphs show the average response of 600 simulated model responses where each simulation starts from a cross-section of $4 \cdot 10^5$ firms.

[36]Equation (21) implies that λ is a function of the aggregate state μ_i alone if marginal vacancy costs are constant.

To study the model's business cycle properties, we solve the model as outlined in Section 4.2. Aggregate productivity attains five equally distant values in the interval $[z_{min}, 2 - z_{min}]$, and the Markov process for z is a mean-reverting process with transition probability ψ, as described in Appendix C of Shimer (2005b). The two parameters (z_{min}, ψ) are set to target a quarterly standard deviation and autocorrelation of labor productivity around trend of 0.02 and 0.85. Starting from a stationary firm distribution, we simulate the evolution of these firms over 2000 months, using the policy functions from the numerical solution of (18) and (19). In every simulation period, the number of firm entrants is obtained as a residual of the economy's resource constraint. We compare two calibrations. In the first, the entry cost K does not vary with aggregate productivity. With this specification, firm entry turns out to be more than 10 times as volatile as in the data. Therefore, we consider a second calibration where the entry costs are allowed to vary procyclically to match the empirical standard deviation of job creation at opening firms.[37] In BED data, job gains at opening firms (1992Q3-2010Q1, logged and HP filtered with $\lambda = 10^5$) have a standard deviation of 7.4 percent. To match this target, we set the elasticity of K with respect to z to 0.44.

Table 4 shows the outcome of this exercise for volatility and comovement with aggregate output. For both calibrations, the model clearly has too low amplification: all labor market variables are less volatile than in the data, as is the case in Shimer's (2005b) calibration of the search and matching model with homogeneous (constant return) firms and socially efficient job creation. One way to understand low amplification is the gap between productivity and the opportunity cost of work; the larger this gap is, the smaller should be the response of job creation to productivity shocks (see Hagedorn and Manovskii (2008), Hall and Milgrom (2008)). In fact, in our calibration, aggregate labor productivity (which is obviously identical to the employment-weighted average product of labor across firms) is 0.45 and the employment-weighted marginal product is 0.315, so that the opportunity cost of work (parameter b) is just 44% of average product and 66% of marginal product. When we double parameter b to 0.4 (and adjust the operating cost to $f = 0.19$ so as to make sure that again only firms with $x = x_{min}$ leave the market), average firm size falls, the average and marginal products of labor increase (albeit by a factor less than two), so that b is at 55% of average product and at 78% of marginal product. In the calibration with variable entry costs, the relative volatility of the vacancy-unemployment ratio nearly doubles (from 1.2 to 2.1). In alternative search and matching models with large firms and intra-firm bargaining, Krause and Lubik (2007), Faccini and Ortigueira (2010) and Hawkins (2011) also find little amplification of neutral technology shocks, whereas Elsby and Michaels (2010) obtain more volatility. Their models differ from ours in several dimensions. Which of those is the reason for the variation between the results remains the subject of future research.

Despite low amplification, our model generates a correlation pattern with aggregate output which is consistent with the data. On the one hand, the model captures a downward-sloping Beveridge curve, that is, a strong negative comovement of unemployment and vacancies. This

[37] There are many possibilities why entry costs vary with the business cycle, e.g. procyclical rental rates or capital prices. In our framework, entry costs would be procyclical if firms are created by entrepreneurs whose opportunity costs (e.g. market work) are higher in upswings.

is despite the feature that job destruction is endogenous in this model. On the other hand, in the calibration with procyclical entry costs, the job-finding rate is strongly procyclical and more volatile than the separation rate which correlates negatively with output.

Table 4: Business cycle statistics

	Data		Model (Fixed entry cost)		Model (Variable entry cost)	
	Relative volatility	Corr. w. output	Relative volatility	Corr. w. output	Relative volatility	Corr. w. output
Productivity	0.764	0.686	0.969	0.999	0.964	0.999
Unemployment rate	7.665	-0.888	0.948	-0.464	0.544	-0.918
Vacancies	7.333	0.762	1.016	0.949	0.715	0.984
job-finding rate	4.398	0.819	0.568	0.842	0.446	0.995
Separation rate	2.685	-0.646	0.537	0.051	0.104	-0.252

Notes: All variables are logged and HP filtered with parameter 10^5. Relative volatility measures the standard deviation of a variable divided by the standard deviation of output. Data are for the U.S. labor market (1948Q1–2007Q1, Vacancies: 1951Q1–2006Q3); the job-finding rate and separation rate series were constructed by Robert Shimer (see Shimer (2007) and his webpage http://sites.google.com/site/robertshimer/research/flows). The model statistics are obtained from a simulation of $4 \cdot 10^5$ firms over a period of 2000 months where the first 40 months are discarded. Monthly series are converted into quarterly series by time averaging.

6 Conclusion

The introduction of multi-worker firms into labor-search models bridges the separate literatures on firm dynamics and labor market frictions. It has the potential to address issues in both fields, and most importantly to create new insights into the interplay between firm heterogeneity, worker and job flows, and the levels and fluctuations in employment. This particular project proposes a wage formation process for such environments that incorporates standard competitive elements adjusted for the fact that search frictions prevent perfect market clearing. The model turns out to be tractable, it generates several stylized facts regarding firm growth, pay and recruitment strategies, and it implements socially efficient allocations both in and out of steady state. It can be viewed as a benchmark against which to judge actual labor market allocations.

To conclude, it is worthwhile to note that this framework is flexible for extensions. On the one hand, it is easy to allow for variable capital investment, as long as the firm-level production functions retain decreasing returns in all variable inputs. On the other hand, it is straightforward to introduce worker heterogeneity provided that vacancies are specific to the type of worker that the firm searches for.[38] An extension along this line would yield insights into the sorting of

[38] Cahuc, Marque, and Wasmer (2008) analyze such a model with random search and bargaining.

different worker types across heterogeneous firms and the interactions between the employment dynamics of different worker types. Another interesting extension concerns the emergence of endogenous worker heterogeneity due to on-the-job learning. When workers build up human capital on the job, firms (and the planner) are no longer indifferent between firing workers with longer or with shorter job tenure, as they are in this paper. Workers hired in later stages of a firm's life must then be compensated for the higher risk of job loss in the event of a downturn.

A further extension is to allow for risk aversion and incomplete markets in our framework. In constant-return environments with exogenous separation rates, Acemoglu and Shimer (1999a) and Rudanko (2010) introduce long-term contracting and analyze the implications for risk sharing, unemployment insurance and labor market dynamics. Our model with exogenous separations could also be augmented along these lines. If workers are risk averse and have no access to capital markets, risk neutral firms offer flat wage contracts. Similar to our exposition in Section 3, firms solve a recursive problem subject to a workers' participation constraint which takes into account $u(w)$ instead of w. But different from our results, equilibrium ceases to be socially efficient, provided that the planner is allowed to redistribute income to the unemployed. Lack of unemployment insurance induces workers to search too much for low-paying but easy-to-get jobs (as in Acemoglu and Shimer (1999a)), and should lead to excess employment in low-productivity firms and therefore to a misallocation of labor between heterogeneous firms. Overall, the current setup still provides the relevant benchmark: if perfect risk sharing is available, workers care only about expected income values as analyzed in this paper and competitive search equilibrium achieves socially efficient firm dynamics.

Appendix

Proof of Proposition 1: Note that our cost function fulfills the conditions $C_1 > 0$, $C_{11} > 0$, $C_{12} \leq 0$, and $C_{13} \geq 0$, and that $xF(L) - C(V, L, x)$ is concave in (V, L) and supermodular in (L, x). Additional requirements that are listed in Lemma 2 below are also satisfied for our cost function.

The observation that the firm's objective and constraints in problem (3) are separable in L and W and linear in W suggests that the value function takes the form $J^x(L, W) = -BW + G(L, x; \rho)$ for some constant $B > 0$. The envelope condition with respect to W yields

$$B = 1 + \beta(1-\eta)B \ , \tag{33}$$

which implies $B = 1/[1 - \beta(1-\eta)]$. Further, $G(L, x; \rho)$ satisfies the recursive problem

$$\begin{aligned} G(L, x; \rho) \quad &= \max_{(\lambda,V)\geq 0} xF(L) - C(V, L, x) - \beta[\lambda\rho + (1-\delta)Bm(\lambda)(b + \beta\rho)]V + \beta(1-\delta)G(\hat{L}, x; \rho) \\ &\text{s.t. } \hat{L} = L(1-s) + m(\lambda)V \ , \ V \geq 0. \end{aligned} \tag{34}$$

This problem is equivalently defined on a compact state space $L \in [0, \overline{L}]$ where $\overline{L}$ is so large that it never binds. This is possible because of the Inada condition $\lim_{L\to\infty} F'(L) = 0$. The RHS in problem (34) defines an operator T which maps a continuous function $G_0(L, x; \rho)$, defined on $[0, \overline{L}] \times [0, \overline{x}] \times [0, \overline{\rho}]$ into a continuous function $G_1(L, x; \rho) = T(G_0)(L, x; \rho)$ defined on the same domain. This operator is a contraction and it maps functions which are increasing in L and x and decreasing in ρ into functions with the same property. Moreover, the fixed point must be decreasing in ρ and strictly increasing in x, which follows from differentiation of G with respect to ρ and x. Differentiability follows from straightforward application of the approach by Benveniste and Scheinkman.

To prove concavity and supermodularity of the value function, we rewrite (34) in terms of hirings $H = m(\lambda)V$, noting that for a given level of hiring only the second and third term on the right hand side of (34) depend on the remaining choice variables and capture the hiring costs. Dropping argument ρ from G, we can equivalently write (34) as

$$G(L,x) = \max_H xF(L) - \mathcal{C}(H,L,x) + \beta(1-\delta)G(L(1-s)+H,x) \tag{35}$$

where

$$\mathcal{C}(H,L,x) \equiv \min_\lambda C\Big(\frac{H}{m(\lambda)},L,x\Big) + \beta(1-\delta)BRH + \beta\rho\frac{\lambda}{m(\lambda)}H \ . \tag{36}$$

As will become clear, the per period return $xF(L)-\mathcal{C}(H,L,x)$ is supermodular in (L,H) but for $h>0$ (which implies $C_{13}>0$) strictly submodular in (H,x) and in $(\hat{L},x)$ when one writes $H=\hat{L}-(1-s)L$, which renders standard tools to prove supermodularity (e.g., Amir (1996)) inapplicable. To proceed, the optimality condition for problem (36) is

$$C_1\Big(\frac{H}{m(\lambda)},L,x\Big) = \beta\rho\frac{m(\lambda)-\lambda m'(\lambda)}{m'(\lambda)} \ . \tag{37}$$

Differentiate this equation to obtain

$$\begin{aligned}
\frac{d\lambda}{dH} &= \frac{C_{11}}{C_{11}H\frac{m'}{m} - \frac{\beta\rho m'' m^2}{(m')^2}} > 0 \ , && (38)\\
\frac{d\lambda}{dL} &= \frac{C_{12}m}{C_{11}H\frac{m'}{m} - \frac{\beta\rho m'' m^2}{(m')^2}} = \frac{C_{12}m}{C_{11}}\frac{d\lambda}{dH} \le 0 \ , && (39)\\
\frac{d\lambda}{dx} &= \frac{C_{13}m}{C_{11}H\frac{m'}{m} - \frac{\beta\rho m'' m^2}{(m')^2}} = \frac{C_{13}m}{C_{11}}\frac{d\lambda}{dH} \ge 0 \ . && (40)
\end{aligned}$$

Therefore, we can express the derivatives of cost function $\mathcal{C}$ as

$$\begin{aligned}
\mathcal{C}_1 &= \frac{\beta\rho}{m'(\lambda)} + \beta(1-\delta)BR \ , \\
\mathcal{C}_2 &= C_2 \ , \\
\mathcal{C}_{11} &= -\frac{\beta\rho m''}{(m')^2}\frac{d\lambda}{dH} > 0 \ , && (41)\\
\mathcal{C}_{12} &= -\frac{\beta\rho m''}{(m')^2}\frac{d\lambda}{dL} \le 0 \ , && (42)\\
\mathcal{C}_{22} &= C_{22} - C_{12}\frac{Hm'}{m^2}\frac{d\lambda}{dL} \ , && (43)\\
\mathcal{C}_{13} &= -\frac{\beta\rho m''}{(m')^2}\frac{d\lambda}{dx} \ge 0 \ , && (44)\\
\mathcal{C}_{23} &= C_{23} - C_{12}\frac{Hm'}{m^2}\frac{d\lambda}{dx} \ . && (45)
\end{aligned}$$

Lemma 1: The recursive problem (35) defines a value function $G(L,x)$ which is

(a) concave in L if the following condition holds:

$$\mathcal{C}_{12}^2 + \mathcal{C}_{11}[xF'' - \mathcal{C}_{22}] \leq 0 \ . \tag{46}$$

(b) concave in L and supermodular in (x, L) if (46) and the following condition hold:

$$\mathcal{C}_{12}\mathcal{C}_{13} + \mathcal{C}_{11}[F' - \mathcal{C}_{23}] \geq 0 \ . \tag{47}$$

Lemma 2:

(a) Condition (46) holds under the following condition on the original cost function C:

$$C_{12}^2 + C_{11}[xF'' - C_{22}] \leq 0 \ . \tag{48}$$

(b) Condition (47) holds under the following condition on the original cost function C:

$$C_{12}C_{13} + C_{11}[F' - C_{23}] \geq 0 \ . \tag{49}$$

Proof of Lemma 1: Write (35) as $G(L, x) = (TG)(L, x)$ where T is an operator mapping continuous functions on $[0, \overline{L}] \times R_+$ into another continuous function on the same domain, which has the same properties as the one described above.

Part (a). Suppose that G is a concave and twice differentiable function of L. Then TG is also twice differentiable, and a policy function exists and is differentiable. Differentiate TG twice with respect to L to obtain

$$\frac{d^2(TG)}{dL^2} = xF'' - \mathcal{C}_{22} + \beta(1-\eta)(1-s)G_{11} + \Big[- \mathcal{C}_{12} + \beta(1-\eta)G_{11}\Big] \frac{dH}{dL} \ . \tag{50}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)G_1$ with respect to L to obtain

$$\frac{dH}{dL} = \frac{\beta(1-\eta)G_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ . \tag{51}$$

Substitute this into (50) to obtain

$$\frac{d^2(TG)}{dL^2} = xF'' - \mathcal{C}_{22} + \frac{\beta(1-\eta)(1-s)G_{11}\mathcal{C}_{11} + \mathcal{C}_{12}^2 - 2\beta(1-\eta)G_{11}\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving G_{11} are negative; hence the numerator is smaller than $\mathcal{C}_{12}^2$. Therefore,

$$\frac{d^2(TG)}{dL^2} \leq xF'' - \mathcal{C}_{22} + \frac{\mathcal{C}_{12}^2}{\mathcal{C}_{11}} \ ,$$

which is non-positive under (46). Hence, T maps a concave and twice differentiable function into a function with the same properties. As concavity is preserved in the limit of value function iteration, the unique fixed point of T is concave.

Part (b). Suppose that G is a concave, supermodular and twice differentiable function of (L, x). Again, TG is twice differentiable and a differentiable policy function exists. Differentiate TG twice with respect to L and x to obtain

$$\frac{d^2(TG)}{dLdx} = F' - \mathcal{C}_{23} + \beta(1-\eta)G_{12} + \Big[- \mathcal{C}_{12} + \beta(1-\eta)G_{11}\Big] \frac{dH}{dx} \ . \tag{52}$$

Differentiate the FOC $\mathcal{C}_1 = \beta(1-\delta)G_1$ with respect to x to obtain

$$\frac{dH}{dx} = \frac{\beta(1-\delta)G_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ . \tag{53}$$

Substitute this into (52) to obtain

$$\frac{d^2(TG)}{dLdx} = F' - \mathcal{C}_{23} + \frac{\beta(1-\eta)G_{12}\mathcal{C}_{11} + \mathcal{C}_{12}\mathcal{C}_{13} - \beta(1-\delta)G_{12}\mathcal{C}_{12} - \beta(1-\eta)G_{11}\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ .$$

In the last term, the denominator is positive and larger than $\mathcal{C}_{11}$. In the numerator, all terms involving G_{11} and G_{12} are non-negative; hence the numerator is greater than $\mathcal{C}_{12}\mathcal{C}_{13} \leq 0$. Therefore,

$$\frac{d^2(TG)}{dLdx} \geq F' - \mathcal{C}_{23} + \frac{\mathcal{C}_{12}\mathcal{C}_{13}}{\mathcal{C}_{11}} \ ,$$

which is non-negative under (47). Hence, T maps a supermodular, concave and twice differentiable function into a function with the same properties. Because concavity and supermodularity are preserved in the limit of value function iteration, the unique fixed point is concave and supermodular. □

Proof of Lemma 2: From (39), (40), (41), (42) and (44) follows that

$$\mathcal{C}_{12} = \frac{\mathcal{C}_{11}C_{12}m}{C_{11}} \ , \tag{54}$$

$$\mathcal{C}_{13} = \frac{\mathcal{C}_{11}C_{13}m}{C_{11}} \ . \tag{55}$$

Furthermore, substituting (42) into (39), and substituting (44) into (40) to eliminate $\frac{\beta\rho m''}{(m')^2}$ imply that

$$\mathcal{C}_{22} = C_{22} - \frac{C_{12}^2}{C_{11}} + \frac{mC_{12}}{C_{11}}\mathcal{C}_{12} \ , \tag{56}$$

$$\mathcal{C}_{23} = C_{23} - \frac{C_{12}}{C_{11}}\Big[C_{13} - m\mathcal{C}_{13}\Big] \ . \tag{57}$$

Part (a): Rewrite (46) using (54) and (56) to obtain the equivalent condition

$$xF'' - C_{22} + \frac{C_{12}^2}{C_{11}} \leq 0 \ .$$

Because of $C_{11} > 0$, this condition is equivalent to (48).

Part (b): Rewrite (47) using (54), (55) and (57) to obtain the equivalent condition

$$F' - C_{23} + \frac{C_{12}C_{13}}{C_{11}} \geq 0 \ .$$

Because of $C_{11} > 0$, this condition is equivalent to (49). □

It follows from Lemma 1 and 2 that the value function $G(L,x)$ is concave in L and supermodular in (L,x) because our cost function satisfies both (48) and (49).

Because of strict concavity of problem (34), policy functions $\lambda^x(L)$ and $V^x(L,\lambda^x(L))$ exist. To derive (4) and (5), consider the first-order conditions for problem (34) with respect to V and λ,

$$C_1(V,L,x) \geq \beta(1-\delta)\Big\{m(\lambda)G_1(\hat{L},x) - B\Big[\lambda\rho\frac{1-\beta(1-\eta)}{1-\delta} + m(\lambda)(b+\beta\rho)\Big]\Big\} \ , \ V \geq 0 \ , \tag{58}$$

$$0 \geq m'(\lambda)VG_1(\hat{L},x) - \Big[\rho\frac{1-\beta(1-\eta)}{1-\delta} + m'(\lambda)(b+\beta\rho)\Big]VB \ , \ \lambda \geq 0, \tag{59}$$

which are both satisfied with complementary slackness. The envelope condition for problem (34) is

$$G_1(L,x) = xF'(L) - C_2(V,L,x) + \beta(1-\eta)G_1(\hat{L},x) \ , \tag{60}$$

It is without loss of generality to impose (59) as equality,[39] and substituting it into (58) and using (33) directly yields the complementary-slackness condition (4). Condition (5) follows immediately from (59) and (60). The properties of V^x stated in Proposition 1 were already established in the main text.

To see how $\lambda^x(L)$ depends on L, use (39) and (51) to get

$$\frac{d\lambda^x(L)}{dL} = \frac{d\lambda(H,L,x)}{dL} + \frac{d\lambda(H,L,x)}{dH}\frac{dH}{dL} = \frac{d\lambda}{dH}\Big[\frac{C_{12}m}{C_{11}} + \frac{\beta(1-\eta)G_{11} - \mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}}\Big] \ .$$

Because of

$$\frac{C_{12}m}{C_{11}} = \frac{\mathcal{C}_{12}}{\mathcal{C}_{11}} \le \frac{\mathcal{C}_{12}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ ,$$

the term in [.] is negative, and so is $d\lambda^x/(dL)$.

To verify that λ is increasing in x, use (40) and (53) to get

$$\frac{d\lambda^x(L)}{dx} = \frac{d\lambda(H,L,x)}{dx} + \frac{d\lambda(H,L,x)}{dH}\frac{dH}{dx} = \frac{d\lambda}{dH}\Big[\frac{C_{13}m}{C_{11}} + \frac{\beta(1-\delta)G_{12} - \mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}}\Big] \ .$$

Because of

$$\frac{C_{13}m}{C_{11}} = \frac{\mathcal{C}_{13}}{\mathcal{C}_{11}} \ge \frac{\mathcal{C}_{13}}{\mathcal{C}_{11} - \beta(1-\delta)G_{11}} \ ,$$

the term in [.] is positive, and so is $d\lambda^x/(dx)$. □

Proof of Corollary 2: Because of exogenous separations, the growth rate of a firm, $[(m(\lambda)V - sL]/L$ is perfectly correlated with the job-creation rate,

$$\text{JCR}(x,L) = m(\lambda^x(L))\frac{V^x(L,\lambda^x(L))}{L} \ .$$

Differentiation of the job-creation rate with respect to x implies

$$\frac{d\text{JCR}}{dx} = m'\frac{d\lambda^x}{dx}\frac{V^x}{L} + \frac{m(\lambda^x)}{L}\frac{dV^x}{dx} + \frac{m(\lambda^x)}{L}\frac{dV^x}{d\lambda}\frac{d\lambda^x}{dx} \ .$$

In this expression, the first and the third term are strictly positive. The second term is zero when $h = 0$, and negative but small if h is small. Thus, $d\text{JCR}/(dx)$ is positive if h is sufficiently small.

Differentiation of the job-creation rate with respect to L implies

$$\frac{d\text{JCR}}{dL} = m'\frac{d\lambda^x}{dL}\frac{V^x}{L} + \frac{m(\lambda^x)}{L}\frac{dV^x}{dL} + \frac{m(\lambda^x)}{L}\frac{dV^x}{d\lambda}\frac{d\lambda^x}{dL} - m\frac{V^x}{L^2} \ .$$

In this expression, the first, the third and the fourth term are strictly negative. The second term is zero when $h = 0$, and positive but small if h is small. Thus, $d\text{JCR}/(dL)$ is negative if h is sufficiently small. □

Lemma 3: Equation (7) has a unique steady state solution $\lambda^* > 0$ if, and only if,

$$h < \frac{\beta(1-\delta)\overline{m}}{1-\beta(1-\eta)} \ , \tag{61}$$

[39] If V is zero then (59) trivially holds with equality. If $V > 0$ then (58) implies $\lambda > 0$ and again (59) has to hold with equality.

with $\overline{m} = \lim_{\lambda\to\infty} m(\lambda) - \lambda m'(\lambda) > 0$. Under this condition, any sequence $\lambda_t > 0$ satisfying this equation converges to λ^*.

Proof of Lemma 3: A steady state λ^* must satisfy the condition

$$\beta\rho[m(\lambda) - \lambda m'(\lambda)] = \frac{\rho h[1-\beta(1-\eta)]}{1-\delta} + \Big[(b+\beta\rho)h + c\Big]m'(\lambda)\ . \tag{62}$$

The LHS is strictly increasing and goes from 0 to $\beta\rho\overline{m}$ as λ goes from 0 to $+\infty$. The RHS is decreasing in λ with limit $\rho h(1-\beta(1-\eta))/(1-\delta)$ for $\lambda \to \infty$. Hence, a unique steady state λ^* exists iff (61) holds.[40] Furthermore, differentiation of (7) at λ^* implies that

$$\frac{d\lambda_{t+1}}{d\lambda_t}\bigg|_{\lambda^*} = \frac{h}{\beta(1-\delta)m(\lambda^*) + h\beta(1-\eta)}\ ,$$

which is positive and smaller than one iff

$$h < \frac{\beta(1-\delta)m(\lambda^*)}{1-\beta(1-\eta)}\ .$$

But this inequality must be true because (62) implies

$$h = \frac{\beta\rho[m(\lambda^*) - \lambda^* m'(\lambda^*)] - cm'(\lambda^*)}{\dfrac{\rho[1-\beta(1-\eta)]}{1-\delta} + (b+\beta\rho)m'(\lambda^*)} < \frac{\beta(1-\delta)m(\lambda^*)}{1-\beta(1-\eta)}\ .$$

Therefore, the steady state λ^* is locally stable. Moreover, equation (7) defines a continuous, increasing relation between λ_{t+1} and λ_t which has only one intersection with the 45-degree line. Hence, $\lambda_{t+1} > \lambda_t$ for any $\lambda_t < \lambda^*$ and $\lambda_{t+1} < \lambda_t$ for any $\lambda_t > \lambda^*$, which implies that λ_t converges to λ^* from any initial value $\lambda_\tau > 0$. □

Proof of Proposition 2:

It remains to prove existence and uniqueness. From Proposition 1 follows that the entrant's value function $J^x(0,0)$ is decreasing and continuous in ρ. Hence the expected profit prior to entry,

$$\Pi^*(\rho) \equiv \sum_{x\in X} \pi(x) J^x(0,0)$$

is a decreasing and continuous function of ρ. Moreover, the function is strictly decreasing in ρ whenever it is positive. This also follows from the proof of Proposition 1 which shows that $G(0,x;\rho)$ is *strictly* decreasing in ρ when the new firm x recruits workers ($V^x(0,\lambda) > 0$). If no new firm recruits workers, expected profit of an entrant cannot be positive. Hence, equation (8) can have at most one solution for any $K > 0$. This implies uniqueness, with entry of firms if (8) can be fulfilled or without entry of firms otherwise. A solution to (8) exists provided that K is sufficiently small. To see this, $\Pi^*(0)$ is strictly positive because of $F'(0) = \infty$: some entrants will recruit workers since the marginal product $G_1(m(\lambda)V, x; \rho)$ is sufficiently large relative to the cost of recruitment and relative to the wage cost which are, for $\rho = 0$, equal to $m(\lambda)Vb$ (see equation (34)). But when $\Pi^*(0) > 0$, a sufficiently small value of K guarantees that (8) has a solution since $\lim_{\rho\to\infty} \Pi^*(\rho) = 0$. □

Proof of Proposition 3:

We will show that the first-order conditions that uniquely characterize the decentralized allocation are also first order conditions to the planner's problem. The same auxiliary problem that we employ

[40] If this condition fails, firms cannot profitably recruit workers even when $\lambda \to \infty$.

in the proof of Proposition 4 part (b) then establishes that the planner cannot improve upon this allocation. We denote by $S_{N,a}$ the derivative of S with respect to N_a and by $S_{L,a,x}$ the derivative of S with respect to L_a^x. The multiplier on the resource constraint is $\mu \geq 0$. First-order conditions with respect to N_0, V_a^x, and λ_a^x, $a \geq 0$, are

$$\sum_{x \in X} \pi(x)\Big[xF(0) - C(V_0^x, 0, x)\Big] - K + \beta(1-\delta)S_{N,1} - \mu \sum_{x \in X} \pi(x)\lambda_0^x V_0^x = 0 \ , \tag{63}$$

$$-N_a\pi(x)\Big[C_1(V_a^x, L_a^x, x) + \mu\lambda_a^x\Big] + \beta S_{L,a+1,x}m(\lambda_a^x) \leq 0 \ , \ V_a^x \geq 0 \ , \tag{64}$$

$$\beta S_{L,a+1,x}m'(\lambda_a^x) - \mu N_a \pi(x) = 0 \ . \tag{65}$$

Here condition (64) holds with complementary slackness. The envelope conditions are, for $a \geq 1$ and $x \in X$,

$$S_{L,a,x} = N_a\pi(x)\Big[xF'(L_a^x) - C_2'(V_a^x, L_a^x, x) - b - \mu\Big] + \beta(1-s)S_{L,a+1,x} \ , \tag{66}$$

$$S_{N,a} = \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - C(V_a^x, L_a^x, x) - bL_a^x\Big] - \mu \sum_{x \in X} \pi(x)\Big(L_a^x + \lambda_a^x V_a^x\Big) + \beta(1-\delta)S_{N,a+1} \tag{67}$$

Use (65) to substitute $S_{L,a,x}$ into (66) to obtain

$$xF'(L_{a+1}^x) - C_2(V_{a+1}^x, L_{a+1}^x, x) - b - \mu = \frac{\mu}{\beta}\Big[\frac{1}{(1-\delta)m'(\lambda_a^x)} - \frac{\beta(1-s)}{m'(\lambda_{a+1}^x)}\Big] \ .$$

This equation describes the planner's optimal recruitment policy; it coincides with equation (5) for $\mu = \beta\rho$. This is intuitive: when the social value of an unemployed worker μ coincides with the surplus value that an unemployed worker obtains in search equilibrium, the firm's recruitment policy is efficient. Next substitute (65) into (64) to obtain the socially optimal vacancy creation, for $a \geq 0$ and $x \in X$,

$$C_1(V_a^x, L_a^x, x) \geq \mu\Big[\frac{m(\lambda_a^x)}{m'(\lambda_a^x)} - \lambda_a^x\Big] \ , \ V_a^x \geq 0 \ . \tag{68}$$

Again for $\mu = \beta\rho$, this condition coincides with the firm's choice of vacancy postings in competitive search equilibrium, equation (4). Lastly, it remains to verify that entry is socially efficient when the value of a jobless worker is $\mu = \beta\rho$. The planner's choice of firm entry, condition (63), together with the recursive equation for the marginal firm surplus $S_{N,a}$, equation (67), shows that

$$K = \sum_{a \geq 0}[\beta(1-\delta)]^a \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) - \mu(L_a^x + \lambda_a^x V_a^x)\Big] \ . \tag{69}$$

On the other hand, the expected profit value of a new firm is

$$\sum_{x \in X} \pi(x)J^x(0,0) = \sum_{a \geq 0}[\beta(1-\delta)]^a \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - W_a^x - C(V_a^x, L_a^x, x)\Big] \ .$$

Hence, the free-entry condition in search equilibrium, equation (8), coincides with condition (69) for $\mu = \beta\rho$ if, for all $x \in X$,

$$\sum_{a \geq 0}[\beta(1-\delta)]^a\Big[(b+\mu)L_a^x + \mu\lambda_a^x V_a^x - W_a^x\Big] = 0 \ . \tag{70}$$

Now after substitution of

$$
\begin{aligned}
L_a^x &= \sum_{k=0}^{a-1} (1-s)^{a-1-k} m(\lambda_k^x) V_k^x \ , \text{ and} \\
W_a^x &= \sum_{k=0}^{a-1} (1-s)^{a-1-k} V_k^x \Big[\frac{\rho \lambda_k^x}{(1-\delta)B} + m(\lambda_k^x)(b + \beta\rho) \Big]
\end{aligned}
$$

into (70), it is straightforward to see that the equation is satisfied for $\mu = \beta\rho$. □

Proof of Proposition 4:

Part (a): The RHS in the system of equations in (16) defines an operator T which maps a sequence of bounded functions $G = (G_t)_{t \geq 0}$, with $G_t : [0, \overline{L}] \times X \times Z^t \to \mathbb{R}$ such that $\|G\| \equiv \sup_t \|G_t\| < \infty$, into another sequence of bounded functions $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$ with $\|\tilde{G}\| = \sup_t \|\tilde{G}_t\| < \infty$. Here $\overline{L}$ is sufficiently large such that the bound $\hat{L} \leq \overline{L}$ does not bind for any $L \in [0, \overline{L}]$. The existence of $\overline{L}$ follows from the Inada condition for F: the marginal product of an additional worker $xzF'(\hat{L}) - b$ must be negative for any $x \in X$, $z \in Z$, for any $\hat{L} \geq \overline{L}$ with sufficiently large $\overline{L}$; hence no hiring will occur beyond $\overline{L}$. Because the operator satisfies Blackwell's sufficient conditions, it is a contraction in the space of bounded function sequences G. Hence, the operator T has a unique fixed point which is a sequence of bounded functions.

Part (b): Take first a solution $\mathbf{X}$ of the planning problem, and write $\beta^t \psi(z^t) \mu(z^t) \geq 0$ for the multipliers on constraints (15). Then $\mathbf{X}$ maximizes the Lagrange function

$$
\begin{aligned}
\mathcal{L} \;=\; \max \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ &- K N_0(z^t) + \sum_{a \geq 0, x^a} N(x^a, z^t) \Bigg[x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) \\
&- f - C(V(x^a, z^t), \overline{L}(x^a, z^t), x_a) - \mu(z^t) \Big[(1 - s(x^a, z^t)) L(x^{a-1}, z^{t-1}) + \lambda(x^a, z^t) V(x^a, z^t) \Big] \Bigg] \Bigg\}
\end{aligned}
$$

For each individual firm, this problem is the sequential formulation of the recursive problem (16) with multipliers $\mu(z^t)$. Hence, firm policies also solve the recursive problem; furthermore, the maximum of the Lagrange function is the same as the sum of the social values of entrant firms plus the social values of firms which already exist at $t = 0$, namely,

$$
\begin{aligned}
\mathcal{L} \;=\; & \max_{N_0(.)} \sum_{t, z^t} \beta^t \psi(z^t) N_0(z^t) \Big[-K + \sum_x \pi_0(x) G_t(0, x, z^t) \Big] \\
&+ \sum_{z \in Z} \psi(z^0) \sum_{a \geq 1, x^{a-1}, x_a} \pi(x_a | x_{a-1}) N(x^{a-1}, .) G_0(L(x^{a-1}, .), x_a, z^0) \ .
\end{aligned}
$$

This also proves that the complementary-slackness condition (17) describes optimal entry.

To prove the converse, suppose that $\mathbf{X}$ solves for every firm the recursive problem (16) with multipliers $\mu(z^t)$, and that (17) and the resource constraints (15) are satisfied. Define an auxiliary problem (AP) as an extension of the original planning problem (14) which allows the planner to rent additional workers (or to rent out existing workers) at rental rate $\mu(z^t)$ in period t. Formally, the (AP) differs from the original problem in that the resource constraint (15) is replaced by

$$
\sum_{a \geq 0, x^a} N(x^a, z^t) \Big[\overline{L}(x^a, z^t) + \lambda(x^a, z^t) V(x^a, z^t) \Big] \leq M(z^t) \ , \tag{71}
$$

with $M(z^t)-1>0$ workers hired or $M(z^t)-1<0$ workers hired out. Further, the rental cost (rental income) term $-\mu(z^t)[M(z^t)-1]$ is added into the braces in the objective function (14). Then it follows immediately that the multiplier on constraint (71) is equal to $\mu(z^t)$. We further claim that allocation $\mathbf{X}$ solves problem (AP), and hence also solves the original planning problem. To see this, suppose that there is an allocation $(\mathbf{X}',\mathbf{M})$ which is feasible for problem (AP) and which strictly dominates $\mathbf{X}$. Write

$$O(x^a,z^t) \equiv x_a z_t F(L(x^a,z^t)) - bL(x^a,z^t) - f - C(V(x^a,z^t),\overline{L}(x^a,z^t),x_a)$$

for the net output created by firm (x^a,z^t) in allocation $\mathbf{X}$ and write $O'(x^a,z^t)$ for the same object in allocation $\mathbf{X}'$. Further, write S for the total surplus value in allocation $(\mathbf{X},\mathbf{1})$ and write $S'>S$ for the surplus value in allocation $(\mathbf{X}',\mathbf{M})$. Then

$$\begin{aligned}
S' &= \sum_{t\geq 0,z^t} \beta^t\psi(z^t)\Big\{ -KN_0'(z^t) + \sum_{a\geq 0,x^a} N'(x^a,z^t)O'(x^a,z^t) - \mu(z^t)[M(z^t)-1]\Big\} \\
&\leq \sum_{t\geq 0,z^t} \beta^t\psi(z^t)\Big\{ -KN_0'(z^t) + \mu(z^t) \\
&\quad + \sum_{a\geq 0,x^a} N'(x^a,z^t)\Big[O'(x^a,z^t) - \mu(z^t)\Big(\overline{L}'(x^a,z^t) + \lambda'(x^a,z^t)V'(x^a,z^t)\Big)\Big]\Big\} \\
&\leq \sum_{t\geq 0,z^t} \beta^t\psi(z^t)N_0'(z^t)\Big[-K + \sum_x \pi_0(x)G_t(0,x,z^t)\Big] \\
&\quad + \sum_{z\in Z}\psi(z^0) \sum_{a\geq 1,x^{a-1},x_a} \pi(x_a|x_{a-1})N(x^{a-1},.)G_0(L(x^{a-1},.),x_a,z^0) + \sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t) \\
&\leq \sum_{t\geq 0,z^t} \beta^t\psi(z^t)N_0(z^t)\Big[-K + \sum_x \pi_0(x)G_t(0,x,z^t)\Big] \\
&\quad + \sum_{z\in Z}\psi(z^0) \sum_{a\geq 1,x^{a-1},x_a} \pi(x_a|x_{a-1})N(x^{a-1},.)G_0(L(x^{a-1},.),x_a,z^0) + \sum_{t,z^t}\beta^t\psi(z^t)\mu(z^t) = S\ .
\end{aligned}$$

Here the first inequality follows from resource constraint (71). The second inequality follows since the discounted sum of surplus values for an individual new firm, namely

$$\sum_{\tau\geq t}\beta^{\tau-t}E_t\Big[O'(x^{\tau-t},z^\tau) - \mu(z^\tau)[\overline{L}'(x^{\tau-t},z^\tau) + \lambda'(x^{\tau-t},z^\tau)V'(x^{\tau-t},z^\tau)]\Big]\ ,$$

is bounded above $G_t(0,x_0,z_t)$ (for new firms) or by $G_0(L(x^{a-1},.),x_a,z^0)$ (for firms existing at $t=0$) by definition of G_t. The third inequality follows from the complementary-slackness condition (17): either the term $-K+\sum_x \pi_0(x)G_t(0,x,z^t)$ is zero in which case the first summand is zero on both sides of the inequality; or it is strictly negative in which case $N_0(z^t)=0$ and $N_0'(z^t)\geq 0$. The last equality follows from the definition of surplus value S and the assumption that allocation $\mathbf{X}$ solves problem (16) at the level of each individual firm. This proves $S'\leq S$ and hence contradicts the hypothesis $S'>S$. □

Proof of Proposition 5:

Part (a): For the multipliers defined by $\mu(z^t)=\mu_i$ for $z_t=z_i$, the unique solution of (16) coincides with the one of (18), i.e. $G_t(L,x,z^t)=G^i(L,x,\mu)$ for $z_t=z_i$, and also the firm-level policies coincide. If they give rise to an allocation $\mathbf{X}$ with positive entry in all aggregate states z^t, (19) implies that (17) holds for all z^t. Hence Proposition 4(b) implies that $\mathbf{X}$ is a solution of the planning problem.

Part (b): Solving (18) in the stationary case involves to find a single value function $G(L, x, \mu)$. Application of the contraction mapping theorem implies that such a solution exits, is unique, and is continuous and non-increasing in μ and strictly decreasing in μ when $G(.) > 0$.

Therefore, the function $\Gamma(\mu) \equiv \sum_x \pi_0(x) G(0, x, \mu) \geq 0$ is continuous, strictly decreasing when positive, and zero for large enough μ. Furthermore, when f and b are sufficiently small, $\Gamma(0) > 0$; hence when $K > 0$ is sufficiently small, there exists a unique $\mu \geq 0$ satisfying equation (19).

For any given vector $(\mu_1, \ldots, \mu_n) \in R^n_+$, the system of recursive equations (18) has a unique solution $G = (G^i)$. Again this follows from the application of the contraction-mapping theorem. Furthermore, G is differentiable in μ, and all elements of the Jacobian $(dG^i/(d\mu_j))$ are non-positive. The RHS of (18) defines an operator mapping a function $G^i(L, x, \mu)$ with a strictly diagonally dominant Jacobian matrix $(dG^i/(d\mu_j))$ into another function $\tilde{G}_j$ whose Jacobian matrix $(d\tilde{G}_i/(d\mu_j))$ is diagonally dominant. This follows since the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and since all elements of $(d\tilde{G}_i/(d\mu_j))$ have the same (non-positive) sign. Therefore, the unique fixed point has a strictly diagonally dominant Jacobian. Now suppose that $(z_1, \ldots, z_n)$ is close to $(z, \ldots, z)$ and consider the solution $\mu_1 = \ldots = \mu_n = \mu$ from part (a). Since the Jacobian matrix $dG^i(0, x, \mu)/(d\mu_j)$ is strictly diagonally dominant, it is invertible. By the implicit function theorem, a unique solution to equation (19) exits. □

Proof of Proposition 6: The proof proceeds in two steps. First, substitute the participation constraint (30) into the firm's problem and make use of the contracts' recursive equations (23) to show that the firms' recursive profit maximization problem is identical to the maximization of the social surplus of a firm. Second, show that the competitive equilibrium is socially optimal.

First, define the social surplus of a firm with history (x^a, z^t) and with predetermined contracts and employment levels as follows:

$$G_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] \equiv J_a\Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] + \sum_{\tau=0}^{a-1} L_\tau \Big[W(\mathcal{C}_\tau, x^a, z^t) - U(z^t)\Big] . \tag{72}$$

Using (23) with $\varphi_a(x^a, z^t) = 0$ and the participation constraint (30), the wage in the hiring period can be expressed as

$$w_a(x^a, z^t) = b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda \rho(z^t)}{m(\lambda)} - \beta E_{x_a, z_t} W(\mathcal{C}_a, x^{a+1}, z^{t+1}) .$$

Now substitute this equation, (23) and (26) into (72), and write

$$S \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] \text{ and } \hat{S} \equiv \Big[(\mathcal{C}_\tau)_{\tau=0}^{a}, (\hat{L}_\tau)_{\tau=0}^{a}, x^{a+1}, z^{t+1}\Big] ,$$

with $\hat{L}_\tau$ as defined in (27), to obtain

$$\begin{aligned} G_a(S) \;\; = \;\; & \max_{\delta, \lambda, V, \mathcal{C}_a} (1-\delta) \Bigg\{ x_a z_t F(\hat{L}) - f - C(V, \overline{L}, x) \\ & - \sum_{\tau=0}^{a-1} \frac{1 - \varphi_\tau(x^a, z^t)}{1-\delta} L_\tau w_\tau(x^a, z^t) - m(\lambda) V \Big[b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda \rho(z^t)}{m(\lambda)} \Big] \end{aligned} \tag{73}$$

$$
\begin{aligned}
&+\beta E_{x_a,z_t}\Big[J_{a+1}(\hat{S})+m(\lambda)VW(\mathcal{C}_a,x^{a+1},z^{t+1})\Big]\Bigg\}\\
&+\sum_{\tau=0}^{a-1}L_\tau(1-\varphi_\tau(x^a,z^t))\Bigg[w_\tau(x^a,z^t)-U(z^t)+\beta E_{x_a,z_t}W(\mathcal{C}_\tau,x^{a+1},z^{t+1})\Bigg]\\
=\;&\max_{\delta,\lambda,V,\mathcal{C}_a}(1-\delta)\Bigg\{x_az_tF(\hat{L})-b\hat{L}-f-\rho(z^t)\lambda V-\rho(z^t)\sum_{\tau=0}^{a-1}L_\tau(1-\varphi_\tau(x^a,z^t))\\
&-C(V,\overline{L},x)+\beta E_{x_a,z_t}\Big[J_{a+1}(\hat{S})+\sum_{\tau=0}^{a}\hat{L}_\tau\Big(W(\mathcal{C}_\tau,x^{a+1},z^{t+1})-U(z^{t+1})\Big)\Big]\Bigg\}\\
=\;&\max_{\delta,\lambda,V,\mathcal{C}_a}(1-\delta)\Bigg\{x_az_tF(\hat{L})-f-C(V,\overline{L},x)-\rho(z^t)\Big[\lambda V+\hat{L}-m(\lambda)V\Big]-b\hat{L}\\
&+\beta E_{x_a,z_t}G_{a+1}(\hat{S})\Bigg\}\;.
\end{aligned}
$$

Here maximization is subject to (27) and (28), and the second equation makes use of (25). This shows that the firm solves a surplus maximization problem which is identical to the one of the planner specified in (16) provided that $\rho(z^t)=\mu(z^t)$ holds for all z^t, where μ is the social value of an unemployed worker as defined in section 4.2. The only difference between the two problems is that the firm commits to cohort-specific separation probabilities, whereas the planner chooses in every period an identical separation probability for all workers (and he clearly has no reason to do otherwise). Nonetheless, both problems have the same solution: they are dynamic optimization problems of a single decision maker in which payoff functions are the same (with $\rho(z^t)=\mu(z^t)$) and the decision sets are the same. Further, time inconsistency is not an issue since there is no strategic interaction and since discounting is exponential. Hence solutions to the two problems, with respect to firm exit, layoffs and hiring strategies, are identical. In both problems the decision maker could discriminate between different cohorts in principal. Because such differential treatment does not raise social firm value, there is also no reason for competitive search to produce such an outcome. Nonetheless, there can be equilibria where different cohorts have different separation probabilities, but these equilibria must also be socially optimal because they maximize social firm value.

It remains to verify that competitive search gives indeed rise to socially efficient firm entry. When $\mu(z^t)=\rho(z^t)$, $G_0(x,z^t)$ as defined in (72) coincides with $G_0(0,x,z^t)$, as defined in (16). Hence, the free-entry condition (31) coincides with the condition for socially optimal firm entry (17). Because of aggregate resource feasibility (32), the planner's resource constraint (15) is also satisfied. Since the allocation of a competitive search equilibrium satisfies all the requirements of Proposition 4(b), it is socially optimal. □

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