A theory of Conscientiousness*

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Abstract

We provide an axiomatic foundation for a personality trait which has important implications for economic behavior, Conscientiousness, and two aspects of that factor, the inhibitive and the proactive. We refer to these two aspects here with the names, probably more intuitive for economists, of control and motivation. The first aspect is commonly associated in analysis of individual behavior with the ability to override impulses and distractions when pursuing a goal. The second is usually associated with the inclination to set ambitious goals.

Our setup and analysis closely follow those of standard decision theoretic analysis. In our model an individual is characterized by a preference order over acts, which are maps from states to lotteries over prizes. In the framework of Drèze, we allow the possibility that the individual can affect the probability of the state which is realized, at some cost. The differences in this cost of control make formal the differences in conscientiousness among individuals: a higher cost of control over the probability corresponds to a lower degree of inhibitive side of conscientiousness. The utility in each state deriving from the realization of an outcome is state dependent. An important part of the research reported here is an axiomatic foundation of preferences with moral hazard and state dependent preferences, first treated in Drèze. This utility evaluated by the individual in reference to a subjective benchmark, or aspiration level. An extreme and simple example is given by an individual who sets an aspiration level, which is a point in a partially ordered space, and derives a positive utility when his outcome is larger than the set level, and does not when the outcome is lower. The level of the aspiration set by the individual corresponds to his motivation, which corresponds to the proactive side of Conscientiousness.

Very preliminary version !

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1 Introduction

Personality theory and decision theory Personality theory and decision theory are two distinct theories of human behavior. Although very different in historical origin and conceptual structure, both try to identify stable characteristics of individuals, able to predict patterns of behavior of that individual in different environments and circumstances. Personality theory wants to identify traits of an individual, defined as relatively stable patterns of affect, behavior and cognition in a wide variety of circumstances. Factors are, as in the statistical process of determining them, a limited number of latent variables are able to explain a large number of possible patterns. If one accepts this definition, then decision theory can be considered a theory of personality with a restricted domain, because it also tried to identify and provide the conceptual structure for stable patterns of behavior in choices involving dated stream of rewards under uncertainty. If one takes it as a search for factors, one may define Decision Theory as the hypothesis that a large set of possible different patterns of choice on the domain of such rewards can be explained by two simple factors, the attitude to risk and that to delay.

So far these two theories have been largely independent. An integration of the two theories seems necessary because several findings in the past years have shown that the predictive power of classical decision theory is significantly increased once we introduce some measurement of personalty traits of individuals, for example the Big Five factors. A first step in this integration is setting the analysis of personality traits on the same formal level of analysis as classical decision theory, and the axiomatic method seems the natural approach. The purpose of this paper is to provide such a foundation for one of the factors, Conscientiousness, and its two aspects, motivation and control.

Why Conscientiousness? There are several reasons to out this trait at the forefront of the analysis. First, Conscientiousness has been found to be important in explaining successful outcomes for several economic variables. In the influential meta-analysis papers of Barrick and Mount ([2]) and Barrick et. al ([3]), Conscientiousness was found to be a good predictor of job performance. In the survey paper of Roberts et al. ([24]), several longitudinal studies are examined, reporting the comparative effect of IQ, Socioeconomic Status (SES) and personality traits at a later time. The dependent variables considered are mortality, divorce, educational and occupational attainment. For each, Conscientiousness has important and significant effects, comparable to those of IQ and SES. The effects on Mortality are larger than those of other traits. For career success Judge et al. ([16]) find that Conscientiousness has the largest standardized coefficient among the Big Five factors. In the study [25], Personality traits and economic preferences compared side by side, and the correlation with economic and social outcomes are are estimated for each. A higher score in Conscientiousness is correlated in larger ability to hold a job, to reduce number of accidents in drivers, and with Credit Score.

Second, Conscientiousness is a very specific human trait, in that it is based on the prominence of long-term goals in directing purposeful behavior which is typically absent in other species. For example in a review of 19 studies on 12 different mammalian and non-mammalian species Gosling and John ([14]) reported factors similar to Extraversion, Agreeableness and Neuroticism in most species, with Openness less represented than these are other three. A factor similar to Conscientiousness has been reported only in chimpanzees, in the study by King and Figueredo [19]. They asked 53 raters to rate 100 chimpanzees at 12 zoological parks. Raters had to choose a score on a 7 points scale for 43 adjectives, which were first briefly described. For example the definition of cautious stated that Subject often seems attentive to possible harm or danger from its actions. Subject avoids risky behaviors. Factor analysis of the ratings show the existence of six latent factors, five of them close to the Big Five factors in human personality literature, including Conscientiousness. The sixth factor was related to Dominance.

Third, Conscientiousness is among the Big Five personality traits the one that is more remote form the conceptual structure of economic analysis, in spite of its importance for the analysis of economic behavior. It is harder to model within standard economic and decision theoretic tradition. This is not the case for the other four traits. For example the two traits of Neuroticism and Extraversion may be mapped into the different attitudes to gains and losses as conceptualized in Prospect Theory ([17]. Similarly, Intelligence can be modeled as a productive skill, and Openness/Intellect can be modeled in a dynamic model under uncertainty as the ability to process information. And Agreeableness as the inclination of an individual to consider into his utility the material payoff of others.

Conscientiousness and its facets Conscientiousness "appears to reflect the tendency to maintain motivational stability within the individual, to make plans and carry them out in an organized and industrious manner" ([11]. In their systematization of the NEO Personality Inventory (NEO-PI) Costa McCrae and Dye ([8]) conceptualize Conscientiousness as "having both proactive and inhibitive aspects." Since a good understanding of these two sides is crucial for the rest of our model, and the conclusions, we will elaborate. Each of the Big Five factors is characterized by six facets. In the case Conscientiousness, the six facets proposed in [8] are Order, Dutifulness, Self Discipline, Achievement Striving, Competence, Deliberation. The first three pertain to the aspect that Costa et alii call the inhibitive side of Conscientiousness, seen "in moral scrupulousness and cautiousness" . The other three pertain to the proactive side, "seen most clearly in the need for achievement and commitment to work."

The two aspects have been proposed several times in further analysis of the Conscientiousness factor. Hough ([18] and Mount and Barrick ([22] propose achievement and dependability as subdomains. Similarly, Roberts et al. [23] and (with slightly different terms) De Young et al. ([10]) suggest industriousness and orderliness. These two sides, that we are going to call here with the terms of motivation and control, are the focus of our analysis.

Conscientiousness and Moral Hazard We approach the modeling of Conscientiousness from a very familiar direction in economic analysis: moral hazard. The amount of care taken in avoiding accidents is a simple and effective index of conscientiousness. So a model where individuals can affect the probability of outcomes is a very natural choice. The setup we adopt is a classical decision theoretic setup, that of Anscombe-Aumann ([1]): Individuals have preferences over acts, which are maps from states to probability distributions over prizes. As in Drèze ([13]) the probability overs states is at least partially under control of the individual, who can affect it at some utility cost. More formally, let S be a finite *state space*, and Z a finite set of *sure outcomes*. Denote by $X = \Delta(Z)$ the set of all probability measures on Z, called *random outcomes*. The set $\mathcal{F} = X^S$ of all maps from S to X, is the set of *acts*; the lottery at state s induced by a generic act is denoted by $f(\cdot, s)$. A binary relation $\succsim$ on $\mathcal{F}$ representing the agent's choices, called *preference*.

The general representation we want to characterize is:

$$V(f) = \max_{p \in \Delta(S)} \left[\sum_{S,Z} p(s) U(z - \phi(s)) f(z, s) - c(p) \right] \tag{1}$$

where we interpret ϕ as the function setting the aspiration level at state s and c is the cost to control the occurrence of a state. In the representation (1) the utility of the outcome z in state s is measured with reference to a state dependent benchmark represented by $\phi(s)$.

We will be in particular interested in the case in which the benchmark is an aspiration level, so that a positive utility is reached when and only when the aspiration level is reached, and the level of utility is otherwise independent of the outcome. For each $f \in \mathcal{F}$ and each $s \in S$, the image $f(\cdot, s)$ of s under f is a probability measure on denoted by

$$f(A \,|s) \quad \forall A \subseteq Z.$$

Let for every state s the aspiration level be denoted by A_s. The evaluation of an act in this case is:

$$V(f) = \max_{p \in \Delta(S)} \left[\sum_{s \in S} p(s) f(A_s \,|s) - c(p) \right] \tag{2}$$

Outline In the rest of the paper, we will first define precisely in section 2 the environment we consider. The form of our general representation 1 clearly suggests that an important intermediate step in the process is the model of moral hazard with state dependent utilities and moral hazard. This is the the main technical task of the paper, which we develop in section 3. Proofs are gathered in section 5.

2 Setup and Basic definitions

The setup we have described earlier is called a *genuine Anscombe-Aumann framework*. It is often convenient to consider the generalization of this framework obtained by replacing $\Delta(Z)$ with an arbitrary convex set X of some arbitrary vector space, we refer to the obtained framework as a *(generalized) Anscombe-Aumann framework*. The latter more general framework is used in most formal statements and proofs, while we will focus our attention on the former in terms of interpretation.

As usual the symbols $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$. If $f, g \in \mathcal{F}$ and $s \in S$, then $f_s g \in \mathcal{F}$ is the act yielding $f(s)$ in state s and $g(s')$ for all $s' \neq s$. Note that clearly $g_s g = g$. If $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $f \alpha g$ is a shorthand for $\alpha f + (1-\alpha) g$. The constant act $x_S \in \mathcal{F}$ yielding $x \in X$ in every state is simply denoted by x.

For each $s \in S$, $\succsim_s$ is the binary relation defined on $\mathcal{F}$ by

$$f \succsim_s g \Longleftrightarrow f_s h \succsim g_s h \quad \forall h \in \mathcal{F} \tag{3}$$

it is called *conditional preference*;

2.1 Moral hazard

The variational state dependent preferences we are after are characterized by a functional V of the form

$$V(f) = \max_{p \in \Delta(S)} \left[\sum_S p(s) U(f(\cdot, s), s) - c(p) \right] \tag{4}$$

Let us consider the two elements separately. The utility function $U : Z \times S \to \mathbb{R}$ is a real valued function, extended to $X \times S$ by linearity, that is

$$u(x, s) = \sum_{z \in Z} x(z) u(z, s) \quad \forall (x, s) \in X \times S \tag{5}$$

The cost function $c : \Delta(S) \to [0, \infty]$ is a lower semi continuous, convex, and grounded ($\min_{p \in \Delta(S)} c(p) = 0$) function.

This representation can be considered as the extension of the moral hazard model of Drèze [13] to the case of a general cost function instead of the cost function which is the indicator of a closed and convex set of probabilities. In fact, for $c(p) = \delta_C(p)$, (4) takes the form:

$$V(f) = \max_{p \in C} \sum_S p(s) U(f(\cdot, s), s) \tag{6}$$

where C is a closed and convex subset of $\Delta(S)$.

2.2 Aspiration

For the model of aspiration, we assume in what follows that Z is a subset of a partially ordered linear space W. An important special case is the one in which Z is a Riesz space and

$$u(z, s) = v(z - \phi(s))$$

where $v : Z \to \mathbb{R}$ is an increasing function and $\phi : S \to Z$ represents the the agent's (*state dependent*) *goal*. Actually, we aim at the very special case

$$v = 1_{W^+}$$

where Z^{+} is the positive cone of Z. In this case,

$$u(z,s) = 1_{Z^{+}}(z - \phi(s)) = \begin{cases} 1 & z \geq \phi(s) \\ 0 & \text{otherwise} \end{cases} = 1_{\phi(s)+W^{+}}(z) \tag{7}$$

if $f(s)$ is a degenerate lottery then

$$u(f(s),s) = \begin{cases} 1 & f(s) \geq \phi(s) \\ 0 & \text{otherwise} \end{cases}$$

else

$$\begin{aligned} u(f(s),s) &= \sum_{z\in Z} f(z|s)\, u(z,s) = \sum_{z\in Z: z\geq\phi(s)} f(z|s) \\ &= f(\{z \in Z : z \geq \phi(s)\} |s) = f\left(\phi(s) + Z^{+}|s\right) \\ &= \Pr\{\text{surpassing the goal in state } s \text{ induced by the choice of } f\}. \end{aligned}$$

If $p \in \Delta(S)$ is the model that generates the occurrence of states,

$$\begin{aligned} \int_S u(f(s),s)dp(s) &= \sum_{s\in S} f\left(\phi(s) + W^{+}|s\right) p(s) \\ &= \Pr\{\text{surpassing the goal by the choice of } f \text{ under } p\}. \end{aligned}$$

A natural extension of (7) that does not require a Riesz space structure, but just a lattice structure, is

$$u(z,s) = 1_{A_s}(z) = \begin{cases} 1 & z \in A_s \\ 0 & \text{otherwise} \end{cases}$$

where A_s is the aspiration level (set) in state S, that is any *proper* subset of Z, that is any set A such that

$$a \in A,\ z \in A, \text{ and } z \geq a \text{ implies } z \in A.$$

Since $f(s)$ is, in general a lottery then

$$\begin{aligned} u(f(s),s) &= \sum_{z\in Z} f(z|s)\, u(z,s) = \sum_{z\in A_s} f(z|s) = f(A_s|s) \\ &= \Pr\{\text{achieving aspirations in state } s \text{ induced by the choice of } f\}. \end{aligned}$$

If $p \in \Delta(S)$ is the model that generates the occurrence of states,

$$\begin{aligned} \int_S u(f(s),s)dp(s) &= \sum_{s\in S} f(A_s|s)\, p(s) \\ &= \Pr\{\text{achieving aspirations by the choice of } f \text{ under } p\}. \end{aligned}$$

Several different representations are possible. For example:

1. The agent has no control over the probability: this s the expected utility model, where $p = p_0$ is given

$$\begin{aligned} V(f) &= \int_S f(A_s | s)\, dp_0(s) \\ &= \Pr\{\text{achieving aspirations by the choice of } f\} \end{aligned}$$

2. The agent has partial (resp. full) control over the probability: he can chose p, but he is constrained to the elements of $C \subset \Delta(S)$ (resp. $C = \Delta(S)$)

$$\begin{aligned} V(f) &= \max_{p \in C} \int_S f(A_s | s)\, dp(s) \\ &= \max_{p \in C} \Pr\{\text{achieving aspirations by the choice of } f \text{ under } p\} \end{aligned}$$

3. The agent has costly control over the probability: he can chose p at a cost $c(p)$

$$V(f) = \max_{p \in \Delta(S)} \left[\int_S f(A_s | s)\, dp(s) - c(p) \right]. \tag{8}$$

2.3 Strategy of proof

Let V be defined for all as in (4) for all $f \in \mathcal{F}$. For each $s \in S$ denote by u_s the s-section of u and by U_s the s-section of the extension of u defined in (5), that is

$$\begin{aligned} U_s : \quad X &\to \mathbb{R} \\ x &\mapsto u(x,s) = \sum_{z \in Z} x(z)\, u(z,s) = \sum_{z \in Z} x(z)\, u_s(z) \end{aligned}$$

for all $x \in X$. All functions $U_s : X \to \mathbb{R}$, for $s \in S$, are affine, and so is their cartesian product $U : X^S \to \mathbb{R}^S$ defined by

$$\begin{aligned} U : \quad X^S &\to \mathbb{R}^S \\ f = [f_s]_{s \in S} &\mapsto U(f) = [U_s(f_s)]_{s \in S} \end{aligned}$$

for all $f \in X^S = \mathcal{F}$, that associates to every $f = (f_s)_{s \in S} \in X^S$ the vector $f^u = (u_s(f(s)))_{s \in S} \in \mathbb{R}^S$. It is easy to show that $(\cdot)^u$ is affine and takes values in $\Pi_{s \in S} u_s(X) = \Phi$.

Notice that Φ is a convex subset of $\mathbb{R}^S$ and the map $I : \Phi \to \mathbb{R}$ defined by

$$I(\varphi) = \max_{p \in \Delta(S)} \left(\langle \varphi, p \rangle - c(p) \right)$$

is a convex niveloid (see the section 7 below), and we can then set the cost function to be:

$$c(p) = \sup_{\varphi \in \Phi} \left(\langle \varphi, p \rangle - I(\varphi) \right) = I^*(p) \quad \forall p \in \Delta(S).$$

3 Moral Hazard and State Dependent Utility

3.1 Axioms

In the sequel, we make use of the following properties of $\succsim$.

Axiom A. 1 (Weak Order) *The binary relation $\succsim$ is complete and transitive.*

Axiom A. 2 (Continuity) *For each $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.*

Axiom A. 3 (Conditional Preference) *For each $s \in S$, if $f, g, h \in \mathcal{F}$, then*

$$f_s g \succsim g \Longrightarrow f_s h \succsim g_s h.$$

Axiom A. 4 (Conditional Independence) *For each $s \in S$, if $f, g, h \in \mathcal{F}$, then*

$$f \sim_s g \Longrightarrow \frac{1}{2}f + \frac{1}{2}h \sim_s \frac{1}{2}g + \frac{1}{2}h.$$

Axiom A. 5 (Conditional Nontriviality) *For each $s \in S$, $\succsim_s$ is nontrivial.*

Axiom A. 6 (Value of Information) *If $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, then*

$$f \sim g \Longrightarrow f \succsim \alpha f + (1-\alpha) g.$$

Axiom A. 7 (Best-worst Independence) *If $f, g, b, w \in \mathcal{F}$, $\alpha \in [0,1]$, and*

$$\alpha f + (1-\alpha)(b\lambda w) \succsim \alpha g + (1-\alpha)(b\lambda w)$$

holds for some $\lambda \in [0,1]$, then it holds for all $\lambda \in [0,1]$, provided $b \succsim_s h \succsim_s w$ for all $s \in S$ and all $h \in \mathcal{F}$.

3.2 Representation Theorems

Let K be any non-empty subset of $\mathbb{R}^S$. A function $I : K \to \mathbb{R}$ is:

1. *normalized* if $I(t1_S) = t$ for all $t \in \mathbb{R}$ such that $t1_S \in X$;
2. *strictly increasing* if $I(\varphi) > I(\psi)$ for all $\varphi, \psi \in K$ such that $\varphi \geq \psi$ and $\varphi \neq \psi$;
3. a *niveloid* if $I(\varphi) - I(\psi) \leq \max_{s \in S}(\varphi(s) - \psi(s))$ for all $\varphi, \psi \in K$.[1]

[1] See Dolecki and Greco [11] and our [20].

Theorem 1 *In a genuine Anscombe-Aumann framework, a binary relation* $\succsim$ *on* $\mathcal{F}$ *satisfies A.1-A.5 if and only if there exist* $U : X \times S \to [0,1]$*, affine and onto with respect to the first component, and a normalized, strictly increasing, and continuous function* $I : [0,1]^S \to [0,1]$ *such that*

$$V(f) = I(U(f(\cdot), \cdot)) \quad \forall f \in \mathcal{F}$$

represents $\succsim$*.*
Moreover, in this case,

1. $f \sim bV(f)w$ *for all* $f \in \mathcal{F}$*, provided* $b \succsim_s h \succsim_s w$ *for all* $s \in S$ *and* $h \in \mathcal{F}$*.*

2. $\succsim$ *satisfies A.6 if and only if* I *is quasiconvex.*

3. $\succsim$ *satisfies A.7 if and only if* I *is a niveloid.*

Both U *and* I *are unique.*

Using the general results on Niveloids presented in the section 7 below, we can now conclude the general representation theorem for variational preferences with state dependent utility. This theorem is an extension of the characterization given by Drèze of state dependent preferences with moral hazard, where the moral hazard had the form of the choice out of an element of a convex set admissible probabilities.

Theorem 2 *In a genuine Anscombe-Aumann framework, a binary relation* $\succsim$ *on* $\mathcal{F}$ *satisfies A.1-A.7 if and only if there exist* $U : X \times S \to [0,1]$*, affine and onto with respect to the first component, and a cost function* c *such that*

$$V(f) = \max_{p \in \Delta(S)} [\sum_s p(s) U(f(\cdot, s), s) - c(p)]$$

represents $\succsim$*. Moreover, in this case, the* U *function is unique and there is a unique maximal cost function* c*.*

4 Aspiration and Control

We now introduce the axiom that gives to the state dependent preferences the

Axiom A. 8 (Aspiration Sets) *For each* $s \in S$ *there is a set* $A_s \subseteq Z$ *such that, for every pair* $z, z' \in Z$*:*

1. $z \succ z'$ *if* $z \in A_s, z' \notin A_s$*,*

2. $z \sim z'$ *if* $z, z' \in A_s$*, or* $z, z' \notin A_s$

The next theorem summarizes the results we have and provides the final representation including both aspiration and control:

Theorem 3 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies A.1- A.8 and if and only if there exists a cost function c such that*

$$V(f) = \max_{p\in\Delta(S)}[\sum_{s\in S, z\in Z} p(s)1_{A_s}(z)f(z,s) - c(p)] \tag{9}$$

represents $\succsim$.

5 Proofs

Denote by f, g, h generic elements of $\mathcal{F}$, by x, y generic elements of X, by α, λ, μ generic elements of $[0,1]$, by φ, ψ generic elements of $[0,1]^S$.

The constant element $x_S \in X^S$ yielding $x \in X$ in every state is simply denoted by x. Analogously, 1_S and 0_S in $\mathbb{R}^S$ are sometimes denoted by 1 and 0, respectively. While $1_s \in \mathbb{R}^S$ is the indicator of state s, for each $s \in S$.

Lemma 1 *If $\succsim$ satisfies A.1-A.3, then, for each $s \in S$, the binary relation $\succsim_s$ satisfies A.1-A.2, and*

$$\begin{aligned} f \succsim_s g &\iff f_s g \succsim g \iff \exists h \in \mathcal{F} : f_s h \succsim g_s h \\ f \succsim_s g &\iff f(s) \succsim_s g(s) \\ f \succ_s g &\iff f_s g \succ g \iff \exists h \in \mathcal{F} : f_s h \succ g_s h \iff f_s h \succ g_s h \quad \forall h \in \mathcal{F} \\ f \succ_s g &\iff f(s) \succ_s g(s). \end{aligned}$$

Moreover, if $f, g \in \mathcal{F}$ and $f \succsim_s g$ for all $s \in S$ then $f \succsim g$, and $f \succ g$ if additionally $f \succ_{\bar{s}} g$ for some $\bar{s} \in S$.

Proof. Arbitrarily choose $s \in S$.

If $f \succsim_s g$, then, by definition (3), $f_s g \succsim g_s g = g$, and so $f_s h \succsim g_s h$ for some $h \in \mathcal{F}$. Conversely, if

$$f_s \bar{h} \succsim g_s \bar{h} \quad \text{for some } \bar{h} \in \mathcal{F} \tag{10}$$

set $\bar{g} = g_s \bar{h}$; then $f_s \bar{h} = f_s \bar{g}$, and (10) amounts to $f_s \bar{g} \succsim \bar{g}$, by A.3 this means $f_s h \succsim \bar{g}_s h = g_s h$ for all $h \in \mathcal{F}$. This proves the chain of characterizations of $\succsim_s$.

Notice that

$$f_s h = f(s)_s h \text{ and } g_s h = g(s)_s h$$

for all $f, g, h \in \mathcal{F}$. Then, if $f, g \in \mathcal{F}$,

$$f \succsim_s g \iff f_s h \succsim g_s h \quad \forall h \in \mathcal{F} \iff f(s)_s h \succsim g(s)_s h \quad \forall h \in \mathcal{F} \iff f(s) \succsim_s g(s).$$

For each $f, g \in \mathcal{F}$, choose $\bar{h} \in \mathcal{F}$, by completeness of $\succsim$, either $f_s \bar{h} \succsim g_s \bar{h}$ or $g_s \bar{h} \succsim f_s \bar{h}$, that is, either $f \succsim_s g$ or $g \succsim_s f$, and $\succsim_s$ is complete. Transitivity of $\succsim_s$ immediately follows by definition (3) and transitivity of $\succsim$. That is, $\succsim_s$ satisfies A.1.

As a consequence of completeness of $\succsim_s$,

$$\begin{aligned} f \succ_s g &\iff g \not\succsim_s f \\ &\iff \neg\left(g_s h \succsim f_s h \quad \forall h \in \mathcal{F}\right) \\ &\iff \exists h \in \mathcal{F} : g_s h \not\succsim f_s h \\ &\iff \exists h \in \mathcal{F} : f_s h \succ g_s h \end{aligned}$$

moreover, using the equivalent definitions of $\succsim_s$,

$$\begin{aligned} g \not\succsim_s f &\iff \neg\left(\exists h \in \mathcal{F} : g_s h \succsim f_s h\right) \\ &\iff g_s h \not\succsim f_s h \quad \forall h \in \mathcal{F} \\ &\iff f_s h \succ g_s h \quad \forall h \in \mathcal{F}. \end{aligned}$$

Summing up,

$$f \succ_s g \iff \exists h \in \mathcal{F} : f_s h \succ g_s h \iff f_s h \succ g_s h \quad \forall h \in \mathcal{F}$$

and so $f_s g \succ g$ is also equivalent to $f \succ_s g$. This proves the chain of characterizations of $\succ_s$ (which, by definition, is the asymmetric part of $\succsim_s$).

The final equivalence for $\succ_s$ follows from

$$f \succ_s g \iff g \not\succsim_s f \iff g(s) \not\succsim_s f(s) \iff f(s) \succ_s g(s).$$

For each $f, g, h \in \mathcal{F}$, choose $\bar{h} \in \mathcal{F}$, the set

$$\begin{aligned} \{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim_s h\} &= \left\{\alpha \in [0,1] : (\alpha f + (1-\alpha) g)_s \bar{h} \succsim h_s \bar{h}\right\} \\ &= \left\{\alpha \in [0,1] : \alpha f_s \bar{h} + (1-\alpha) g_s \bar{h} \succsim h_s \bar{h}\right\} \end{aligned}$$

is closed and so is $\{\alpha \in [0,1] : h \succsim_s \alpha f + (1-\alpha) g\}$, that is, $\succsim_s$ satisfies A.2.

Finally, assume $f, g \in \mathcal{F}$ and $f \succsim_s g$ for all $s \in S$. Set $S = \{s_1, s_2, ..., s_n\}$ and recall that $f_{s_i} h \succsim g_{s_i} h$ for all $h \in \mathcal{F}$ and $i = 1, 2, ..., n$, that is,

$$\begin{bmatrix} h(s_1) \\ \vdots \\ h(s_{i-1}) \\ f(s_i) \\ h(s_{i+1}) \\ \vdots \\ h(s_n) \end{bmatrix} \succsim \begin{bmatrix} h(s_1) \\ \vdots \\ h(s_{i-1}) \\ g(s_i) \\ h(s_{i+1}) \\ \vdots \\ h(s_n) \end{bmatrix}.$$

Moreover, if $f \succ_{s_j} g$ for some $j = 1, 2, ..., n$, then $f_{s_j} h \succ g_{s_j} h$ for all $h \in \mathcal{F}$. Therefore

$$\begin{bmatrix} f(s_1) \\ f(s_2) \\ f(s_3) \\ \vdots \\ f(s_{n-1}) \\ f(s_n) \end{bmatrix} \succsim \begin{bmatrix} g(s_1) \\ f(s_2) \\ f(s_3) \\ \vdots \\ f(s_{n-1}) \\ f(s_n) \end{bmatrix} \succsim \begin{bmatrix} g(s_1) \\ g(s_2) \\ f(s_3) \\ \vdots \\ f(s_{n-1}) \\ f(s_n) \end{bmatrix} \succsim \cdots \succsim \begin{bmatrix} g(s_1) \\ g(s_2) \\ g(s_3) \\ \vdots \\ g(s_{n-1}) \\ f(s_n) \end{bmatrix} \succsim \begin{bmatrix} g(s_1) \\ g(s_2) \\ g(s_3) \\ \vdots \\ g(s_{n-1}) \\ g(s_n) \end{bmatrix}$$

and transitivity delivers $f \succsim g$. Moreover, if $f \succ_{s_j} g$ for some $j = 1, 2, ..., n$, then

$$\begin{bmatrix} g(s_1) \\ \vdots \\ g(s_{j-1}) \\ f(s_j) \\ f(s_{j+1}) \\ \vdots \\ f(s_n) \end{bmatrix} \succ \begin{bmatrix} g(s_1) \\ \vdots \\ g(s_{j-1}) \\ g(s_j) \\ f(s_{j+1}) \\ \vdots \\ f(s_n) \end{bmatrix}$$

and $f \succ g$. ■

Lemma 2 *If $\succsim$ satisfies A.1-A.4, then for each $s \in S$, there exists an affine function $U_s : X \to \mathbb{R}$ such that*

$$f \succsim_s g \Longleftrightarrow U_s(f(s)) \geq U_s(g(s)).$$

Moreover,

1. *if $X = \Delta(Z)$ for some finite set Z, then $\succsim$ satisfies A.9;*

2. *if A.9 is satisfied, then $U_s(X)$ is a compact interval for all $s \in S$.*

Proof. The result follows from Hernstein and Milnor [15] and Lemma 1. ■

Lemma 3 *If $U_s : X \to \mathbb{R}$ is affine for all $s \in S$, then the function*

$$v : \mathcal{F} \to \mathbb{R}^S$$

defined for each $s \in S$ by

$$v_s(f) = U_s(f(s))$$

is affine.

Proof. Trivial. ■

Axiom A. 9 (Conditional Boundedness) *For each $s \in S$, $\succsim_s$ admits a maximum and a minimum element.*

Proposition 1 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies A.1-A.5 and A.9 if and only if there exist $U : X \times S \to [0,1]$, affine and onto with respect to the first component, and a normalized, strictly increasing, and continuous function $I : [0,1]^S \to [0,1]$ such that*

$$V(f) = I(U(f(\cdot), \cdot)) \quad \forall f \in \mathcal{F} \tag{11}$$

represents $\succsim$.
Moreover, in this case,

1. *$f \sim bV(f)w$ for all $f \in \mathcal{F}$, provided $b, w \in \mathcal{F}$ are such that $b \succsim_s h \succsim_s w$ for all $s \in S$ and $h \in \mathcal{F}$.*

2. *$\succsim$ satisfies A.6 if and only if I is quasiconvex.*

3. *$\succsim$ satisfies A.7 if and only if I is a niveloid.*

4. *$U_s : X \to [0,1]$ represents $\succsim_s$ on X for all $s \in S$.*

Both U and I are unique.

Proof of Proposition 1 and Theorem 1. We first assume $\succsim$ on $\mathcal{F}$ satisfies A.1-A.5 and either $X = \Delta(Z)$ for some finite set Z or A.9 is satisfied.

By Lemma 2, for each $s \in S$, there exists an affine function $U_s : X \to \mathbb{R}$ such that

$$f \succsim_s g \Longleftrightarrow U_s(f(s)) \geq U_s(g(s)). \tag{12}$$

For each $s \in S$, $U_s(X)$ is a compact interval $[r_s, t_s] \subseteq \mathbb{R}$ and A.5 implies that $r_s < t_s$. The assumption $r_s = 0$ and $t_s = 1$ causes no loss in generality and makes U_s *the only* affine function for which (12) holds.

In particular, the function $v : \mathcal{F} \to [0,1]^S$ defined in Lemma 3 is affine and such that

$$f \succsim_s g \Longleftrightarrow v_s(f) \geq v_s(g). \tag{13}$$

The proof now proceeds through several steps.

Step 1. v is onto and $b, w \in \mathcal{F}$ are such that

$$b \succsim_s h \succsim_s w \quad \forall s \in S, h \in \mathcal{F}$$

if and only if $v(b) = 1_S$ and $v(w) = 0_S$.
Proof of the Step. Let $\varphi \in [0,1]^S$, for each $s \in S$, there exists $x_s \in X$ such that $U_s(x_s) = \varphi_s$ because $U_s : X \to [0,1]$ is onto. Set

$$f(s) = x_s \quad \forall s \in S$$

to obtain $v_s(f) = U_s(f(s)) = \varphi_s$ for all $s \in S$, that is $v(f) = \varphi$. Therefore v is onto.

In particular, there exist $b, w \in \mathcal{F}$ such that

$$v(b) = 1_S \text{ and } v(w) = 0_S. \tag{14}$$

If $b, w \in \mathcal{F}$ satisfy (14), for each $s \in S$ and each $h \in \mathcal{F}$,

$$v_s(b) = 1 \geq v_s(h) \geq 0 = v_s(w)$$

therefore, by (13)

$$b \succsim_s h \succsim_s w. \tag{15}$$

Conversely, if $\bar{b}, \bar{w} \in \mathcal{F}$ satisfy $\bar{b} \succsim_s h \succsim_s \bar{w}$ for all $s \in S$ and $h \in \mathcal{F}$, choose $b, w \in \mathcal{F}$ that satisfy (14). It follows that $\bar{b} \succsim_s b, w \succsim_s \bar{w}$ and, by (15), $b \succsim_s \bar{b}, \bar{w} \succsim_s w$, so that $\bar{b} \sim_s b$ and $\bar{w} \sim_s w$ for all $s \in S$. By (13)

$$v_s(\bar{b}) = v_s(b) = 1 \text{ and } v_s(\bar{w}) = v_s(w) = 0 \quad \forall s \in S$$

that is, $v(\bar{b}) = 1_S$ and $v(\bar{w}) = 0_S$. □

In the next five steps denote by b and w *any* two elements of $\mathcal{F}$ such that $b \succsim_s h \succsim_s w$ for all $s \in S$ and $h \in \mathcal{F}$.[2] Affinity of v delivers

$$v(b\lambda w) = \lambda v(b) + (1-\lambda) v(w) = \lambda 1_S \quad \forall \lambda \in [0,1]. \tag{16}$$

Step 2. For each $f \in \mathcal{F}$ there exists a unique $V(f) \in [0,1]$ such that $f \sim bV(f)w$ and it does not depend on the choice of b and w.[3]

Proof of the Step. Lemma 1 implies that $b \succsim f \succsim w$ for each $f \in \mathcal{F}$. A.1 and A.2 imply the existence of λ such that $f \sim b\lambda w$. If $f \sim b\mu w$ for some $\mu > \lambda$, by (16) and (13),

$$b\mu w \succ_s b\lambda w \quad \forall s \in S,$$

a contradiction, since Lemma 1 would then imply $b\mu w \succ b\lambda w$. □

Step 3. $V : \mathcal{F} \to [0,1]$ represents $\succsim$ and is mixture continuous.

Proof of the Step. By (16) and (13), $b\lambda w \succsim a\mu w$ if and only if $\lambda \geq \mu$.[4] This implies that

$$f \succsim g \iff bV(f)w \succsim bV(g)w \iff V(f) \geq V(g),$$

[2] Such elements exist since v is onto.

[3] In particular, $V(b\lambda w) = \lambda$ for all $\lambda \in [0,1]$.

[4] If $\lambda \geq \mu$, then $v_s(b\lambda w) = \lambda \geq \mu = v_s(b\mu w)$ for all $s \in S$, whence $b\lambda w \succsim_s b\mu w$ for all $s \in S$, and Lemma 1 delivers $b\lambda w \succsim b\mu w$. The converse is shown at the end of the previous step, that is, $\lambda \ngeq \mu$ implies $b\lambda w \not\succsim b\mu w$.

that is, V represents $\succsim$. As for mixture continuity, it is sufficient to observe that for each $\tau \in \mathbb{R}$ and $f, g \in \mathcal{F}$, the set

$$\{\alpha \in [0,1] : V(\alpha f + (1-\alpha) g) \leq \tau\} = \begin{cases} [0,1] & \tau > 1 \\ \{\alpha \in [0,1] : b\tau w \succsim \alpha f + (1-\alpha) g\} & 0 \leq \tau \leq 1 \\ \varnothing & \tau < 0 \end{cases}$$

is closed, and so is $\{\alpha \in [0,1] : V(\alpha f + (1-\alpha) g) \geq \tau\}$. □

Step 4. If $v(f) \geq v(g)$ (resp. $v(f) > v(g)$), then $f \succsim g$ (resp. $f \succ g$).

Proof of the Step. Finally, $v(f) \geq v(g)$, by definition means

$$v_s(f) \geq v_s(g) \quad \forall s \in S$$

by (13),

$$f \succsim_s g \quad \forall s \in S$$

which implies $f \succsim g$ by Lemma 1. □

Step 5. The correspondence I from $[0,1]^S$ to $[0,1]$, defined for each $\varphi \in [0,1]^S$ by

$$I : \varphi \mapsto V(f) \quad \text{if } f \in \mathcal{F} \text{ and } v(f) = \varphi$$

is a normalized and strictly increasing function $I : [0,1]^S \to [0,1]$ such that

$$I(v(f)) = V(f) \quad \forall f \in \mathcal{F}.$$

Proof of the Step. Since v is onto, for each $\varphi \in [0,1]^S$, there exists $f \in \mathcal{F}$ such that $v(f) = \varphi$ (the correspondence has nonempty values in $[0,1]$). Moreover, if $f, g \in \mathcal{F}$ are such that $v(f) = \varphi = v(g)$, then $f \sim g$ by Step 4, and $V(f) = V(g)$. Therefore, the correspondence I is a function since it has singleton values in $[0,1]$.

In general, for each $\varphi \in [0,1]^S$, set

$$b\varphi w(s) = \varphi(s) b(s) + (1-\varphi(s)) w(s)$$

then

$$\begin{aligned} v_s(b\varphi w) &= U_s(\varphi(s) b(s) + (1-\varphi(s)) w(s)) = \varphi(s) U_s(b(s)) + (1-\varphi(s)) U_s(w(s)) \\ &= \varphi(s) v_s(b) + (1-\varphi(s)) v_s(w(s)) = \varphi(s) \end{aligned}$$

that is

$$\varphi = v(b\varphi w)$$

and hence

$$I(\varphi) = V(b\varphi w).$$

In particular, for all $\lambda \in [0,1]$, $I(\lambda 1_S) = V(b\lambda w) = \lambda$ and I is normalized. Finally, $\varphi > \psi$ imply $v(f) > v(g)$ if $f, g \in \mathcal{F}$ are such that $v(f) = \varphi$ and $v(g) = \psi$, by Step 4, $f \succ g$; therefore $V(f) > V(g)$ and $I(\varphi) > I(\psi)$. □

Step 6. I is continuous.

Proof of the Step. Consider $\varphi, \psi \in [0,1]^S$ and $\tau \in \mathbb{R}$. Let $f, g \in \mathcal{F}$ be such that $v(f) = \varphi$ and $v(g) = \psi$. Then,

$$\begin{aligned}\{\alpha \in [0,1] : I(\alpha\varphi + (1-\alpha)\psi) \leq \tau\} &= \{\alpha \in [0,1] : I(\alpha v(f) + (1-\alpha) v(g)) \leq \tau\} \\ &= \{\alpha \in [0,1] : I(v(\alpha f + (1-\alpha) g)) \leq \tau\} \\ &= \{\alpha \in [0,1] : V(\alpha f + (1-\alpha) g) \leq \tau\}.\end{aligned}$$

is closed since V is mixture continuous, and so is $\{\alpha \in [0,1] : I(\alpha\varphi + (1-\alpha)\psi) \geq \tau\}$. Since I is increasing, by Proposition 43 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6], it follows that I is continuous. □

Steps 1-6 show that the axioms are sufficient for the representation, with $U(x,s) = U_s(x)$ for all $(x,s) \in X \times S$, and the representation satisfies Point 1 of the statement.

Conversely, assume that $\succsim$ is represented as in (11). For each $s \in S$, set $U_s(x) = U(x,s)$ for all $x \in X \times S$. Since U_s is affine for all $s \in S$, so is v defined as in Lemma 3, moreover, $V(f) = I(v(f))$ for all $f \in \mathcal{F}$.

Step 7. $\succsim$ satisfies A.1 and A.2.

Proof of the Step. A.1 follows from existence of the representation V. Moreover, if $f, g, h \in \mathcal{F}$ then

$$\begin{aligned}\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\} &= \{\alpha \in [0,1] : I(v(\alpha f + (1-\alpha) g)) \geq I(v(h))\} \\ &= \{\alpha \in [0,1] : I(\alpha v(f) + (1-\alpha) v(g)) \geq I(v(h))\}.\end{aligned}$$

Since I is continuous, it follows that $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ is closed (see, e.g., [6, Proposition 43, page 1306]), and so is $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$, that is, $\succsim$ satisfies A.2. □

Step 8. v is onto and such that

$$f \succsim_s g \Longleftrightarrow v_s(f) \geq v_s(g). \tag{17}$$

In particular, $b, w \in \mathcal{F}$ are such that

$$b \succsim_s h \succsim_s w \quad \forall s \in S, h \in \mathcal{F} \tag{18}$$

if and only if $v(b) = 1_S$ and $v(w) = 0_S$.

Proof of the Step. See Step 1 for the proof of the fact that v is onto. In particular, there exist $b, w \in \mathcal{F}$ such that

$$v(b) = 1_S \text{ and } v(w) = 0_S. \tag{19}$$

Let $s \in S$. If $f \succsim_s g$, then $f_s w \succsim g_s w$ then, by (11), $I(v_s(f) 1_s) = I(v(f_s w)) \geq I(v(g_s w)) = I(v_s(g) 1_s)$. Since I is strictly increasing $v_s(f) < v_s(g)$ would imply $I(v_s(f) 1_s) < I(v_s(g) 1_s)$, therefore $v_s(f) \geq v_s(g)$. Conversely, $v_s(f) \geq v_s(g)$ implies $v(f_s h) = v_s(f) 1_s + v(h) - v_s(h) 1_s \geq v_s(g) 1_s + v(h) - v_s(h) 1_s = v(g_s h)$ and monotonicity of I implies $I(v(f_s h)) \geq I(v(g_s h))$ for all $h \in \mathcal{F}$, and $f \succsim_s g$. This proves (17).

If $b, w \in \mathcal{F}$ satisfy (19), for each $s \in S$ and each $h \in \mathcal{F}$,

$$v_s(b) = 1 \geq v_s(h) \geq 0 = v_s(w)$$

therefore, by (17)

$$b \succsim_s h \succsim_s w. \tag{20}$$

Conversely, if $\bar{b}, \bar{w} \in \mathcal{F}$ satisfy $\bar{b} \succsim_s h \succsim_s \bar{w}$ for all $s \in S$ and $h \in \mathcal{F}$, choose $b, w \in \mathcal{F}$ that satisfy (19). It follows that $\bar{b} \succsim_s b, w \succsim_s \bar{w}$ and, by (15), $b \succsim_s \bar{b}, \bar{w} \succsim_s w$, so that $\bar{b} \sim_s b$ and $\bar{w} \sim_s w$ for all $s \in S$. By (17)

$$v_s(\bar{b}) = v_s(b) = 1 \text{ and } v_s(\bar{w}) = v_s(w) = 0 \quad \forall s \in S$$

that is, $v(\bar{b}) = 1_S$ and $v(\bar{w}) = 0_S$. □

Denote by b and w *any* two elements of $\mathcal{F}$ $v(b) = 1_S$ and $v(w) = 0_S$. Such elements exist since v is onto and (18) guarantees that $\succsim$ satisfies A.9. On the other hand (17) delivers $b \succ_s w$ for all $s \in S$, and $\succsim$ satisfies A.5 Affinity of v delivers

$$v(b \lambda w) = \lambda v(b) + (1 - \lambda) v(w) = \lambda 1_S \quad \forall \lambda \in [0, 1]. \tag{21}$$

Also notice that, for each $s \in S$,

$$x \succsim_s y \iff v_s(x) \geq v_s(y) \iff U_s(x) \geq U_s(y).$$

Therefore $U_s : X \to [0, 1]$ represents $\succsim_s$ on X, which together with $\max_{x \in X} U_s(x) = 1 = 1 + \min_{x \in X} U_s(x)$ makes U_s unique.

Step 9. $\succsim$ satisfies A.4.

Proof of the Step. Since v is affine and so is the projection on the s-th component,

$$\begin{aligned} f \sim_s g \quad &\Rightarrow \quad v_s(f) = v_s(g) \Rightarrow \frac{1}{2} v_s(f) + \frac{1}{2} v_s(h) = \frac{1}{2} v_s(g) + \frac{1}{2} v_s(h) \\ &\Rightarrow \quad v_s\left(\frac{1}{2} f + \frac{1}{2} h\right) = v_s\left(\frac{1}{2} g + \frac{1}{2} h\right) \Rightarrow \frac{1}{2} f + \frac{1}{2} h \sim_s \frac{1}{2} g + \frac{1}{2} h. \end{aligned}$$

The arbitrary chioce of $s \in S$ and $f, g, h \in \mathcal{F}$ allows to deduce that $\succsim$ satisfies A.4. □

Step 10. $\succsim$ satisfies A.3.

Proof of the Step. Consider $s \in S$ and $f, g, h \in \mathcal{F}$, and assume $f_s g \succsim g$. Then $v(f_s g) = v_s(f) 1_s + v(g) - v_s(g) 1_s = v(g) + (v_s(f) - v_s(g)) 1_s$. By strict monotonicity $v_s(f) - v_s(g) \geq 0$ (otherwise $I(v(f_s g)) < I(v(g))$ contradicting $f_s g \succsim g$). Therefore, $v(f_s h) = v_s(f) 1_s + v(h) - v_s(h) 1_s \geq$

$v_s(g)\,1_s + v(h) - v_s(h)\,1_s = v(g_s h)$ and monotonicity of I implies $I(v(f_s h)) \geq I(v(g_s h))$ for all $h \in \mathcal{F}$, that is $f_s h \succsim g_s h$. □

Steps 8-10 show that the axioms are necessary for the representation. Normalization of I and (21) deliver

$$V(bV(f)\,w) = I(v(bV(f)\,w)) = I(V(f)\,1_S) = V(f).$$

Therefore, the representation satisfies Point 1 of the statement.

Let $W : X \times S \to [0,1]$ be another function – affine and onto with respect to the first component – and $J : [0,1]^S \to [0,1]$ be another function – normalized, strictly increasing, and continuous – such that the value function defined by

$$V'(f) = J(W(f(\cdot),\cdot)) \quad \forall f \in \mathcal{F}$$

represents $\succsim$.

The arguments used in the proof of necessity of the axioms for the representation show that $W_s : X \to [0,1]$ represents $\succsim_s$ on X for each $s \in S$, which together with $\max_{x\in X} W_s(x) = 1 = 1 + \min_{x\in X} W_s(x)$ makes W_s unique. Thus $W = U$.

Moreover, for each $\varphi \in [0,1]^S$, let $f \in \mathcal{F}$ be such that $\varphi = v(f) = U(f(\cdot),\cdot) = W(f(\cdot),\cdot)$. By Point 1 of the statement, and taking $b, w \in \mathcal{F}$ such that $b \succsim_s h \succsim_s w$ for all $s \in S$,

$$f \sim bV'(f)\,w$$

then, by normalization of I and (21),

$$I(\varphi) = I(v(f)) = I\big(v\big(bV'(f)\,w\big)\big) = V'(f) = J(W(f(\cdot),\cdot)) = J(\varphi).$$

That is $J = I$.

Uniqueness is established.

Finally, assume representation (11) holds and let v be defined as in Lemma 3.

Proof of Point 2. Assume $\succsim$ satisfies A.6. Consider $\varphi, \psi \in [0,1]^{|S|}$ such that $I(\varphi) = I(\psi)$ and $\alpha \in [0,1]$. Let $f, g \in \mathcal{F}$ be such that $\varphi = v(f)$ and $\psi = v(g)$ (which exist by Step 8). Then $I(v(f)) = I(v(g))$, $f \sim g$, and $f \succsim \alpha f + (1-\alpha)\,g$ by A.6. Therefore,

$$I(\varphi) = V(f) \geq V(\alpha f + (1-\alpha)\,g) = I(v(\alpha f + (1-\alpha)\,g)) = I(\alpha\varphi + (1-\alpha)\,\psi).$$

Since φ, ψ and α were arbitrarily chosen,

$$I(\varphi) \geq I(\alpha\varphi + (1-\alpha)\,\psi) \quad \forall \alpha \in [0,1] \text{ and } \forall \varphi, \psi \in [0,1]^{|S|} : I(\varphi) = I(\psi).$$

Since I is continuous, it follows that I is quasiconvex.

Conversely, assume I is quasiconvex. Consider $f, g \in \mathcal{F}$ such that $f \sim g$ and $\alpha \in [0,1]$. Then $I(v(f)) = I(v(g))$ and

$$V(\alpha f + (1-\alpha) g) = I(v(\alpha f + (1-\alpha) g)) = I(\alpha v(f) + (1-\alpha) v(g)) \leq I(v(f)) = V(f)$$

where the inequality is a consequence of quasiconvexity and implies $f \succsim \alpha f + (1-\alpha) g$. Since f, g and α were arbitrarily chosen, it follows that $\succsim$ satisfies A.6 □

Proof of Point 3. Assume $\succsim$ satisfies A.7. Since I is monotonic and normalized, the characterization results of [20], guarantee that, in order to show that I is a niveloid, it sufficies to prove that

$$I(\alpha\varphi + (1-\alpha)\mu) = I(\alpha\varphi + (1-\alpha)\frac{1}{2}) + (1-\alpha)\left(\mu - \frac{1}{2}\right) \tag{22}$$

for each $\varphi \in [0,1]^{|S|}$ and $\alpha, \mu \in [0,1]$. Consider $f \in \mathcal{F}$ such that $v(f) = \varphi$ and recall $v(b\mu w) = \mu$, by (21).

Notice that

$$\alpha 1_S + (1-\alpha)\frac{1}{2} \geq \alpha\varphi + (1-\alpha)\frac{1}{2} \geq \alpha 0_S + (1-\alpha)\frac{1}{2}$$

then, by continuity of $t \mapsto I\left(\alpha t 1_S + (1-\alpha)\frac{1}{2}\right)$ from $[0,1] \to \mathbb{R}$ and monotonicity of I, there is $\lambda \in [0,1]$ such that

$$I\left(\alpha\lambda 1_S + (1-\alpha)\frac{1}{2}\right) = I\left(\alpha\varphi + (1-\alpha)\frac{1}{2}\right). \tag{23}$$

Normalization of I delivers

$$I\left(\alpha\varphi + (1-\alpha)\frac{1}{2}\right) = \alpha\lambda + (1-\alpha)\frac{1}{2}. \tag{24}$$

By (23) and since $v\left(\alpha f + (1-\alpha) b\frac{1}{2}w\right) = \alpha\varphi + (1-\alpha)\frac{1}{2}$, it follows that

$$\alpha f + (1-\alpha) b\frac{1}{2}w \sim b\left(\alpha\lambda + (1-\alpha)\frac{1}{2}\right)w = \alpha b\lambda w + (1-\alpha) b\frac{1}{2}w$$

By A.7, this implies that

$$\alpha f + (1-\alpha) b\mu w \sim \alpha(b\lambda w) + (1-\alpha) b\mu w = b(\alpha\lambda + (1-\alpha)\mu) w$$

for all $\mu \in [0,1]$, that is

$$I(\alpha\varphi + (1-\alpha)\mu) = \alpha\lambda + (1-\alpha)\mu. \tag{25}$$

Subtracting (24) from (25) delivers (22).

Conversely, assume I is a niveloid. If $f, g \in \mathcal{F}$, $\alpha \in [0,1]$, and

$$\alpha f + (1-\alpha)(b\lambda w) \succsim \alpha g + (1-\alpha)(b\lambda w)$$

holds for some $\lambda \in [0,1]$. Then,

$$I(\alpha v(f) + (1-\alpha)\lambda) \geq I(\alpha v(g) + (1-\alpha)\lambda)$$

since I is a niveloid this amounts to $I(\alpha v(f)) + (1-\alpha)\lambda \geq I(\alpha v(g)) + (1-\alpha)\lambda$ that is

$$I(\alpha v(f)) \geq I(\alpha v(g))$$

which in turn delivers $I(\alpha v(f)) + (1-\alpha)\mu \geq I(\alpha v(g)) + (1-\alpha)\mu$ for all $\mu \in [0,1]$, and the properties of niveloids deliver

$$I(\alpha v(f) + (1-\alpha)\mu) \geq I(\alpha v(g) + (1-\alpha)\mu)$$

or

$$\alpha f + (1-\alpha)(b\mu w) \succsim \alpha g + (1-\alpha)(b\mu w)$$

for all $\mu \in [0,1]$. That is $\succsim$ satisfies A.7. ■

6 Conclusions

We have provided an axiomatic foundation of the Big Five factor Conscientiousness. The theory provides condition under which two characteristics, the aspiration level and the cost of control, can be uniquely identified from the choice behavior of the individual. These two characteristic are reasonably accurate correspondent of the two aspects of Conscientiousness that have been repeatedly identified in the literature, and labeled in the NEO-PI inventory proactive and inhibitive side of the factor ([8]).

We consider as main contribution of the paper the successful extension of the axiomatic method to Personality Theory. The model we have given here of the two aspects of conscientiousness is very stylized, but it has the advantage of giving a clean correspondence between two parameters that are in principle identifiable from behavior with general description of individual attitude that have been so far based on factor analysis of survey questions.

This step is important in two respects. First, the analysis of the consequences of individual personality traits on behavior and performance is fast developing, and just as a sound axiomatization of concepts such as risk aversion was needed to put the analysis of choice under uncertainty on a sound quantitative basis, so we hope models like ours will provide a similar basis for the empirical and experimental analysis of individual behavior. Second, genetic and neuroeconomic analysis of individuals differences is also progressing, and needs a precise definition of the phenotype (in genetic analysis) and of the characteristics of the decision process that may be summarized by a simple and operational measurement. Our model may provide the conceptual structure for such measurement.

7 Niveloids

Let (S,Σ) be a measurable space. Denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions and by $B(\Sigma)$ its supnorm closure (in the space of all real-valued bounded functions on S). $B_0(\Sigma,K)$ (resp. $B(\Sigma,K)$) is the set of all functions in $B_0(\Sigma)$ (resp. $B(\Sigma)$) taking values in the interval $K\subseteq\mathbb{R}$.

When endowed with the supnorm, $B_0(\Sigma)$ is a normed vector space and $B(\Sigma)$ is a Banach space. The norm dual of $B_0(\Sigma)$ (resp. $B(\Sigma)$) is the space $ba(\Sigma)$ of all bounded and finitely additive set functions $\mu:\Sigma\to\mathbb{R}$ endowed with the total variation norm, the duality being

$$\langle\varphi,\mu\rangle=\int\varphi d\mu$$

for all $\varphi\in B_0(\Sigma)$ (resp. $B(\Sigma)$) and all $\mu\in ba(\Sigma)$ (see, e.g., [**?**, p. 258]). As it is well known, on $\Delta(\Sigma)$ the $\sigma(ba(\Sigma),B_0(\Sigma))$-topology coincides with the $\sigma(ba(\Sigma),B(\Sigma))$-topology, and they go under the name of *weak* topology*; moreover a subset of $\Delta^\sigma(\Sigma)$ is weakly* compact iff it is weakly compact (i.e. compact in the weak topology of the Banach space $ba(\Sigma)$).

For $\varphi,\psi\in B(\Sigma)$ we write $\varphi\geq\psi$ (resp. $\varphi>\psi$) if $\varphi(s)\geq\psi(s)$ (resp. $\varphi(s)>\psi(s)$) for all $s\in S$.

Let Φ be any nonempty collection of elements of $B(\Sigma)$, and Φ_c the constant functions in Φ.[5] We call Φ a *tube* if $\Phi=\Phi+\mathbb{R}$.[6]

Given a functional $I:\Phi\to\mathbb{R}$, we say that I is:

(i) *normalized* if $I(k)=k$ for all $k\in\Phi_c$;

(ii) *monotonic* if $\varphi\geq\psi$ implies $I(\varphi)\geq I(\psi)$ for all $\varphi,\psi\in\Phi$;

(iii) *vertically invariant* if $I(\varphi+c)=I(\varphi)+c$ for all $\varphi\in\Phi$ and $c\in\mathbb{R}$ such that $\varphi+c\in\Phi$;

(iv) a *niveloid* if $I(\varphi)-I(\psi)\leq\sup(\varphi-\psi)$ for all $\varphi,\psi\in\Phi$.[7]

Remark 1 Notice that I is a niveloid iff $I(\psi)-I(\varphi)\geq-\sup(\varphi-\psi)=\inf(\psi-\varphi)$ for all $\varphi,\psi\in\Phi$ iff $\inf(\psi-\varphi)\leq I(\psi)-I(\varphi)\leq\sup(\psi-\varphi)$ for all $\psi,\varphi\in\Phi$. Clearly a niveloid is Lipschitz continuous of rank 1 in the supnorm ($\sup(\varphi-\psi)\leq\|\varphi-\psi\|$).

Remark 2 For all $I:\Phi\to\mathbb{R}$, define $\bar{I}:-\Phi\to\mathbb{R}$ by $\bar{I}(\varphi)=-I(-\varphi)$. It is easy to check that: $\overline{(\bar{I})}=I$; I is normalized iff $\bar{I}$ is normalized; I is monotonic iff $\bar{I}$ is monotonic; I is vertically invariant iff $\bar{I}$ is vertically invariant; I is a niveloid iff $\bar{I}$ is a niveloid.

[5] As usual, we write k both for the real number k and for the constant function $k1_S\in B_0(\Sigma)$.

[6] Clearly, if Φ is not a tube, then $\Phi+\mathbb{R}$ is the smallest tube containing Φ.

[7] Dolecki and Greco [**?**] call *niveloid* a monotonic and vertically invariant functional $T:[-\infty,\infty]^S\to[-\infty,\infty]$. Their Corollary 1.3 and our Lemma 7 explain why we chose to abuse this term.

7.1 Vertically invariant functionals

Next Lemma provides a useful condition for vertical invariance.

Lemma 4 *Let Φ be a convex subset of $B_0(\Sigma)$ (or $B(\Sigma)$) with $0 \in \Phi$ and $I : \Phi \to \mathbb{R}$ be a functional that satisfies*

$$I(\alpha\varphi + (1-\alpha)k) = I(\alpha\varphi) + (1-\alpha)k \tag{26}$$

for all $\varphi \in \Phi$, $k \in \Phi_c$, and $\alpha \in (0,1)$. Then I is vertically invariant provided one of the following conditions holds:

- *Φ is open,*
- *I is continuous and $0 \in \operatorname{int}(\Phi)$,*
- *$\Phi = B_0(\Sigma, K)$ for some interval $K \subseteq \mathbb{R}$ such that $0 \in \operatorname{int}(K)$.*

Proof. If $c = 0$ then $I(\varphi + c) = I(\varphi) + c$ for all $\varphi \in \Phi$. It is sufficient to prove that $I(\varphi + c) = I(\varphi) + c$ for all $\varphi \in \Phi$ and $c > 0$ such that $\varphi + c \in \Phi$.[8]

Let $\varphi, \varphi + c \in \Phi$ and $c > 0$.

Step 1. If $\varphi, \varphi + c \in \operatorname{int}(\Phi)$, then $I(\varphi + c) = I(\varphi) + c$.

There exists $\alpha \in (0,1)$ such that $\varphi/\alpha, (\varphi + c)/\alpha \in \operatorname{int}(\Phi)$. Hence $(\varphi + t)/\alpha \in \operatorname{int}(\Phi)$ for each $t \in [0, c]$. In fact, there exists $\gamma \in [0,1]$ such that $t = \gamma c$ and

$$\frac{\varphi + t}{\alpha} = \frac{\varphi + \gamma c}{\alpha} = \gamma \frac{\varphi + c}{\alpha} + (1-\gamma)\frac{\varphi}{\alpha} \in \operatorname{int}(\Phi).$$

[8] If $c < 0$, set $\psi = \varphi + c$, and $d = -c$. This yields $\psi, \psi + d \in \Phi$ and $d > 0$, then $I(\psi + d) = I(\psi) + d$, that is $I(\varphi) = I(\varphi + c) - c$.

Choose $n \geq 2$ such that $\frac{c/n}{1-\alpha} \in \Phi_c$.[9] Then

$$\begin{aligned}
I(\varphi + c) &= I\left(\varphi + \frac{c}{n} + \ldots + \frac{c}{n}\right) \\
&= I\left(\varphi + \frac{c(n-1)}{n} + \frac{c}{n}\right) \\
&= I\left(\alpha\left(\frac{\left(\varphi + \frac{c(n-1)}{n}\right)}{\alpha}\right) + (1-\alpha)\frac{c/n}{1-\alpha}\right) \\
&= I\left(\alpha\left(\frac{\left(\varphi + \frac{c(n-1)}{n}\right)}{\alpha}\right)\right) + (1-\alpha)\frac{c/n}{1-\alpha} \\
&= I\left(\varphi + \frac{c(n-1)}{n}\right) + \frac{c}{n} \\
&= \cdots \\
&= I\left(\varphi + \frac{c}{n}\right) + \frac{c(n-1)}{n} \\
&= I(\varphi) + \frac{c}{n} + \frac{c(n-1)}{n} \\
&= I(\varphi) + c,
\end{aligned}$$

as wanted. □

Step 1 proves the lemma if Φ is open. If I is continuous and $0 \in \operatorname{int}(\Phi)$, since $\varphi, \varphi + c \in \Phi$, then $\left(1 - \frac{1}{n}\right)\varphi$ and $\left(1 - \frac{1}{n}\right)(\varphi + c) \in \operatorname{int}(\Phi)$ for all $n \geq 1$. But, by Step 1, I is vertically invariant on $\operatorname{int}(\Phi)$ and hence

$$\begin{aligned}
I(\varphi + c) &= \lim_{n\to\infty} I\left(\left(1 - \frac{1}{n}\right)(\varphi + c)\right) = \lim_{n\to\infty} I\left(\left(1 - \frac{1}{n}\right)\varphi + \left(1 - \frac{1}{n}\right)c\right) \\
&= \lim_{n\to\infty} I\left(\left(1 - \frac{1}{n}\right)\varphi\right) + \left(1 - \frac{1}{n}\right)c = I(\varphi) + c.
\end{aligned}$$

It remains to prove the last case when $K = [a, b)$ or $(a, b]$ or $[a, b]$ with $-\infty \leq a < 0 < b \leq \infty$.[10]

Step 2. Assume K contains b (and hence $b < \infty$) and $a < \varphi < \varphi + c \leq b$, then $I(\varphi + c) = I(\varphi) + c$.

Choose $n \geq 2$ such that $b - \frac{c}{n} > 0$ and $\varphi / \left(\frac{b - \frac{c}{n}}{b}\right) > a$. Set $\alpha = \frac{b - \frac{c}{n}}{b} \in (0, 1)$. Notice that for all $t \in \left[\frac{c}{n}, \frac{c(n-1)}{n}\right]$

$$\varphi < \varphi + \frac{c}{n} \leq \varphi + t \leq \varphi + \frac{c(n-1)}{n} = \varphi + c - \frac{c}{n} \leq b - \frac{c}{n} < b.$$

Divide all the terms by α to obtain

$$a < \varphi/\alpha < \left(\varphi + \frac{c}{n}\right)/\alpha \leq (\varphi + t)/\alpha \leq \left(\varphi + \frac{c(n-1)}{n}\right)/\alpha \leq b$$

[9] This is possible since in any case $0 \in \operatorname{int}(\Phi)$.
[10] If $K = (a, b)$, then $B_0(\Sigma, K)$ is open in $B_0(\Sigma)$.

and hence $(\varphi + t)/\alpha \in B_0(\Sigma, K)$ for each $t \in \left[\frac{c}{n}, \frac{c(n-1)}{n}\right]$. Moreover $\frac{c/n}{1-\alpha} = b \in K$, and

$$\begin{aligned}
I(\varphi + c) &= I\left(\varphi + \frac{c(n-1)}{n} + \frac{c}{n}\right) \\
&= I\left(\alpha\left(\frac{\left(\varphi + \frac{c(n-1)}{n}\right)}{\alpha}\right) + (1-\alpha)\frac{c/n}{1-\alpha}\right) \\
&= I\left(\alpha\left(\frac{\left(\varphi + \frac{c(n-1)}{n}\right)}{\alpha}\right)\right) + (1-\alpha)\frac{c/n}{1-\alpha} \\
&= I\left(\varphi + \frac{c(n-1)}{n}\right) + \frac{c}{n}.
\end{aligned}$$

But $a < \varphi < \varphi + \frac{c(n-1)}{n} < b$ implies $\varphi, \varphi + \frac{c(n-1)}{n} \in \operatorname{int}(\Phi)$, and Step 1 guarantees $I\left(\varphi + \frac{c(n-1)}{n}\right) = I(\varphi) + \frac{c(n-1)}{n}$ whence $I(\varphi + c) = I(\varphi) + c$. □

Step 2 concludes the proof if $K = (a, b]$.

Step 3. Assume K contains a (and hence $a > -\infty$) and $a \leq \varphi < \varphi + c < b$, then $I(\varphi + c) = I(\varphi) + c$.

Consider $-K$ and notice that $-b < -\varphi - c < -\varphi \leq -a$, then $\psi = -\varphi - c \in B_0(\Sigma, -K)$, $c > 0$ and $\psi + c = -\varphi \in B_0(\Sigma, -K)$. Moreover, it is immediate to show that $\bar{I}$ satisfies (26), by Step 2

$$\begin{aligned}
I(\varphi + c) &= -\bar{I}(-\varphi - c) = -\bar{I}(\psi) \\
&= -\left(\bar{I}(\psi + c) - c\right) = -\bar{I}(-\varphi) + c \\
&= I(\varphi) + c,
\end{aligned}$$

as wanted. □

Step 3 concludes the proof if $K = [a, b)$.

If $K = [a, b]$, then $-\infty < a < b < \infty$. If $\varphi, \varphi + c \in B_0(\Sigma, K)$, then $a \leq \varphi < \varphi + \frac{c}{2} < b$ and $a < \varphi + \frac{c}{2} < \varphi + c \leq b$, thus applying Step 2 and Step 3 we obtain

$$I(\varphi + c) = I\left(\varphi + \frac{c}{2}\right) + \frac{c}{2} = I(\varphi) + c.$$

■

Lemma 5 *Let Φ be a convex subset of $B_0(\Sigma)$ (or $B(\Sigma)$) and $I : \Phi \to \mathbb{R}$ a vertically invariant functional that satisfies*

$$I(\alpha\psi + (1-\alpha)\varphi) \geq I(\varphi) \tag{27}$$

for all $\varphi, \psi \in \Phi$ such that $I(\psi) = I(\varphi)$ and $\alpha \in (0, 1)$. Then Φ is concave provided one of the following conditions holds:

- *I is continuous and $\operatorname{int}(\Phi)$ is not empty,*

- Φ *is a tube.*

Proof. Assume I is continuous and $\text{int}(\Phi)$ is not empty. Let $\varphi_0 \in \text{int}(\Phi)$, there exist $\varepsilon > 0$ such that

$$\begin{aligned} N(\varphi_0, \varepsilon) &= \{\psi \in B_0(\Sigma) : \|\varphi_0 - \psi\| \leq \varepsilon\} \\ &= \{\psi \in B_0(\Sigma) : \varphi_0 - \varepsilon \leq \psi \leq \varphi_0 + \varepsilon\} \end{aligned}$$

is contained in $\text{int}(\Phi)$. Moreover - by continuity - there exists $\rho \in \left(0, \frac{1}{3}\right)$ such that $\|\varphi - \varphi_0\| \leq \rho\varepsilon$ implies $|I(\varphi) - I(\varphi_0)| \leq \frac{\varepsilon}{3}$. Then if $\varphi, \psi \in N(\varphi_0, \rho\varepsilon)$, we have

$$|I(\varphi) - I(\psi)| \leq |I(\varphi) - I(\varphi_0)| + |I(\varphi_0) - I(\psi)| \leq \frac{2}{3}\varepsilon$$

and $-\frac{2}{3}\varepsilon \leq I(\varphi) - I(\psi) \leq \frac{2}{3}\varepsilon$. Setting $t = I(\varphi) - I(\psi)$, we get $-\frac{2}{3}\varepsilon \leq t \leq \frac{2}{3}\varepsilon$. Notice that $-\frac{1}{3}\varepsilon \leq -\rho\varepsilon \leq \psi - \varphi_0 \leq \rho\varepsilon \leq \frac{1}{3}\varepsilon$ and $\varphi_0 - \frac{1}{3}\varepsilon \leq \psi \leq \varphi_0 + \frac{1}{3}\varepsilon$. Summing up,

$$\varphi_0 - \varepsilon \leq \psi + t \leq \varphi_0 + \varepsilon$$

and $\psi + t \in \text{int}(\Phi)$. Since $\psi \in \text{int}(\Phi)$ too, then $I(\psi + t) = I(\psi) + t = I(\varphi)$, so that

$$I(\alpha(\psi + t) + (1 - \alpha)\varphi) \geq I(\varphi). \tag{28}$$

Hence,

$$\begin{aligned} I(\varphi) &\leq I(\alpha(\psi + t) + (1 - \alpha)\varphi) = I(\alpha\psi + (1 - \alpha)\varphi + \alpha t) \\ &= I(\alpha\psi + (1 - \alpha)\varphi) + \alpha t \\ &= I(\alpha\psi + (1 - \alpha)\varphi) + \alpha(I(\varphi) - I(\psi)) \end{aligned}$$

and

$$I(\alpha\psi + (1 - \alpha)\varphi) \geq \alpha I(\psi) + (1 - \alpha)I(\varphi). \tag{29}$$

We conclude that I is concave in $N(\varphi_0, \rho\varepsilon)$. As the choice of $N(\varphi_0, \varepsilon)$ was arbitrary, we conclude that I is locally concave on $\text{int}(\Phi)$. A standard result from convex analysis yields concavity on $\text{int}(\Phi)$. Finally, the continuity of I implies its concavity on the whole Φ. This proves the first case. To prove the second, for all $\varphi, \psi \in \Phi$ and $\alpha \in (0, 1)$, set $t = I(\varphi) - I(\psi)$. Since Φ is a tube, $\psi + t \in \Phi$, and $I(\psi + t) = I(\psi) + t = I(\varphi)$. Repeat the argument leading from (28) to (29). ■

Lemma 6 *Let Φ be a nonempty subset of $B(\Sigma)$ and $I : \Phi \to \mathbb{R}$ be a vertically invariant functional. Then there exists a unique vertically invariant functional $\tilde{I} : \Phi + \mathbb{R} \to \mathbb{R}$ extending I to the tube $\Phi + \mathbb{R}$ generated by Φ. Moreover, if Φ is convex and I is concave, then ($\Phi + \mathbb{R}$ is convex and) $\tilde{I}$ is concave.*

Proof. If there exists a vertically invariant functional $\tilde{I}:\Phi+\mathbb{R}\to\mathbb{R}$ extending I on $\Phi+\mathbb{R}$, then for all $\varphi+d\in\Phi+\mathbb{R}$ with $\varphi\in\Phi$ and $d\in\mathbb{R}$, it satisfies

$$\tilde{I}(\varphi+d)=\tilde{I}(\varphi)+d=I(\varphi)+d. \tag{30}$$

In particular it is unique. Next we show that Eq. (30) defines a vertically invariant functional (that obviously extends I). If $\varphi,\psi\in\Phi$, $d,c\in\mathbb{R}$, and $\varphi+d=\psi+c$, then $\varphi=\psi+c-d$. In particular, $\psi\in\Phi$ and $c-d\in\mathbb{R}$ are such that $\psi+(c-d)\in\Phi$, and

$$\begin{aligned}I(\varphi)+d&=I(\psi+c-d)+d\\&=I(\psi)+c-d+d=I(\psi)+c.\end{aligned}$$

This proves that $\tilde{I}$ is well defined. If $\varphi+d\in\Phi+\mathbb{R}$ (with $\varphi\in\Phi$ and $d\in\mathbb{R}$) and $c\in\mathbb{R}$, then

$$\tilde{I}((\varphi+d)+c)=\tilde{I}(\varphi+d+c)=I(\varphi)+d+c=\tilde{I}(\varphi+d)+c,$$

that is, $\tilde{I}$ is vertically invariant.

Assume I is concave. Let $\varphi+d,\psi+t\in\Phi+\mathbb{R}$ with $\varphi,\psi\in\Phi$ and $d,t\in\mathbb{R}$ and $\alpha\in(0,1)$. If $\tilde{I}(\varphi+d)=\tilde{I}(\psi+t)=c$, then $I(\varphi)=c-d$ and $I(\psi)=c-t$. Therefore $I(\alpha\varphi+(1-\alpha)\psi)\geq\alpha I(\varphi)+(1-\alpha)I(\psi)=\alpha(c-d)+(1-\alpha)(c-t)$, that is

$$\begin{aligned}\tilde{I}(\alpha(\varphi+d)+(1-\alpha)(\psi+t))&=\tilde{I}((\alpha\varphi+(1-\alpha)\psi)+\alpha d+(1-\alpha)t)\\&=I(\alpha\varphi+(1-\alpha)\psi)+\alpha d+(1-\alpha)t\geq c.\end{aligned}$$

By Lemma 5, since $\Phi+\mathbb{R}$ is a tube, this means that $\tilde{I}$ is concave. ■

7.2 Extensions of niveloids

In this section we obtain some novel results on the extension of niveloids (the first results on this subject appear in [**?**]).

Lemma 7 *Let Φ be a nonempty subset of $B(\Sigma)$ and $I:\Phi\to\mathbb{R}$. The following statements are equivalent:*

(i) I is vertically invariant and its unique vertically invariant extension $\tilde{I}$ to $\Phi+\mathbb{R}$ is monotonic.

(ii) I is a niveloid.

In particular, if Φ is a tube, then $I:\Phi\to\mathbb{R}$ is a niveloid iff it is vertically invariant and monotonic (see [**?**, Cor 1.3]).

Proof. Let I be vertically invariant and $\tilde{I}$ be monotonic. For all $\varphi,\psi\in\Phi$, $\varphi\leq\psi+\sup(\varphi-\psi)$, but $\varphi,\psi+\sup(\varphi-\psi)\in\Phi+\mathbb{R}$, then $\tilde{I}(\varphi)\leq\tilde{I}(\psi+\sup(\varphi-\psi))$ that is

$$I(\varphi)\leq I(\psi)+\sup(\varphi-\psi).$$

Conversely, if I is a niveloid, for all $\varphi \in \Phi$ and $c \in \mathbb{R}$ such that $\varphi + c \in \Phi$

$$c = \inf ((\varphi + c) - \varphi) \leq I(\varphi + c) - I(\varphi) \leq \sup ((\varphi + c) - \varphi) = c$$

that is $I(\varphi + c) = I(\varphi) + c$, and I is vertically invariant. Moreover, if $\varphi, \psi \in \Phi$ and $d, t \in \mathbb{R}$ are such that $\psi + t \geq \varphi + d$, then $I(\psi) - I(\varphi) \geq \inf(\psi - \varphi)$ implies

$$\begin{aligned} \tilde{I}(\psi + t) - \tilde{I}(\varphi + d) &= I(\psi) - I(\varphi) + t - d \\ &\geq \inf(\psi - \varphi) + t - d \\ &= \inf((\psi + t) - (\varphi + d)) \\ &\geq 0, \end{aligned}$$

that is $\tilde{I}$ is monotonic. ■

Lemma 8 *A vertically invariant and monotonic functional* $I : B_0(\Sigma, K) \to \mathbb{R}$ *is a niveloid.*

Proof. In view of Lemma 7, we just have to show that $\tilde{I}$ is monotonic. Let $\varphi, \psi \in B_0(\Sigma, K)$ and $d, t, c \in \mathbb{R}$ be such that $\psi + t \geq \varphi + d$. We want to show that $I(\psi) + t \geq I(\varphi) + d$, i.e., that $\psi + c \geq \varphi$ implies $I(\psi) + c \geq I(\varphi)$.

Assume $\sup K = b < \infty$ is not attained. If $c < b - \sup \psi$, then $\varphi \leq \psi + c \leq \sup \psi + c < b$, then $\psi + c \in B_0(\Sigma, K)$ and $I(\varphi) \leq I(\psi + c) = I(\psi) + c$. Else $c \geq b - \sup \psi \geq 0$ and there exists $\varepsilon > 0$ such that $\varphi < b - \varepsilon < b$. A fortiori $c > (b - \varepsilon) - \sup \psi$. There are two subcases:

- $c > (b - \varepsilon) - \inf \psi$, then $I(\psi) + c \geq I(\inf \psi) + c = \tilde{I}(0) + \inf \psi + c > \tilde{I}(0) + \inf \psi + (b - \varepsilon) - \inf \psi \geq \tilde{I}(0) + b - \varepsilon = I(b - \varepsilon) \geq I(\varphi)$.
- $c \leq (b - \varepsilon) - \inf \psi$ (that is $\inf \psi \leq (b - \varepsilon) - c < \sup \psi$), then $\psi + c \geq \varphi$ implies $(\psi + c) \wedge (b - \varepsilon) \geq \varphi$, but $(\psi + c) \wedge (b - \varepsilon) \in B_0(\Sigma, K)$ and $(\psi + c) \wedge (b - \varepsilon) = \min\{\psi + c, b - \varepsilon\} - c + c = \min\{\psi, b - \varepsilon - c\} + c = (\psi \wedge (b - \varepsilon - c)) + c$. Notice that also $\psi \wedge (b - \varepsilon - c) \in B_0(\Sigma, K)$ since $(b - \varepsilon - c) \in [\inf \psi, \sup \psi) \subseteq K$. Therefore

$$\begin{aligned} I(\psi) + c &\geq I(\psi \wedge (b - \varepsilon - c)) + c \\ &= I((\psi \wedge (b - \varepsilon - c)) + c) \\ &= I((\psi + c) \wedge (b - \varepsilon)) \\ &\geq I(\varphi), \end{aligned}$$

  as desired.

Assume that $\sup K = b < \infty$ is attained. If $c \leq b - \sup \psi$, then $\varphi \leq \psi + c \leq \sup \psi + c \leq b$, then $\psi + c \in B_0(\Sigma, K)$ and $I(\varphi) \leq I(\psi + c) = I(\psi) + c$. Else $c > b - \sup \psi \geq 0$ while $\varphi \leq b$. There are two subcases:

- $c \geq b - \inf \psi$, then $I(\psi) + c \geq I(\inf \psi) + c = \tilde{I}(0) + \inf \psi + c \geq \tilde{I}(0) + \inf \psi + b - \inf \psi = I(b) \geq I(\varphi)$.
- $c < b - \inf \psi$ (that is $\inf \psi < b - c < \sup \psi$), then $\psi + c \geq \varphi$ implies $(\psi + c) \wedge b \geq \varphi$, but $(\psi + c) \wedge b \in B_0(\Sigma, K)$ and $(\psi + c) \wedge b = \min\{\psi + c, b\} - c + c = \min\{\psi, b - c\} + c = (\psi \wedge (b - c)) + c$. Notice that also $\psi \wedge (b - c) \in B_0(\Sigma, K)$ since $(b - c) \in (\inf \psi, \sup \psi) \subseteq K$. Therefore

$$\begin{aligned} I(\psi) + c &\geq I(\psi \wedge (b - c)) + c \\ &= I((\psi \wedge (b - c)) + c) \\ &= I((\psi + c) \wedge b) \\ &\geq I(\varphi), \end{aligned}$$

  as desired.

Finally, if $\sup K = \infty$, and $\varphi \leq \psi + c$, then $\psi + c \in B_0(\Sigma, K)$ and $I(\varphi) \leq I(\psi + c) = I(\psi) + c$. ■

Lemma 9 *Let $I : \Phi \to \mathbb{R}$ be a niveloid on a nonempty subset Φ of $B(\Sigma)$, and set*

$$\mathcal{L} = \left\{\varphi \in \Phi + \mathbb{R} : \tilde{I}(\varphi) \geq 0\right\} + B\left(\Sigma, \mathbb{R}^+\right).$$

The functional defined on $B(\Sigma)$ by

$$\hat{I}(\varphi) = \sup\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} \quad \forall \varphi \in B(\Sigma)$$

is the minimum niveloid on $B(\Sigma)$ that extends I. Moreover, if Φ is convex and I is concave, then $\hat{I}$ is concave.

Before entering the proof's details, notice that if I is a niveloid on a tube Φ, then for all $\varphi \in \Phi$, $I(\varphi) = \sup\{c \in \mathbb{R} : c \leq I(\varphi)\} = \sup\{c \in \mathbb{R} : \varphi - c \in \{I \geq 0\}\}$, where $\{I \geq 0\} = \{\varphi \in \Phi : I(\varphi) \geq 0\}$ (see also [**?**, p.160]).

Proof. If $\varphi \in \Phi + \mathbb{R}$, and $\varphi \in \mathcal{L}$, then $\varphi \geq \psi$ for some $\psi \in \left\{\tilde{I} \geq 0\right\}$, whence $\tilde{I}(\varphi) \geq \tilde{I}(\psi) \geq 0$, that is $\varphi \in \left\{\tilde{I} \geq 0\right\}$. This proves that $\varphi \in \Phi + \mathbb{R}$ belongs to $\mathcal{L}$ iff it belongs to $\left\{\tilde{I} \geq 0\right\}$. As a consequence, for all $\varphi \in \Phi + \mathbb{R}$, we have

$$\begin{aligned} \tilde{I}(\varphi) &= \sup\left\{c \in \mathbb{R} : \varphi - c \in \left\{\tilde{I} \geq 0\right\}\right\} \\ &= \sup\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} \\ &= \hat{I}(\varphi). \end{aligned}$$

Then $\hat{I} : B(\Sigma) \to [-\infty, \infty]$ extends $\tilde{I}$, *a fortiori* I.

Notice that:

- If $\varphi \in \mathcal{L}$ and $\psi \geq \varphi$, then $\psi \in \mathcal{L} + B(\Sigma, \mathbb{R}^+) = \mathcal{L}$.

- If $\varphi \notin \mathcal{L}$ and $\psi \leq \varphi$, then $\psi \notin \mathcal{L}$.

- If $\psi_0 \in \Phi$, then $\psi_0 + d \in \mathcal{L}$ iff $\tilde{I}(\psi_0 + d) \geq 0$ iff $d \geq -I(\psi_0)$. In particular, $\psi_0 - I(\psi_0) \in \mathcal{L}$ and $\psi_0 - I(\psi_0) - 1 \notin \mathcal{L}$.

Let $\psi_0 \in \Phi$. For all $\varphi \in B(\Sigma)$, $\varphi - (\inf \varphi - \sup \psi_0 + I(\psi_0)) \geq \psi_0 - I(\psi_0) \in \mathcal{L}$, hence $\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} \neq \emptyset$; therefore $\hat{I}(\varphi) > -\infty$. For all $\varphi \in B(\Sigma)$ and all $c \geq \sup \varphi - \inf \psi_0 + I(\psi_0) + 1$, $\varphi - c \leq \varphi - (\sup \varphi - \inf \psi_0 + I(\psi_0) + 1) \leq \psi_0 - I(\psi_0) - 1 \notin \mathcal{L}$ implies $\varphi - c \notin \mathcal{L}$, and $\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\}$ is bounded above; therefore $\hat{I}(\varphi) < \infty$. We conclude that $\hat{I} : B(\Sigma) \to \mathbb{R}$.

If $\psi \geq \varphi$ and $\varphi - c \in \mathcal{L}$, then $\psi - c \geq \varphi - c$ implies $\psi - c \in \mathcal{L}$. It follows that $\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} \subseteq \{c \in \mathbb{R} : \psi - c \in \mathcal{L}\}$ and $\hat{I}(\psi) \geq \hat{I}(\varphi)$, i.e., $\hat{I}$ is monotonic.

Let $d \in \mathbb{R}$ and $\varphi \in B(\Sigma)$, $\varphi - c \in \mathcal{L}$ iff $(\varphi + d) - (c + d) \in \mathcal{L}$, that is

$$\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} + d = \{t \in \mathbb{R} : (\varphi + d) - t \in \mathcal{L}\}$$

and

$$\begin{aligned}
\hat{I}(\varphi + d) &= \sup \{t \in \mathbb{R} : (\varphi + d) - t \in \mathcal{L}\} \\
&= \sup (\{c \in \mathbb{R} : \varphi - c \in \mathcal{L}\} + d) \\
&= \hat{I}(\varphi) + d.
\end{aligned}$$

That is $\hat{I}$ is a niveloid.

Notice that $\left\{\hat{I} \geq 0\right\} = \overline{\mathcal{L}}$. In fact, if $\hat{I}(\varphi) \geq 0$, then for all $\varepsilon > 0$ we have $\hat{I}(\varphi + \varepsilon) > 0$, i.e.

$$\sup \{c \in \mathbb{R} : \varphi + \varepsilon - c \in \mathcal{L}\} > 0,$$

therefore there exists $c > 0$ such that $\varphi + \varepsilon - c \in \mathcal{L}$. This implies $\varphi + \varepsilon \in \mathcal{L}$ (since $\varphi + \varepsilon \geq \varphi + \varepsilon - c \in \mathcal{L}$). Since this is true for all ε, it follows $\varphi = \lim_n \left(\varphi + \frac{1}{n}\right) \in \overline{\mathcal{L}}$, and we conclude $\left\{\hat{I} \geq 0\right\} \subseteq \overline{\mathcal{L}}$. Conversely, for all $\varphi \in \mathcal{L}$, $\varphi - 0 \in \mathcal{L}$ guarantees $\hat{I}(\varphi) \geq 0$; the continuity of $\hat{I}$ implies $\overline{\mathcal{L}} \subseteq \left\{\hat{I} \geq 0\right\}$.

Let $\hat{J}$ be a niveloid on $B(\Sigma)$ that extends I, then $\hat{J}$ coincides with $\tilde{I}$ on $\Phi + \mathbb{R}$. For all $\psi \in \mathcal{L}$ there exists $\varphi \in \left\{\tilde{I} \geq 0\right\}$ such that $\psi \geq \varphi$, therefore $\hat{J}(\psi) \geq \hat{J}(\varphi) = \tilde{I}(\varphi) \geq 0$. Then $\left\{\hat{I} \geq 0\right\} = \overline{\mathcal{L}} \subseteq \left\{\hat{J} \geq 0\right\}$, and this implies that for all $\varphi \in B(\Sigma)$

$$\begin{aligned}
\hat{I}(\varphi) &= \sup \left\{c \in \mathbb{R} : \varphi - c \in \left\{\hat{I} \geq 0\right\}\right\} \\
&\leq \sup \left\{c \in \mathbb{R} : \varphi - c \in \left\{\hat{J} \geq 0\right\}\right\} = \hat{J}(\varphi).
\end{aligned}$$

This shows that $\hat{I}$ is the minimum niveloid on $B(\Sigma)$ that extends I.

Assume Φ is convex and I is concave, then $\tilde{I}$ is concave and $\left\{\tilde{I} \geq 0\right\}$ is convex. So $\mathcal{L} = \left\{\tilde{I} \geq 0\right\} + B(\Sigma, \mathbb{R}^+)$ and $\left\{\hat{I} \geq 0\right\} = \overline{\mathcal{L}}$ are convex. This implies that $\hat{I}$ is concave. In fact, for all

$\varphi, \psi \in B(\Sigma)$ such that $\hat{I}(\varphi) = \hat{I}(\psi) = c$, and $\alpha \in (0,1)$, since $\varphi - c, \psi - c \in \left\{\hat{I} \geq 0\right\}$, then

$$\begin{aligned}\hat{I}(\alpha\varphi + (1-\alpha)\psi) - \hat{I}(\varphi) &= \hat{I}(\alpha\varphi + (1-\alpha)\psi - c) \\ &= \hat{I}(\alpha(\varphi - c) + (1-\alpha)(\psi - c)) \geq 0,\end{aligned}$$

and Lemma 5 guarantees concavity. ■

Inspection of the proof shows that for a non-empty subset Φ of $B_0(\Sigma)$ setting

$$\mathcal{L}_0 = \left\{\varphi \in \Phi : \tilde{I}(\varphi) \geq 0\right\} + B_0\left(\Sigma, \mathbb{R}^+\right)$$

we could obtain the minimum niveloid extending I to $B_0(\Sigma)$.

7.3 Fenchel conjugates of concave niveloids

Remark 3 If $I : B(\Sigma) \to \mathbb{R}$ is a concave niveloid, direct application of the Fenchel Moreau Theorem (see, e.g., [**?**, p. 42]) guarantees

$$I(\varphi) = \min_{\mu \in ba(\Sigma)} \left(\langle \varphi, \mu \rangle - I^*(\mu)\right)$$

where $I^*(\mu) = \inf_{\psi \in B(\Sigma)} \left(\langle \psi, \mu \rangle - I(\psi)\right)$ is the *Fenchel conjugate* of I. If μ is not positive, there exists $\varphi \geq 0$ such that $\langle \varphi, \mu \rangle < 0$, then $\langle \alpha\varphi, \mu \rangle - I(\alpha\varphi) \leq \alpha \langle \varphi, \mu \rangle - I(0)$ for all $\alpha \geq 0$, whence $I^*(\mu) = -\infty$. If $\mu(S) \neq 1$, choose $\psi \in B(\Sigma)$, then $\langle \psi + c, \mu \rangle - I(\psi + c) = \langle \psi, \mu \rangle - I(\psi) + c(\mu(S) - 1)$ for all $c \in \mathbb{R}$, and so $I^*(\mu) = -\infty$. That is,

$$I(\varphi) = \min_{\mu \in \Delta(\Sigma)} \left(\langle \varphi, \mu \rangle - I^*(\mu)\right).$$

In this section Φ is a (non-empty) convex subset of $B(\Sigma)$ and $I : \Phi \to \mathbb{R}$ is a concave niveloid. We set

$$\partial_\pi I(\varphi) = \{p \in \Delta(\Sigma) : I(\psi) - I(\varphi) \leq \langle \psi - \varphi, p \rangle \text{ for each } \psi \in \Phi\}.$$

Notice that

$$\partial_\pi I(\varphi) = \left\{p \in \Delta(\Sigma) : \tilde{I}(\psi) - \tilde{I}(\varphi) \leq \langle \psi - \varphi, p \rangle \text{ for each } \psi \in \Phi + \mathbb{R}\right\}.$$

Lemma 10 *Let $I : \Phi \to \mathbb{R}$ be a concave niveloid. Then, $\partial_\pi I(\varphi) \neq \varnothing$ for all $\varphi \in \Phi$.*

Proof. By Lemma 9, there exists a concave niveloid $\hat{I}$ on $B(\Sigma)$ such that $\hat{I}_{|\Phi} = I$. Let $\varphi \in \Phi$. Being a niveloid, $\hat{I}$ is Lipschitz continuous, then, its standard superdifferential at φ

$$\partial\hat{I}(\varphi) = \left\{\mu \in ba(\Sigma) : \hat{I}(\psi) - \hat{I}(\varphi) \leq \langle \psi - \varphi, \mu \rangle \text{ for each } \psi \in B(\Sigma)\right\}$$

is nonempty (see, e.g., [**?**, p. 6-7]).

For all $c \in \mathbb{R}$ and $\mu \in \partial\hat{I}(\varphi)$ we have

$$\hat{I}(\varphi) + c = \hat{I}(\varphi + c) \leq \hat{I}(\varphi) + \langle \varphi + c - \varphi, \mu \rangle = \hat{I}(\varphi) + c\mu(S),$$

and so $c \leq c\mu(S)$. This implies $\mu(S) = 1$.

For all $\psi \geq 0$ and $\mu \in \partial \hat{I}(\varphi)$ we have

$$\langle \psi, \mu \rangle = \langle \varphi + \psi, \mu \rangle - \langle \varphi, \mu \rangle \geq \hat{I}(\varphi + \psi) - \hat{I}(\varphi) \geq 0,$$

this implies $\mu \in ba^{+}(\Sigma)$.

Therefore, $\partial \hat{I}(\varphi) \subseteq \partial_\pi I(\varphi)$ and we conclude that $\partial_\pi I(\varphi) \neq \varnothing$. ■

Lemma 11 *Let Φ be a convex subset of $B(\Sigma)$ such that $\Phi_c \neq \varnothing$, and $I : \Phi \to \mathbb{R}$ be a concave and normalized niveloid. Then:*

(i) For each $\varphi \in \Phi$,

$$I(\varphi) = \min_{p \in \Delta(\Sigma)} (\langle \varphi, p \rangle - I^{\star}(p)) = \min_{p \in \bigcup_{\psi \in \Phi} \partial_\pi I(\psi)} (\langle \varphi, p \rangle - I^{\star}(p)) \tag{31}$$

where $I^{\star} : \Delta(\Sigma) \to [-\infty, 0]$ is given by

$$I^{\star}(p) = \inf_{\psi \in \Phi} (\langle \psi, p \rangle - I(\psi)) \quad \forall p \in \Delta(\Sigma).$$

(ii) $I^{\star}$ is the maximal functional $R : \Delta(\Sigma) \to [-\infty, 0]$ such that

$$I(\varphi) = \inf_{p \in \Delta(\Sigma)} (\langle \varphi, p \rangle - R(p)) \quad \forall \varphi \in \Phi. \tag{32}$$

(iii) $I^{\star}$ coincides with the Fenchel conjugate $\hat{I}^{*}$ of $\hat{I}$ on $\Delta(\Sigma)$ and

$$\hat{I}(\varphi) = \min_{p \in \Delta(\Sigma)} (\langle \varphi, p \rangle - I^{\star}(p)) \quad \forall \varphi \in B(\Sigma). \tag{33}$$

(iv) If (32) holds, and $\Psi \subseteq \Phi$ is such that $\sup_{s \in S} \psi(s) - \inf_{s \in S} \psi(s) < c$ for all $\psi \in \Psi$, then

$$I(\psi) = \inf_{\{p \in \Delta(\Sigma) : R(p) \geq -c\}} (\langle \varphi, p \rangle - R(p)) \quad \forall \psi \in \Psi. \tag{34}$$

Proof. Notice that $I^{\star}(p) \leq 0$ for all $p \in \Delta(\Sigma)$. For, if we take a constant $k \in \Phi_c$ we have $\langle k, p \rangle = I(k) = k$.

By definition of $I^{\star}$, for all $\varphi \in \Phi$ and $p \in \Delta(\Sigma)$

$$I(\varphi) \leq \langle \varphi, p \rangle - I^{\star}(p); \tag{35}$$

moreover,

$$\begin{aligned} p \in \partial_\pi I(\varphi) &\Leftrightarrow I(\varphi) \geq I(\psi) - \langle \psi, p \rangle + \langle \varphi, p \rangle \quad \forall \psi \in \Phi \\ &\Leftrightarrow I(\varphi) \geq \sup_{\psi \in \Phi} (I(\psi) - \langle \psi, p \rangle) + \langle \varphi, p \rangle \\ &\Leftrightarrow I(\varphi) \geq \langle \varphi, p \rangle - \inf_{\psi \in \Phi} (\langle \psi, p \rangle - I(\psi)) \\ &\Leftrightarrow I(\varphi) \geq \langle \varphi, p \rangle - I^{\star}(p) \\ &\Leftrightarrow I(\varphi) = \langle \varphi, p \rangle - I^{\star}(p). \end{aligned}$$

Therefore, for all $\varphi \in \Phi$

$$\begin{aligned} I(\varphi) &= \min_{p \in \partial_\pi I(\varphi)} \left(\langle \varphi, p \rangle - I^\star(p)\right) \geq \inf_{p \in \bigcup_{\psi \in \Phi} \partial_\pi I(\psi)} \left(\langle \varphi, p \rangle - I^\star(p)\right) \\ &\geq \inf_{p \in \Delta(\Sigma)} \left(\langle \varphi, p \rangle - I^\star(p)\right) \geq I(\varphi), \end{aligned}$$

which implies (31). This proves (i). For later use, notice that if P is a subset of $\Delta(\Sigma)$ such that $\partial_\pi I(\varphi) \cap P \neq \emptyset$ for all $\varphi \in \Phi$, then the above argument yields

$$I(\varphi) = \min_{p \in P} \left(\langle \varphi, p \rangle - I^\star(p)\right). \tag{36}$$

Let $R : \Delta(\Sigma) \to [-\infty, 0]$ be such that $I(\varphi) = \inf_{p \in \Delta(\Sigma)} \left(\langle \varphi, p \rangle - R(p)\right)$ for all $\varphi \in \Phi$. Then,

$$R(p) \leq \langle \varphi, p \rangle - I(\varphi) \quad \forall p \in \Delta(\Sigma), \varphi \in \Phi,$$

and hence

$$R(p) \leq \inf_{\varphi \in \Phi} \left(\langle \varphi, p \rangle - I(\varphi)\right) = I^\star(p) \quad \forall p \in \Delta(\Sigma).$$

This proves (ii).

For all $\varphi \in B(\Sigma)$, set $\hat{J}(\varphi) = \inf_{p \in \Delta(\Sigma)} \left(\langle \varphi, p \rangle - I^\star(p)\right)$, $\hat{J}$ is a normalized and concave niveloid on $B(\Sigma)$ that extends I. By (ii) applied to $\hat{J}$, we obtain

$$\begin{aligned} \hat{J}^*(p) = \hat{J}^\star(p) \geq I^\star(p) &= \inf_{\varphi \in \Phi} \left(\langle \varphi, p \rangle - I(\varphi)\right) \\ &\geq \inf_{\varphi \in B(\Sigma)} \left(\langle \varphi, p \rangle - \hat{I}(\varphi)\right) \quad (\text{this is } \hat{I}^*(p)) \\ (\text{since } \hat{I} \leq \hat{J}) \quad &\geq \inf_{\varphi \in B(\Sigma)} \left(\langle \varphi, p \rangle - \hat{J}(\varphi)\right) = \hat{J}^*(p) \end{aligned}$$

that is $\hat{J}^*(p) = I^\star(p) = \hat{I}^*(p)$ for all $p \in \Delta(\Sigma)$. Apply (i) (or Remark 3) to $\hat{I}$ to obtain

$$\hat{I}(\varphi) = \min_{p \in \Delta(\Sigma)} \left(\langle \varphi, p \rangle - \hat{I}^*(p)\right) = \min_{p \in \Delta(\Sigma)} \left(\langle \varphi, p \rangle - I^\star(p)\right) \quad \forall \varphi \in B(\Sigma).$$

This completes the proof of (iii).

Finally, as to (iv), the monotonicity of $\hat{I}$ implies that $\inf_{s \in S} \psi(s) = \hat{I}\left(\inf_{s \in S} \psi(s)\right) \leq I(\psi), \langle \psi, p \rangle \leq \sup_{s \in S} \psi(s) = \hat{I}\left(\sup_{s \in S} \psi(s)\right)$ for all $p \in \Delta(\Sigma)$ and all $\psi \in \Psi$. For each $\psi \in \Psi$, there exists $\varepsilon > 0$ such that $\sup_{s \in S} \psi(s) - \inf_{s \in S} \psi(s) + \varepsilon < c$. For all $p \in \Delta(\Sigma)$ such that $R(p) < -c$, we have

$$\begin{aligned} &R(p) < -\sup_{s \in S} \psi(s) + \inf_{s \in S} \psi(s) - \varepsilon, \text{ and} \\ &\sup_{s \in S} \psi(s) + \varepsilon < \inf_{s \in S} \psi(s) - R(p), \text{ i.e.} \\ &I(\psi) + \varepsilon \leq \sup_{s \in S} \psi(s) + \varepsilon < \inf_{s \in S} \psi(s) - R(p) \leq \langle \varphi, p \rangle - R(p). \end{aligned}$$

On the other hand,

$$\begin{aligned} I(\psi) &= \inf_{p \in \Delta(\Sigma)} \left(\langle \psi, p \rangle - R(p)\right) \\ &= \min\left(\inf_{p \in \{R < -c\}} \left(\langle \psi, p \rangle - R(p)\right), \inf_{p \in \{R \geq -c\}} \left(\langle \psi, p \rangle - R(p)\right)\right) \end{aligned}$$

which concludes the proof, since $\inf_{p\in\{R<-c\}}\left(\langle\varphi,p\rangle-R(p)\right)\geq I(\psi)+\varepsilon$. ■

Remark 4 Inspection of the proof shows that: (i) $\partial_\pi I(\varphi)=\arg\min_{p\in\Delta(\Sigma)}\left(\langle\varphi,p\rangle-I^\star(p)\right)$. (ii) If $k\in\Phi_c$, then $\partial_\pi I(k)=\{I^\star=0\}=\arg\max_{p\in\Delta(\Sigma)}I^\star(p)$. (iii) $I^\star$ is concave and weakly* upper semicontinuous.

Corollary 1 *Let Φ be a convex subset of $B_0(\Sigma)$ (resp. $B(\Sigma)$) such that $\Phi_c\neq\emptyset$ and $\Phi+\mathbb{R}=B_0(\Sigma)$ (resp. $\Phi+\mathbb{R}=B(\Sigma)$),*[11] *and $I:\Phi\to\mathbb{R}$ be a concave and normalized niveloid. Then, $I^\star$ is the Fenchel conjugate of the unique niveloid $\tilde{I}$ extending I to $B_0(\Sigma)$ (resp. $B(\Sigma)$). In this case $I^\star$ is the unique concave and weakly* upper semicontinuous function $R:\Delta(\Sigma)\to[-\infty,0]$ such that*

$$I(\varphi)=\inf_{p\in\Delta(\Sigma)}\left(\langle\varphi,p\rangle-R(p)\right)\quad\forall\varphi\in\Phi.$$

Proof. The equality

$$\begin{aligned}I^\star(p)&=\inf_{\psi\in\Phi}\left(\langle\psi,p\rangle-I(\psi)\right)=\inf_{\substack{\psi\in\Phi\\ c\in\mathbb{R}}}\left(\langle\psi,p\rangle+c-I(\psi)-c\right)\\&=\inf_{\substack{\psi\in\Phi\\ c\in\mathbb{R}}}\left(\langle\psi+c,p\rangle-\tilde{I}(\psi+c)\right)=\inf_{\psi\in B_0(\Sigma)}\left(\langle\psi,p\rangle-\tilde{I}(\psi)\right)\end{aligned}$$

yields the first part of the statement. Let $R:\Delta(\Sigma)\to[-\infty,0]$ be a concave and weakly* upper semicontinuous functional such that $I(\varphi)=\inf_{p\in\Delta(\Sigma)}\left(\langle\varphi,p\rangle-R(p)\right)$ for all $\varphi\in\Phi$. Then, $\tilde{I}(\varphi)=\inf_{p\in\Delta(\Sigma)}\left(\langle\varphi,p\rangle-R(p)\right)$ for all $\varphi\in B_0(\Sigma)$. By the Fenchel-Moreau Theorem

$$\begin{aligned}R(p)=R^{**}(p)&=\inf_{\varphi\in B_0(\Sigma)}\left(\langle\varphi,p\rangle-R^*(\varphi)\right)\\&=\inf_{\varphi\in B_0(\Sigma)}\left(\langle\varphi,p\rangle-\left(\inf_{p\in\Delta(\Sigma)}\left(\langle\varphi,p\rangle-R(p)\right)\right)\right)\\&=\inf_{\varphi\in B_0(\Sigma)}\left(\langle\varphi,p\rangle-\tilde{I}(\varphi)\right)=\tilde{I}^*(p)=I^\star(p),\end{aligned}$$

as desired. ■

[11] E.g. $\Phi=B_0(\Sigma,K)$ with K an unbounded interval.

References

[1] Anscombe, F.J. & Aumann, R., (01963), A Definition of Subjective Probability, *Annals of mathematical Statistics*, 34, 199–205.

[2] Barrick, M. R., & Mount, M. K. (1991). The big five personality dimensions and job performance: A meta-analysis. *Personnel Psychology*, 44, 1–26

[3] Barrick, M.R., Mount, M.K., Judge, T.A. (2001). Personality and performance at the beginning of the new millennium: What do we know and where do we go next? *nternational Journal of Selection and Assessment* 9, 9-30

[4] Castagnoli, E., (1990), Qualche riflessione sull'Utilità Attesa, *Ratio mathematica*, 51–59.

[5] E. Castagnoli, LiCalzi, M., (2006), Benchmarking real-valued acts, *Games and Economic Behavior*, 236-253

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[7] Costa, P. T., Jr., & McCrae, R. R. (1992). Four ways five factors are basic. *Personality and Individual Differences*, 13, 653–665.

[8] Costa, Paul T., Mccrae, Robert R. & Dye, David A., (1991), Facet scales for agreeableness and conscientiousness: A revision of the NEO personality inventory, *Personality and Individual Differences*, 12, 9, 887–898.

[9] DeYoung, C. G. & Gray, J. R. (2009). Personality Neuroscience: Explaining Individual Differences in Affect, Behavior, and Cognition. In P. J. Corr & G. Matthews (Eds.), *Cambridge handbook of personality*, 323–346, New York: Cambridge University Press

[10] DeYoung, C. G., Quilty, L. C., & Peterson, J. B. (2007). Between facets and domains: 10 Aspects of the Big Five, *Journal of Personality and Social Psychology*, 93, 880–896

[11] Dolecki, S. and Greco, G., Niveloids, *Topological Methods in Nonlinear Analysis*, 5, 1–22, 1995.

[12] Depue, R. A. & Collins, P. F. (1999). Neurobiology of the structure of personality: Dopamine, facilitation of incentive motivation, and extraversion. *Behavioral and Brain Sciences*, 22, 491-569.

[13] Drèze, J. H., (1987). Decision Theory with Moral hazard and state-dependent preferences, Chapter 2 in *Essays on Economic Decisions under Uncertainty*, Cambridge University Press, Cambridge, UK

[14] Gosling, S. D. & John, O. P. (1999). Personality dimensions in nonhuman animals: A crossspecies review. Current Directions in Psychological Science, 8, 69–75.

[15] I. N Herstein and J. Milnor, An axiomatic approach to measurable utility, *Econometrica*, 21, 291–297, 1953.

[16] Judge, T. A., Higgins, C., Thoresen, C. J., & Barrick, M. R. (1999). The Big Five personality traits, general mental ability, and career success across the life span. *Personnel Psychology*, 52, 621-652

[17] Kahneman, D. & Tversky, A. (1979) Prospect Theory: An Analysis of Decision under Risk, *Econometrica*, 47, 263-291.

[18] Leatta M. Hough, (1992), The 'Big Five' Personality Variables–Construct Confusion: Description Versus Prediction *Human Performance*, 5, 139–155.

[19] King, J.E., & Figueredo, A.J. (1997). The five-factor model plus dominance in chimpanzee personality. *Journal of Research in Personality*, 31, 271–271.

[20] Maccheroni, F., Marinacci, M, & Rustichini, A. (2006), Niveloids and their extensions, mimeo

[21] P. K. Monteiro, Some results on the existence of utility functions on path connected spaces, *Journal of Mathematical Economics*, 16, 147–156, 1987.

[22] Mount M.K., Barrick, M.R., (1995), The Big Five personality dimansions: Implications for reaserach and practice in human resources management, *Reserach in Personnel and Human Respurces Management*, 13, 153–200.

[23] Roberts, B. W., Chernyshenko, O. S., Stark, S., & Goldberg, L. R. (2005). The structure of Conscientiousness: An empirical investigation based on seven major personality questionnaires. *Personnel Psychology*, 58, 103-139.

[24] Roberts, B. W., Kuncel, N., Shiner, R., N., Caspi, A., & Goldberg, L. R. (2007). The power of personality: The comparative validity of personality traits, socio-economic status, and cognitive ability for predicting important life outcomes. *Perspectives in Psychological Science*, 2, 313–345

[25] Rustichini, A., DeYoung, C. G., Anderson, J. and Burks, S. (2011), Toward the Integration of Personality Theory and Decision Theory in the Explanation of Economic and Health Behavior, University of Minnesota, Mimeo