Perturbation Methods for Markov-Switching Models*

Andrew Foerster[†] Juan Rubio-Ramirez[‡] Dan Waggoner[§] Tao Zha[¶]

December 30, 2011

Abstract

Markov Switching models are a way to consider discrete changes in the economic environment, such as policy changes, and allow agents in the economy to form expectations over these changes. This paper develops a methodology for constructing approximations to the solution of Markov Switching dynamic stochastic general equilibrium (MS-DSGE) models. The method allows for changes in parameters that both do and do not affect the economy's steady state, and enables linear or higher-order approximations. In addition, the paper proves that first-order approximations to a wide class of MS-DSGE models are not certainty equivalent. The numerical procedure handles potentially large systems and considers existence and uniqueness using the concept of mean square stability. Two examples, one Real Business Cycle and one New Keynesian, illustrate the procedure and issues of certainty equivalence and mean square stability.

*The views expressed herein are solely those of the authors and do not necessarily reflect the views of the Federal Reserve Banks of Atlanta or Kansas City or the Federal Reserve System.

[†]Federal Reserve Bank of Kansas City

[‡]Duke University, Federal Reserve Bank of Atlanta, and CEPR

[§]Federal Reserve Bank of Atlanta

[¶]Emory University and Federal Reserve Bank of Atlanta

1 Introduction

Following the introduction of vector autoregressions (VARs) to macroeconomics by Sims (1980) it was quickly realized that it is difficult to find macroeconomic applications for which model parameters remain stable over long periods of time. This problem was not unique to reduced form representations of the data, but was also an issue when more structural approaches were considered. One way to solve the problem, pursued by Clarida *et al.* (2000) and followed up by Lubik & Schorfheide (2004), breaks the sample into sub-periods and estimates the structural models in which one or more of the model's parameters differ across sub-samples. While this approach addresses the parameter instability problem, it fails to consider that forward looking agents living in a world in which parameters are known to change occasionally would be expected to take possible parameter change into account when forming their expectations and, therefore, will affect their optimal decisions.

An alternative approach to parameter instability, suggested by the work of Hamilton (1989) and pursued in Sims & Zha (2006), is to estimate a backward-looking vector autoregression (VAR) with regime dependent parameters. This approach has its limitations since it does not allow for the presence of forward-looking components that are present in a dynamic stochastic general equilibrium (DSGE) model.

A number of authors have recently studied forward looking Markov-switching linear rational expectations (MSLRE) models. Work in this area includes papers by Leeper & Zha (2003), Svensson & Williams (2007), Blake & Zampolli (2006), Davig & Leeper (2007), and Farmer *et al.* (2009). MSLRE models are more complicated than linear rational expectations models since the agents of the model must be allowed to take account of the possibility of future regime changes when forming expectations. The MSLRE literature has made some headway in addressing questions like setting necessary and sufficient conditions to determine if the parameters of a Markov-switching rational expectations model lead to a determinate equilibrium (See Farmer *et al.* (2009)).

There are two main shortcomings with the MSLRE approach. First, most of the analyzed models do not begin from first principles. In other words, researchers consider linear rational expectations (LRE) models where Markov-switching (MS) has been added after the model has been linearized. Second, higher order solutions are not considered. Given that MS parameters add a lot of uncertainty to the model, considering higher order approximations may be potentially important. This paper solves these two shortcomings. In particular, it shows how to use perturbation methods to solve Markov-switching rational expectations (MSRE) models - note the absence of the "linear" - starting from first principles, i.e. from the set of (non-linearized) first order conditions that define equilibrium.

Following Costa *et al.* (2005), Farmer *et al.* (2009), and Farmer *et al.* (2008a), this paper uses the concept of mean square stability (MSS) to characterize stable solutions. The perturbation approach uses the theory of Gröbner Bases to find solutions, and determines existence and uniqueness of MSS solutions. It also allows for a flexible regime-switching specification, including in parameters that affect the steady state of the economy. In particular, the first order approximation of models where switching affects the steady state is not certainty equivalent.

After developing the methodology, the paper presents two example economies that illustrate the methodology and highlight the issues of mean square stability and certainty equivalence. In the first, a simple real business cycle model with stochastic drift shows how to use the methodology and the importance of certainty equivalence. The second, a New Keynesian model, adds sticky prices and a monetary authority with changes in the policy rule, and shows how mean square stability determines existence and uniqueness.

The remainder of the paper is as follows: Section 2 describes a general class of MS-DSGE models and the nature of Markov switching. Sections 3 and 4 discuss the first-order approximation, the former showing how to solve the model, and the latter highlighting the key quadratic equations and how to use Gröbner Bases to solve them. Section 5 has an example RBC economy, Section 6 has an example NK economy, and Section 7 concludes.

2 The Model

Consider a dynamic general equilibrium model in which some of the parameters follow a discrete state Markov chain indexed by s_t with transition matrix $P = (p_{s,s'})$. The element $p_{s,s'}$ represents the probability that $s_{t+1} = s'$ given $s_t = s$ for $s, s' \in \{1, \ldots n_s\}$ where n_s is the number of regimes and when $s_t = s$ the model is said to be in regime s at time t. The vector of changing parameters θ_t has size $n_\theta \times 1$.[1] Given any $x_{t-1}, \varepsilon_t,$ and θ_t, the set of equilibrium conditions of a wide variety of this class of models can be written as

$$\mathbb{E}_t f\left(y_{t+1}, y_t, x_t, x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t\right) = 0 \tag{1}$$

where $\mathbb{E}_t$ denotes the mathematical expectations operator conditional on information available at time t. The vector x_{t-1} of predetermined variables (endogenous and exogenous) is of size $n_x \times 1$, the vector y_t of non-predetermined variables is of size $n_y \times 1$, the vector ε_t of independent innovations to the exogenous predetermined variables with mean equal to zero is of size $n_\varepsilon \times 1$, and χ is the perturbation parameter. The function f maps $\mathbb{R}^{2(n_y+n_x+n_\theta+n_\varepsilon)}$ into $\mathbb{R}^{n_y+n_x}$. Since the parameters, θ_t, in (1) depend on the state of the Markov chain, there are n_s sets of equilibrium conditions, one for each value of the Markov chain, instead of the single set of equilibrium conditions in the constant parameter case.

The solution to the model has the form

$$y_t \equiv g\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) \tag{2}$$

$$y_{t+1} \equiv g\left(x_t, \chi\varepsilon_{t+1}, \chi, s_{t+1}\right) \tag{3}$$

and

$$x_t \equiv h\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) \tag{4}$$

where g maps $\mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots n_s\}$ into $\mathbb{R}^{n_y}$ and h maps $\mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots n_s\}$ into $\mathbb{R}^{n_x}$. The goal is to find the Taylor expansion of the functions g and h around the steady state.

The parameters θ_t depend on the regime in the following way

$$\theta_t \equiv \theta\left(\chi, s_t\right) \text{ and } \theta_{t+1} \equiv \theta\left(\chi, s_{t+1}\right) \tag{5}$$

[1] There may also be a set of non-changing parameters not included in θ_t.

where θ maps $\mathbb{R} \times \{1, \dots n_s\}$ into $\mathbb{R}^{n_\theta}$. The set of parameters θ_t has two subsets θ_{1t} and θ_{2t}:

$$\theta_t = \begin{pmatrix} \theta'_{1t} & \theta'_{2t} \end{pmatrix}' \equiv \begin{pmatrix} \theta_1 (\chi, s_t)' & \theta_2 (\chi, s_t)' \end{pmatrix}', \tag{6}$$

where

$$\theta_1 (\chi, s_t) = \overline{\theta}_1 + \chi \widehat{\theta}_1 (s_t)$$

and

$$\theta_2 (\chi, s_t) = \widehat{\theta}_2 (s_t) .$$

The parameters θ_{t+1} have the same functional forms.[2] Note two things about this specification: first, $\widehat{\theta}_1 (s_t)$ is the deviation from $\overline{\theta}_1$ in regime s_t and, second, θ_{2t} is not a function of the perturbation parameter χ. Hence, perturbation just applies to a subset of the parameters, θ_{1t}, while θ_{2t} is not perturbed. The choice of which parameters to perturb, θ_{1t}, and which ones do not perturb, θ_{2t}, is not unique, but there is one restriction. Define steady state of the model as vectors x_{ss} and y_{ss} such that

$$f \left(y_{ss}, y_{ss}, x_{ss}, x_{ss}, 0, 0, \begin{pmatrix} \overline{\theta}'_1 & \widehat{\theta}_2 (s_{t+1})' \end{pmatrix}', \begin{pmatrix} \overline{\theta}'_1 & \widehat{\theta}_2 (s_t)' \end{pmatrix}' \right) = 0$$

for all s_t and s_{t+1}. Thus, the partition should be such that neither $\theta_2 (0, s_t) = \widehat{\theta}_2 (s_t)$ nor $\theta_2 (0, s_{t+1}) = \widehat{\theta}_2 (s_{t+1})$ enter in the calculation of the steady state since the last expression has to hold for all s_t and s_{t+1}. As mentioned, in general, more than one partition of parameters, θ_t, between θ_{1t} and θ_{2t} accomplishes this objective. In any case, included in is θ_{1t} the minimum set of parameters such that the steady state is defined as described above. Since the steady state depends upon $\bar{\theta}_1$, a natural choice for this point is the mean of the ergodic distribution across θ_{1t}, but again, this selection is not unique. Sections 5 and 6 provide examples of the partition of θ_t and the choice of $\bar{\theta}_1$.

Given the definition of the steady state, it is the case that

$$y_{ss} = g (x_{ss}, 0, 0, s_t) \text{ and } x_{ss} = h (x_{ss}, 0, 0, s_t)$$

[2]These functional forms are not necessary but just convenient for the derivations; any other functional form such that $\theta_1 (0, s_t) = \overline{\theta}_1$ for all s_t holds would also work.

for all s_t and

$$y_{ss} = g\left(x_{ss}, 0, 0, s_{t+1}\right) \text{ and } x_{ss} = h\left(x_{ss}, 0, 0, s_{t+1}\right)$$

for all s_{t+1}.

Using (2), (4), and (5) re-write the function f as

$$\mathcal{F}\left(x_{t-1}, \varepsilon_t, \varepsilon_{t+1}, s_{t+1}, \chi, s_t\right) =$$
$$f\left(\begin{array}{c} g\left(h\left(x_{t-1}, \varepsilon_t, \chi, s_t\right), \chi\varepsilon_{t+1}, \chi, s_{t+1}\right), g\left(x_{t-1}, \varepsilon_t, \chi, s_t\right), \\ h\left(x_{t-1}, \varepsilon_t, \chi, s_t\right), x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta\left(\chi, s_{t+1}\right), \theta\left(\chi, s_t\right) \end{array}\right)$$

for all $x_{t-1}, \varepsilon_t, \varepsilon_{t+1}, s_{t+1}$, and s_t. The function $\mathcal{F}$ maps $\mathbb{R}^{n_x+2n_\varepsilon+1} \times \{1, \ldots n_s\} \times \{1, \ldots n_s\}$ into $\mathbb{R}^{n_y+n_x}$.

Assuming that innovations to the exogenous predetermined variables, ε_t, are independent of the Markov chain, s_t, re-write (1) as

$$\mathbb{G}\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) = \sum_{s'=1}^{n_s} p_{s_t,s'} \int \mathcal{F}\left(x_{t-1}, \varepsilon', \varepsilon_t, s', \chi, s_t\right) \mu\left(\varepsilon'\right) d\varepsilon' = 0 \tag{7}$$

for all x_{t-1}, ε_t, and s_t where μ is the density of the innovations. The function $\mathbb{G}$ maps $\mathbb{R}^{n_x+n_\varepsilon+1} \times \{1, \ldots n_s\}$ into $\mathbb{R}^{n_y+n_x}$.

The remainder of the paper will use the following notation

$$\mathcal{D}\mathbb{G}\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) = \left[\mathcal{D}_j \mathbb{G}^i\left(x_{t-1}, \varepsilon_t, \chi, s_t\right)\right]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq n_x+n_\varepsilon+1}$$

to refer the $(n_y + n_x) \times (n_x + n_\varepsilon + 1)$ matrix of partial derivatives of $\mathbb{G}$ with respect to $(x_{t-1}, \varepsilon_t, \chi)$ evaluated at $(x_{t-1}, \varepsilon_t, \chi, s_t)$. Note the absence of derivatives with respect to s_t, since it is a discrete variable. Equivalently,

$$\mathcal{D}\mathbb{G}\left(x_{ss}, 0, 0, s_t\right) = \left[\mathcal{D}_j \mathbb{G}^i\left(x_{ss}, 0, 0, s_t\right)\right]_{1 \leq i \leq n_y+n_x, 1 \leq j \leq n_x+n_\varepsilon+1}$$

refers to the $(n_y + n_x) \times (n_x + n_\varepsilon + 1)$ matrix of partial derivatives of $\mathbb{G}$ with respect to $(x_{t-1}, \varepsilon_t, \chi)$ evaluated at $(x_{ss}, 0, 0, s_t)$. To simplify notation define

$$\mathcal{D}\mathbb{G}_{ss}\left(s_t\right) \equiv \mathcal{D}\mathbb{G}\left(x_{ss}, 0, 0, s_t\right) \text{ and } \mathcal{D}_j \mathbb{G}_{ss}^i\left(s_t\right) \equiv \mathcal{D}_j \mathbb{G}^i\left(x_{ss}, 0, 0, s_t\right)$$

for all i, j, and s_t. Thus,

$$\mathcal{D}\mathbb{G}_{ss}(s_t) = \left[\mathcal{D}_j\mathbb{G}^i_{ss}(s_t)\right]_{1\leq i\leq n_y+n_x, 1\leq j\leq n_x+n_\varepsilon+1}$$

for all s_t. In the same way,

$$\mathcal{D}f_{ss}(s_{t+1}, s_t) =$$
$$\left[\mathcal{D}_j f^i\left(y_{ss}, y_{ss}, x_{ss}, x_{ss}, 0, 0, \left(\begin{array}{cc}\overline{\theta}_1' & \widehat{\theta}_2(s_{t+1})'\end{array}\right)', \left(\begin{array}{cc}\overline{\theta}_1' & \widehat{\theta}_2(s_t)'\end{array}\right)'\right)\right]_{1\leq i\leq n_y+n_x, 1\leq j\leq 2(n_y+n_x+n_\theta+n_\varepsilon)}$$

is the $(n_y + n_x) \times (2(n_y + n_x + n_\theta + n_\varepsilon))$ matrix of partial derivatives of f with respect to all its components evaluated at $\left(y_{ss}, y_{ss}, x_{ss}, x_{ss}, 0, 0, \left(\begin{array}{cc}\overline{\theta}_1' & \widehat{\theta}_2(s_{t+1})'\end{array}\right)', \left(\begin{array}{cc}\overline{\theta}_1' & \widehat{\theta}_2(s_t)'\end{array}\right)'\right)$ for all s_{t+1} and s_t,

$$\mathcal{D}g(x_{ss}, 0, 0, s_t) = \left[\mathcal{D}_j g^i(x_{ss}, 0, 0, s_t)\right]_{1\leq i\leq n_y, 1\leq j\leq n_x+n_\varepsilon+1}$$

is the $n_y \times (n_x + n_\varepsilon + 1)$ matrix of partial derivatives of g with respect to $(x_{t-1}, \varepsilon_t, \chi)$ evaluated at $(x_{ss}, 0, 0, s_t)$ for all s_t, and

$$\mathcal{D}h(x_{ss}, 0, 0, s_t) = \left[\mathcal{D}_j h^i(x_{ss}, 0, 0, s_t)\right]_{1\leq i\leq n_x, 1\leq j\leq n_x+n_\varepsilon+1}$$

is the $n_x \times (n_x + n_\varepsilon + 1)$ matrix of partial derivatives of h with respect to $(x_{t-1}, \varepsilon_t, \chi)$ evaluated at $(x_{ss}, 0, 0, s_t)$ for all s_t. To simplify notation, define

$$\mathcal{D}g_{ss}(s_t) \equiv \mathcal{D}g(x_{ss}, 0, 0, s_t) \text{ and } \mathcal{D}_j g^i_{ss}(s_t) \equiv \mathcal{D}_j g^i(x_{ss}, 0, 0, s_t),$$

for all i, j, and s_t and

$$\mathcal{D}h_{ss}(s_t) \equiv \mathcal{D}h(x_{ss}, 0, 0, s_t) \text{ and } \mathcal{D}_j h^i_{ss}(s_t) \equiv \mathcal{D}_j h^i(x_{ss}, 0, 0, s_t)$$

for all i, j, and s_t. Thus,

$$\mathcal{D}g_{ss}(s_t) = \left[\mathcal{D}_j g^i_{ss}(s_t)\right]_{1\leq i\leq n_y+n_x, 1\leq j\leq n_x+n_\varepsilon+1}$$

and

$$\mathcal{D}h_{ss}(s_t) = \left[\mathcal{D}_j h^i_{ss}(s_t)\right]_{1\leq i\leq n_y+n_x, 1\leq j\leq n_x+n_\varepsilon+1}$$

for all s_t.

3 First Order Approximation

This section shows how to find the first order Taylor expansions to g and h around the point $(x_{ss}, 0, 0, s_t)$ of the form

$$g\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) - y_{ss} \simeq \left[\mathcal{D}_1 g_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x} g_{ss}\left(s_t\right)\right]\left(x_{t-1} - x_{ss}\right)$$
$$+ \left[\mathcal{D}_{n_x+1} g_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x+n_\varepsilon} g_{ss}\left(s_t\right)\right] \varepsilon_t + \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}\left(s_t\right) \chi$$

and

$$h\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) - x_{ss} \simeq \left[\mathcal{D}_1 h_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x} h_{ss}\left(s_t\right)\right]\left(x_{t-1} - x_{ss}\right)$$
$$+ \left[\mathcal{D}_{n_x+1} h_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x+n_\varepsilon} h_{ss}\left(s_t\right)\right] \varepsilon_t + \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}\left(s_t\right) \chi$$

for all s_t where $\mathcal{D}_j g_{ss}\left(s_t\right)$ is the j^{th} column vector of $\mathcal{D} g_{ss}\left(s_t\right)$ and $\mathcal{D}_j h_{ss}\left(s_t\right)$ is the j^{th} column vector of $\mathcal{D} h_{ss}\left(s_t\right)$. To simply notation, define

$$\begin{aligned}
\mathcal{D}_{n,m} g_{ss}\left(s_t\right) &\equiv \left[\mathcal{D}_n g_{ss}\left(s_t\right), \ldots, \mathcal{D}_m g_{ss}\left(s_t\right)\right] \\
\mathcal{D}_{n,m} h_{ss}\left(s_t\right) &\equiv \left[\mathcal{D}_n h_{ss}\left(s_t\right), \ldots, \mathcal{D}_m h_{ss}\left(s_t\right)\right]
\end{aligned}$$

for all n and m and all s_t.

Hence, the above approximations are equivalent to

$$g\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) - y_{ss} \simeq \mathcal{D}_{1,n_x} g_{ss}\left(s_t\right)\left(x_{t-1} - x_{ss}\right) + \mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}\left(s_t\right) \varepsilon_t + \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}\left(s_t\right) \chi$$

and

$$h\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) - x_{ss} \simeq \mathcal{D}_{1,n_x} h_{ss}\left(s_t\right)\left(x_{t-1} - x_{ss}\right) + \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}\left(s_t\right) \varepsilon_t + \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}\left(s_t\right) \chi$$

The objective is now to find the coefficients

$$\left\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\right\}_{s=1}^{n_s}, \ \left\{\mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s), \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s)\right\}_{s=1}^{n_s},$$
$$\text{and } \left\{\mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s), \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s)\right\}_{s=1}^{n_s}$$

of the above describe expansions. The current setup requires finding a set of n_s policy functions, one for each possible value of the Markov chain, instead of the single set of policy functions in the constant parameter case.

The coefficients of these policy functions are going to be obtained by using the fact that

$$\mathbb{G}\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) = 0$$

for all $x_{t-1}, \varepsilon_t, \chi$, and s_t and, therefore, it must be the case that

$$\mathcal{D}\mathbb{G}\left(x_{t-1}, \varepsilon_t, \chi, s_t\right) = 0$$

for all $x_{t-1}, \varepsilon_t, \chi$, and s_t and, in particular,

$$\mathcal{D}\mathbb{G}_{ss}\left(s_t\right) = 0$$

for all s_t. Thus,

$$\begin{gathered}
\left[\mathcal{D}_1\mathbb{G}_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x}\mathbb{G}_{ss}\left(s_t\right)\right] = 0, \\
\left[\mathcal{D}_{n_x+1}\mathbb{G}_{ss}\left(s_t\right), \ldots, \mathcal{D}_{n_x+n_\varepsilon}\mathbb{G}_{ss}\left(s_t\right)\right] = 0, \\
\mathcal{D}_{n_x+n_\varepsilon+1}\mathbb{G}_{ss}\left(s_t\right) = 0,
\end{gathered} \tag{8}$$

for all s_t where $\mathcal{D}_j\mathbb{G}_{ss}\left(s_t\right)$ is the j^{th} column vector of $\mathcal{D}\mathbb{G}_{ss}\left(s_t\right)$. Again, note that there are a set of n_s derivatives of $\mathbb{G}$, one for each possible value of s_t, instead of the single derivative in the constant parameter case. To simply notation, again, define

$$\mathcal{D}_{n,m}\mathbb{G}_{ss}\left(s_t\right) \equiv \left[\mathcal{D}_n\mathbb{G}_{ss}\left(s_t\right), \ldots, \mathcal{D}_m\mathbb{G}_{ss}\left(s_t\right)\right]$$

therefore, expression (8) can be written as

$$\mathcal{D}_{1,n_x}\mathbb{G}_{ss}\left(s_t\right) = 0,\ \mathcal{D}_{n_x+1,n_x+n_\varepsilon}\mathbb{G}_{ss}\left(s_t\right) = 0,\ \text{and } \mathcal{D}_{n_x+n_\varepsilon+1}\mathbb{G}_{ss}\left(s_t\right) = 0.$$

3.1 Solving for the Derivatives of x

Using (7), taking derivatives with respect to x_{t-1} produces the following expression, where $\mathcal{D}_{1,n_x}\mathbb{G}_{ss}\left(s_t\right) = 0$ for all s_t :

$$\mathcal{D}_{1,n_x}\mathbb{G}_{ss}\left(s_t\right) = \sum_{s'=1}^{n_s} p_{s_t,s'} \int \begin{pmatrix} \mathcal{D}_{1,n_y} f_{ss}\left(s', s_t\right) \mathcal{D}_{1,n_x} g_{ss}\left(s'\right) \mathcal{D}_{1,n_x} h_{ss}\left(s_t\right) \\ +\mathcal{D}_{n_y+1,2n_y} f_{ss}\left(s', s_t\right) \mathcal{D}_{1,n_x} g_{ss}\left(s_t\right) \\ +\mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}\left(s', s_t\right) \mathcal{D}_{1,n_x} h_{ss}\left(s_t\right) \\ +\mathcal{D}_{2n_y+n_x+1,2\left(n_y+n_x\right)} f_{ss}\left(s', s_t\right) \end{pmatrix} \mu\left(\varepsilon'\right) d\varepsilon'$$

for all s_t. Next, taking into account that $\int \mu(\varepsilon') d\varepsilon' = 1$, this expression simplifies to

$$\mathcal{D}_{1,n_x}\mathbb{G}_{ss}(s_t) = \sum_{s'=1}^{n_s} p_{s_t,s'} \left(\begin{array}{c} \mathcal{D}_{1,n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s') \mathcal{D}_{1,n_x} h_{ss}(s_t) \\ +\mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s_t) \\ +\mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} h_{ss}(s_t) \\ +\mathcal{D}_{2n_y+n_x+1,2(n_y+n_x)} f_{ss}(s', s_t) \end{array} \right)$$

for all s_t. Now, rearranging, for each s_t:

$$\mathcal{D}_{1,n_x}\mathbb{G}_{ss}(s_t) = \tag{9}$$
$$\sum_{s'=1}^{n_s} p_{s_t,s'} \left(\begin{array}{c} \left(\mathcal{D}_{1,n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s') + \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t)\right) \mathcal{D}_{1,n_x} h_{ss}(s_t) \\ +\mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s_t) + \mathcal{D}_{2n_y+n_x+1,2(n_y+n_x)} f_{ss}(s', s_t) \end{array} \right).$$

Putting together the n_s versions of (9), one for each value of s_t, yields a system of $(n_x + n_y) n_x n_s$ quadratic equations in the same number of unknowns $\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\}_{s=1}^{n_s}$. Section 4 describes how to solve this system.

3.2 Solving for the Derivatives of ε and χ

After finding $\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\}_{s=1}^{n_s}$, obtaining $\{\mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s), \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s)\}_{s=1}^{n_s}$ and $\{\mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s), \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s)\}_{s=1}^{n_s}$ is simply solving a system of linear equations. Let us first solve for $\{\mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s), \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s)\}_{s=1}^{n_s}$, then $\{\mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s), \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s)\}_{s=1}^{n_s}$.

In order to solve for $\{\mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s), \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s)\}_{s=1}^{n_s}$, obtain the expressions for $\mathcal{D}_{n_x+1,n_x+n_\varepsilon}\mathbb{G}_{ss}(s_t)$

$$\mathcal{D}_{n_x+1,n_x+n_\varepsilon}\mathbb{G}_{ss}(s_t) =$$
$$\sum_{s'=1}^{n_s} p_{s_t,s'} \int \left(\begin{array}{c} \mathcal{D}_{1,n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s') \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s_t) + \\ \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t) \mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s_t) + \\ \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t) \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s_t) + \\ \mathcal{D}_{2(n_y+n_x)+n_\varepsilon+1,2(n_y+n_x+n_\varepsilon)} f_{ss}(s', s_t) \end{array} \right) \mu(\varepsilon') d\varepsilon' = 0$$

for all s_t. Taking into account that $\int \mu(\varepsilon') d\varepsilon' = 1$, this expression simplifies to

$$\mathcal{D}_{n_x+1,n_x+n_\varepsilon} \mathbb{G}_{ss}(s_t) =$$
$$\sum_{s'=1}^{n_s} p_{s_t,s'} \left(\begin{array}{c} \left(\mathcal{D}_{1,n_y} f_{ss}(s', s_t) \mathcal{D}_{1,n_x} g_{ss}(s') + \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t)\right) \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(s_t) \\ \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t) \mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s_t) \\ \mathcal{D}_{2(n_y+n_x)+n_\varepsilon+1,2(n_y+n_x+n_\varepsilon)} f_{ss}(s', s_t) \end{array} \right) \tag{10}$$

for all s_t.

To solve this system, stack (10) for $s_t = 1, ..., n_s$, which produces the matrix equation

$$\left[\begin{array}{cc} \Theta_\varepsilon & \Phi_\varepsilon \end{array} \right] \left[\begin{array}{c} \mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(1) \\ \vdots \\ \mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(n_s) \\ \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(1) \\ \vdots \\ \mathcal{D}_{n_x+1,n_x+n_\varepsilon} h_{ss}(n_s) \end{array} \right] = \Psi_\varepsilon \tag{11}$$

where

$$\Theta_\varepsilon = \sum_{s'=1}^{n_s} \left[\begin{array}{ccc} p_{1,s'} \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', 1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{n_s,s'} \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', n_s) \end{array} \right],$$

$$\Phi_\varepsilon = \sum_{s'=1}^{n_s} \left[\begin{array}{ccc} p_{1,s'} \mathcal{D}_{1,n_y} f_{ss}(s', 1) \mathcal{D}_{1,n_x} g_{ss}(s') & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{n_s,s'} \mathcal{D}_{1,n_y} f_{ss}(s', n_s) \mathcal{D}_{1,n_x} g_{ss}(s') \end{array} \right]$$
$$+ \sum_{s'=1}^{n_s} \left[\begin{array}{ccc} p_{1,s'} \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', 1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{n_s,s'} \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', n_s) \end{array} \right],$$

and

$$\Psi_\varepsilon = -\sum_{s'=1}^{n_s} \left[\begin{array}{c} p_{1,s'} \mathcal{D}_{2(n_y+n_x)+n_\varepsilon+1,2(n_y+n_x+n_\varepsilon)} f_{ss}(s', 1) \\ \vdots \\ p_{n_s,s'} \mathcal{D}_{2(n_y+n_x)+n_\varepsilon+1,2(n_y+n_x+n_\varepsilon)} f_{ss}(s', n_s) \end{array} \right].$$

Thus, given the solution for $\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\}_{s=1}^{n_s}$, the above is a system of $(n_x+n_y)\, n_\varepsilon n_s$ linear equations in the same number of unknowns given by $\{\mathcal{D}_{n_x+1,n_x+n_e} g_{ss}(s), \mathcal{D}_{n_x+1,n_x+n_e} h_{ss}(s)\}_{s=1}^{n_s}$ that can be solved by inverting $\begin{bmatrix} \Theta_\varepsilon & \Phi_\varepsilon \end{bmatrix}$.

Now to find $\{\mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s), \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s)\}_{s=1}^{n_s}$, use the derivative of (7) with respect to χ:

$$\mathcal{D}_{n_x+n_\varepsilon+1}\mathbb{G}_{ss}(s_t) =$$

$$\sum_{s'=1}^{n_s} p_{s_t,s'} \int \begin{pmatrix} \mathcal{D}_{1,n_y} f_{ss}(s', s_t) \begin{bmatrix} \mathcal{D}_{1,n_x} g_{ss}(s')\, \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s_t) \\ +\mathcal{D}_{n_x+1,n_x+n_\varepsilon} g_{ss}(s')\, \varepsilon' + \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s') \end{bmatrix} + \\ \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t)\, \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s_t) + \\ \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t)\, \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s_t) + \\ \mathcal{D}_{2n_y+n_x+1,2n_y+n_x+n_\varepsilon} f_{ss}(s', s_t)\, \varepsilon' + \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s', s_t)\, \mathcal{D}\theta(0, s') + \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)} f_{ss}(s', s_t)\, \mathcal{D}\theta(0, s_t) \end{pmatrix} \mu(\varepsilon')\, d\varepsilon' = 0$$

for all s_t, where $\mathcal{D}\theta(0, s_{t+1})$ is the derivative of $\theta(\chi, s_{t+1})$ with respect to χ

$$\mathcal{D}\theta(\chi, s_t) = \left[\mathcal{D}_j^i \theta(\chi, s_t)\right]_{1\leq i\leq n_\theta, j=1}$$

for all s_t, evaluated at $\chi = 0$.

Taking into account that $\int \mu(\varepsilon')\, d\varepsilon' = 1$ and $\int \varepsilon' \mu(\varepsilon')\, d\varepsilon' = 0$, the above simplifies to

$$\mathcal{D}_{n_x+n_\varepsilon+1}\mathbb{G}_{ss}(s_t) = \tag{12}$$

$$\sum_{s'=1}^{n_s} p_{s_t,s'} \begin{pmatrix} \mathcal{D}_{1,n_y} f_{ss}(s', s_t) \{\mathcal{D}_{1,n_x} g_{ss}(s')\, \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s_t) + \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s')\} + \\ \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t)\, \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s_t) + \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t)\, \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s_t) \\ +\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s', s_t)\, \mathcal{D}\theta(0, s') + \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)} f_{ss}(s', s_t)\, \mathcal{D}\theta(0, s_t) \end{pmatrix} = 0$$

for all s_t.

In matrix notation expression (12) can be written as

$$\begin{bmatrix} \Theta_\chi & \Phi_\chi \end{bmatrix} \begin{bmatrix} \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(1) \\ \vdots \\ \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(n_s) \\ \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(1) \\ \vdots \\ \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(n_s) \end{bmatrix} = \Psi_\chi, \tag{13}$$

where

$$\Theta_\chi = \begin{bmatrix} p_{1,1}\mathcal{D}_{1,n_y} f_{ss}(1,1) + \mathcal{D}_{n_y+1,2n_y} f_{ss}(1,1) & \cdots & p_{1,n_s}\mathcal{D}_{1,n_y} f_{ss}(n_s,1) \\ \vdots & \ddots & \vdots \\ p_{n_s,1}\mathcal{D}_{1,n_y} f_{ss}(1,n_s) & \cdots & p_{n_s,n_s}\mathcal{D}_{1,n_y} f_{ss}(n_s,n_s) + \mathcal{D}_{n_y+1,2n_y} f_{ss}(n_s,n_s) \end{bmatrix},$$

$$\begin{aligned} \Phi_\chi &= \sum_{s'=1}^{n_s} \begin{bmatrix} p_{1,s'}\mathcal{D}_{1,n_y} f_{ss}(s',1)\,\mathcal{D}_{1,n_x} g_{ss}(s') & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{n_s,s'}\mathcal{D}_{1,n_y} f_{ss}(s',n_s)\,\mathcal{D}_{1,n_x} g_{ss}(s') \end{bmatrix} \\ &\quad + \sum_{s'=1}^{n_s} \begin{bmatrix} p_{1,s'}\mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s',1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{n_s,s'}\mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s',n_s) \end{bmatrix}, \end{aligned}$$

and

$$\Psi_\chi = -\sum_{s'=1}^{n_s} \begin{bmatrix} p_{1,s'}\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s',1)\,\mathcal{D}\theta(0,s') + \ldots \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)} f_{ss}(s',1)\,\mathcal{D}\theta(0,1) \\ \vdots \\ p_{n_s,s'}\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s',n_s)\,\mathcal{D}\theta(0,s') + \ldots \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)} f_{ss}(s',n_s)\,\mathcal{D}\theta(0,n_s) \end{bmatrix}.$$

Thus, given the solution for $\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\}_{s=1}^{n_s}$, this is a system of $(n_x+n_y)\,n_s$ linear equations in the same number of unknowns given by the elements of $\{\mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(s), \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(s)\}_{s=1}^{n_s}$ that can be solved by inverting $\begin{bmatrix} \Theta_\chi & \Phi_\chi \end{bmatrix}$.

3.3 Non-Certainty Equivalence of First-Order Approximation

As pointed out by Schmitt-Grohe & Uribe (2004), one important feature of models without Markov switching is certainty equivalence of the first-order approximation. This feature of models implies that first-order approximations are inadequate for analyzing interesting behavior such as response to risk because the approximated decision rules are invariant to changes in volatility. For example, van Binsbergen *et al.* (2008) and Rudebusch & Swanson (2008) note that at least second-order approximations are needed to analyze certain asset pricing implications, such as the yield curve, since second-order approximations are not certainty equivalent, and hence react to changes in volatility. The second-order approximations also imply a degree of difficulty in performing likelihood based estimation, such as Fernández-Villaverde & Rubio-Ramirez (2007) who use the particle filter for estimation. These factors mean that addressing interesting questions with second-order approximations may be necessary but difficult in models without Markov Switching. As shown below, first order approximations to Markov Switching models are not (in general) certainty equivalent. This nice feature opens the door to analyze risk related behaviors using linearly approximated models.

To see the certainty equivalence of the model without Markov switching, consider equation (13) with only one regime, so $n_s = 1$. In this case,

$$\begin{bmatrix} \Theta_\chi & \Phi_\chi \end{bmatrix} \begin{bmatrix} \mathcal{D}_{n_x+n_\varepsilon+1} g_{ss}(1) \\ \mathcal{D}_{n_x+n_\varepsilon+1} h_{ss}(1) \end{bmatrix} = \Psi_\chi \tag{14}$$

where

$$\begin{bmatrix} \Theta_\chi & \Phi_\chi \end{bmatrix} =$$

$$\begin{bmatrix} \mathcal{D}_{1,n_y} f_{ss}(1,1)\, \mathcal{D}_{1,n_x} g_{ss}(1) + \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(1,1) & \mathcal{D}_{1,n_y} f_{ss}(1,1) + \mathcal{D}_{n_y+1,2n_y} f_{ss}(1,1) \end{bmatrix}$$

and

$$\Psi_\chi = -\begin{bmatrix} \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(1,1)\, \mathcal{D}\theta(0,1) \end{bmatrix}.$$

Clearly, in the fixed regime case $\theta(\chi,1) = \bar{\theta}$. Therefore, it is the case that $\mathcal{D}\theta(0,1) = 0$, which implies $\Psi_\chi = 0$. Consequently the system (14) is homogenous. If a unique system exists, then it is given by

$$\mathcal{D}_{n_x+n_\varepsilon+1}g_{ss}(1) = \mathcal{D}_{n_x+n_\varepsilon+1}h_{ss}(1) = 0. \tag{15}$$

Since in the fixed-regime case the only source of uncertainty is ε_{t+1}, solution (15) implies that the linear approximation is certainty equivalent, i.e.

$$\mathcal{D}_{1,n_x}g_{ss}(1)(x_{ss}-x_{ss}) + \mathcal{D}_{n_x+1,n_x+n_\varepsilon}g_{ss}(1)\,0 + \mathcal{D}_{n_x+n_\varepsilon+1}g_{ss}(1) = 0$$

and

$$\mathcal{D}_{1,n_x}h_{ss}(1)(x_{ss}-x_{ss}) + \mathcal{D}_{n_x+1,n_x+n_\varepsilon}h_{ss}(1)\,0 + \mathcal{D}_{n_x+n_\varepsilon+1}h_{ss}(1) = 0.$$

Now, turning to the case of Markov switching, note that if equation (13) is a non-homogenous system, i.e. if $\Psi_\chi \neq 0$, then it will be the case that $\mathcal{D}_{n_x+n_\varepsilon+1}g_{ss}(s) \neq 0$ and $\mathcal{D}_{n_x+n_\varepsilon+1}h_{ss}(1) \neq 0$ if an unique solution exists.

So, consider when $\Psi_\chi \neq 0$. In the expression for Ψ_χ

$$\Psi_\chi = -\sum_{s'=1}^{n_s}\left[\begin{array}{c} p_{1,s'}\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta}f_{ss}(s',1)\,\mathcal{D}\theta(0,s') + \ldots \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)}f_{ss}(s',1)\,\mathcal{D}\theta(0,1) \\ \vdots \\ p_{n_s,s'}\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta}f_{ss}(s',n_s)\,\mathcal{D}\theta(0,s') + \ldots \\ \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)}f_{ss}(s',n_s)\,\mathcal{D}\theta(0,n_s) \end{array}\right],$$

clearly, if $\mathcal{D}\theta(0,s) = 0$ for all s, then $\Psi_\chi = 0$. So a necessary condition for non-certainty equivalence is that $\mathcal{D}\theta(0,s) \neq 0$ for some s. Recalling the form of θ_t :

$$\begin{aligned} \theta_1(\chi,s) &= \overline{\theta}_1 + \chi\widehat{\theta}_1(s) \\ \theta_2(\chi,s) &= \widehat{\theta}_2(s) \end{aligned}$$

then

$$\mathcal{D}\theta(0,s) = \left[\begin{array}{cc} \widehat{\theta}_1(s)' & 0' \end{array}\right]'.$$

Then $\mathcal{D}\theta(0,s) \neq 0$ for some s if and only if $\widehat{\theta}_1(s) \neq 0$. This condition implies that all regimes cannot have identical steady states if they were to occur permanently.

However, the condition that $\mathcal{D}\theta(0,s) \neq 0$ for some s is not sufficient for $\Psi_\chi \neq 0$. In addition, it must be the case that either

$$\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s',s) \neq 0, \text{ or } \mathcal{D}_{2(n_x+n_y+n_\varepsilon)+n_\theta+1,2(n_x+n_y+n_\varepsilon+n_\theta)} f_{ss}(s',s) \neq 0$$

which will be true when the switching parameters do not enter the equilibrium conditions multiplicatively with a variable which expected value equals zero in steady state.

In summary, the necessary and sufficient conditions for no certainty equivalence are (i) that $\mathcal{D}\theta(0,s) \neq 0$ for some s and (ii)

$$\mathcal{D}_{2(n_x+n_y+n_\varepsilon)+1,2(n_x+n_y+n_\varepsilon)+n_\theta} f_{ss}(s',s) \neq 0.$$

4 The Solution to the Quadratic System

As mentioned above, the n_s versions of (9) form a system of $(n_x+n_y)\,n_s n_x$ quadratic equations in the elements of $\{\mathcal{D}_{1,n_x} g_{ss}(s), \mathcal{D}_{1,n_x} h_{ss}(s)\}_{s=1}^{n_s}$. This section describes how to find the solution to this system. Putting (9) into matrix form produces

$$\begin{bmatrix} \sum_{s'=1}^{n_s} p_{s_t,s'} \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s',s_t) & p_{s_t,1} \mathcal{D}_{1,n_y} f_{ss}(1,s_t) & \cdots & p_{s_t,n_s} \mathcal{D}_{1,n_y} f_{ss}(n_s,s_t) \end{bmatrix} \times$$
$$\begin{bmatrix} I \\ \mathcal{D}_{1,n_x} g_{ss}(1) \\ \vdots \\ \mathcal{D}_{1,n_x} g_{ss}(n_s) \end{bmatrix} \mathcal{D}_{1,n_x} h_{ss}(s_t) = \tag{16}$$
$$-\sum_{s'=1}^{n_s} p_{s_t,s'} \begin{bmatrix} \mathcal{D}_{2n_y+n_x+1,2(n_y+n_x)} f_{ss}(s',s_t) & \mathcal{D}_{n_y+1,2n_y} f_{ss}(s',s_t) \end{bmatrix} \begin{bmatrix} I \\ \mathcal{D}_{1,n_x} g_{ss}(s_t) \end{bmatrix}$$

for all s_t. Hence, there are n_s equations of this form.

The quadratic system just described is nothing else than an algebraic system of equations. In a constant regime framework, $n_s = 1$, mapping this system into a generalized eigenvalue problem allows solving it by a singular value decomposition (SVD) type of algorithm. In the case of Markov switching, the fact that $\{\mathcal{D}_{1,n_x} g_{ss}(s)\}_{s=1}^{n_s}$ appear in every of the n_s equations described above makes it impossible to map the algebraic systems of equations into a generalized eigenvalue problem. Instead solutions are found using Gröbner Bases.

4.1 Gröbner Basis

What is a Gröbner basis? A Gröbner basis for a system of polynomials is a set of multivariate polynomials that possesses desirable algorithmic properties. The most important of these features for the current problem is that the system of polynomials in a Gröbner basis have the same collection of roots as the original polynomials. Every set of polynomials can be transformed into a Gröbner basis, although this transformation may not be unique. The transformation process generalizes the familiar techniques of Gaussian elimination for solving linear systems of equations. In general, solving the problem in the system of polynomials in a Gröbner basis is much simpler that in the original system. Also, a fundamental insight and contribution of Gröbner bases theory is that every polynomial system, no matter how complicated, can be transformed into Gröbner basis form (see Buchberger's algorithm). As an example, consider the following system of polynomials of four quadratic equations in four unknowns

$$\begin{aligned} xy + zw + 2 &= 0, \\ xy + yz + 3 &= 0, \\ xz + wx + wy + 6 &= 0, \text{ and} \\ xz + 2xy + 3 &= 0. \end{aligned}$$

A Gröbner basis, with respect to the lexicographic ordering $\{x, y, z, w\}$, is

$$\begin{aligned} &-49 - 19w^2 + 9w^4 + 3w^6, \\ &2w + 9w^3 + 3w^5 + 14z, \\ &-99w + 6w^3 + 9w^5 + 28y, \text{ and} \\ &15w - 6w^3 - 9w^5 + 28x. \end{aligned}$$

Note that the first element of the basis is a polynomial in w only. Given a root w of the first polynomial, the second polynomial is linear in z, the third is linear in y, and the last is linear in x. Solving the first element of the basis produces the following six solutions

$$\{w = -1.55461, w = -1.39592i, w = 1.39592i, w = 0. - 1.86232i, w = 1.86232i, w = 1.55461\}$$

Solving the other three basis, conditional on these solutions, gives the following roots

$$\{z = 4.58328, z = -0.41342i, z = 0.41342i, z = 0.914097i, z = 0.914097i, z = -4.58328\},$$

$$\{y = -1.7728, y = -3.81477i, y = 3.81477i, y = -0.768342i, y = 0.768342i, y = 1.7728\}$$

and

$$\{x = -2.89104, x = -0.372997i, x = 0.372997i, x = -4.81861i, x = 4.81861i, x = 2.89104\}.$$

These roots solve the original system of four quadratic equations in four unknowns.

4.2 Mean Square Stability

Now, having discussed Gröbner bases, the objective is to use them to solve the n_s systems of equations in (16). Defining

$$A(s_t) = \left[\begin{array}{cccc} \sum_{s'=1}^{n_s} p_{s_t,s'} \mathcal{D}_{2n_y+1,2n_y+n_x} f_{ss}(s', s_t) & p_{s_t,1} \mathcal{D}_{1,n_y} f_{ss}(1, s_t) & \cdots & p_{s_t,n_s} \mathcal{D}_{1,n_y} f_{ss}(n_s, s_t) \end{array} \right]$$

and

$$B(s_t) = -\sum_{s'=1}^{n_s} p_{s_t,s'} \left[\begin{array}{cc} \mathcal{D}_{2n_y+n_x+1,2(n_y+n_x)} f_{ss}(s', s_t) & \mathcal{D}_{n_y+1,2n_y} f_{ss}(s', s_t) \end{array} \right]$$

there are n_s versions of

$$A(s_t) \left[\begin{array}{c} I \\ \mathcal{D}_{1,n_x} g_{ss}(1) \\ \vdots \\ \mathcal{D}_{1,n_x} g_{ss}(n_s) \end{array} \right] \mathcal{D}_{1,n_x} h_{ss}(s_t) = B(s_t) \left[\begin{array}{c} I \\ \mathcal{D}_{1,n_x} g_{ss}(s_t) \end{array} \right]. \tag{17}$$

where the unknowns are $\{\mathcal{D}_{1,n_x} h_{ss}(s), \mathcal{D}_{1,n_x} g_{ss}(s)\}_{s=1}^{n_s}$.

To solve these unknowns, stack the systems (17), which gives a set of $n * n_s$ quadratic equations and unknowns. Then, using the ordering on

$$\left\{ vec(\mathcal{D}_{1,n_x} h_{ss}(1))', ..., vec(\mathcal{D}_{1,n_x} h_{ss}(n_s))', vec(\mathcal{D}_{1,n_x} g_{ss}(1)), ..., vec(\mathcal{D}_{1,n_x} g_{ss}(n_s)) \right\}.$$

to construct the Gröbner basis, and solving the resulting quadratic system produces a set of solutions for $\{\mathcal{D}_{1,n_x}h_{ss}(s), \mathcal{D}_{1,n_x}g_{ss}(s)\}_{s=1}^{n_s}$.

In a typical model without Markov switching, determinacy is easily verified by checking whether the number of eigenvalues of the system (17) inside the unit circle equals to the number of state variables. In a model with Markov switching, as the one described here, the problem is more subtle. As shown in Farmer *et al.* (2009), it is possible that the number of stable eigenvalues associated with each of the regimes is equal to the number of states but the system, as a whole, does not have a stable solution under several concepts of stability. The good news is that the Markov switching model can be checked for mean-square stability (MSS), as defined in Costa *et al.* (2005). In particular, MSS requires checking if the following matrix has its eigenvalues inside the unit circle

$$T = \left(P' \otimes I_{n_x^2}\right) diag\left[\mathcal{D}_{1,n_x}h_{ss} \otimes \mathcal{D}_{1,n_x}h_{ss}\right] \tag{18}$$

where

$$diag\left[\mathcal{D}_{1,n_x}h_{ss} \otimes \mathcal{D}_{1,n_x}h_{ss}\right] = \begin{bmatrix} \mathcal{D}_{1,n_x}h_{ss}(1) \otimes \mathcal{D}_{1,n_x}h_{ss}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{D}_{1,n_x}h_{ss}(n_s) \otimes \mathcal{D}_{1,n_x}h_{ss}(n_s) \end{bmatrix}.$$

Thus, with Markov switching, the policy functions $\{\mathcal{D}_{1,n_x}h_{ss}(i)\}_{i=1}^{n_s}$ for all possible solutions must be checked for stability under (18). If only one policy function is stable then the model only has one stable solution. If more than one are stable, the model has multiple stable solutions. If none are stable, the model has no stable solutions.

As mentioned, all possible solutions must be constructed and checked for stability. For each regime, the problem of constructing solutions depends upon the selection of eigenvalues for construction of the matrices $\{\mathcal{D}_{1,n_x}h_{ss}(s)\}_{s=1}^{n_s}$. Each regime has n eigenvalues, and n_x must be used for the matrix $\mathcal{D}_{1,n_x}h_{ss}(s)$. Some eigenvalues are required to be selected: those associated with the exogenous variables must be in $\mathcal{D}_{1,n_x}h_{ss}(s)$ by definition. So if n_{exog} is the number of exogenous variables, then $n_{endo} = n - n_{exog}$ is the number of endogenous variables, and n_x' is the number of non-exogenous predetermined variables, then constructing solutions amounts to choosing $n_x - n'_x$ eigenvalues (those associated with the endogenous predetermined variables) from n_{endo} possible selections. And this must be chosen for each regime, meaning a total of

$$\left(\begin{array}{c} n_{endo} \\ n'_x \end{array}\right)^{n_s} = \left(\frac{n_{endo}!}{n'_x!\,(n_{endo} - n'_x)!}\right)^{n_s}$$

possible solutions.

5 Example 1: RBC Model

This section presents a simple exercise to illustrate the theoretical framework at hand. The perfect vehicle for such pedagogical effort is the real business cycle model. There are two reasons. First, the stochastic neoclassical growth model is the foundation of modern macroeconomics. Even the more complicated New Keynesian models, such as those in Woodford (2003) or Christiano *et al.* (2005), are built around the core of the neoclassical growth model augmented with nominal and real rigidities. Thus, after understanding how to deal with Markov switching in this prototype economy, it will be rather straightforward to extend it to richer environments such as the ones commonly used for policy analysis. Second, the model is so well known, its working so well understood, and its computation so thoroughly explored that the role of time-varying volatility in it will be staggeringly transparent.

5.1 The RBC Model

Consider a real business cycle model where growth in total factor productivity follows a Markov process with only two regimes. In particular, the TFP process will follow a random walk in logs with drift that takes one of two levels, high and low, so the economy experiences high or low growth. The random walk specification helps simplify the number of variables considered in a stationary equilibrium, and is hence the most parsimonious illustrative example. The specification of two regimes will allow a succinct discussion of the methodology, but, as mentioned above, more regimes can be handled easily within the framework.

To get into the substantive questions as soon as possible, the description of the standard features of the prototype economy will be limited to fix notation. There is a representative household in the economy, whose preferences over stochastic sequences of consumption, c_t, are represented by a utility function:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

where $\beta \in (0,1)$. The resource constraint is

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

where k_t is capital and the technological change, z_t, proceeds according to a random walk in logs with drift where the Markov switching is in the drift, i.e.

$$\log z_t = \mu_t + \log z_{t-1} + \sigma \varepsilon_t$$

where the drift takes two values

$$\mu_t = \mu(s_t), \; s_t \in \{1,2\}$$

and the transition matrix is $P = [p_{i,j}]$ where $p_{i,j} = \Pr(s_t = j | s_{t-1} = i)$.

For this model it is natural to work with the solution to the social planner's problem. The optimality conditions are standard:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{c_{t+1}} \left(\alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta \right)$$

and

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1}.$$

Due to the unit root the economy is non-stationary. Thus, define $\omega_t = z_{t-1}^{\frac{1}{1-a}}$, and let $\tilde{c}_t = \frac{c_t}{\omega_t}$, $\tilde{k}_{t-1} = \frac{k_{t-1}}{\omega_t}$, $\tilde{z}_t = \frac{z_t}{z_{t-1}}$. Then the re-scaled equilibrium conditions are

$$\frac{1}{\tilde{c}_t} = \beta \mathbb{E}_t \frac{\tilde{z}_t^{\frac{1}{\alpha-1}}}{\tilde{c}_{t+1}} \left(\alpha \tilde{z}_{t+1} \tilde{k}_t^{\alpha-1} + 1 - \delta \right),$$

$$\tilde{c}_t + \tilde{k}_t \tilde{z}_t^{\frac{1}{1-a}} = \tilde{z}_t \tilde{k}_{t-1}^{\alpha} + (1-\delta) \tilde{k}_{t-1},$$

and,

$$\log \tilde{z}_t = \mu_t + \sigma\varepsilon_t.$$

Substituting the expression for $\tilde{z}_t$, the conditions are then

$$\frac{1}{\tilde{c}_t} = \beta\mathbb{E}_t \frac{1}{\tilde{c}_{t+1}} e^{\frac{\mu_t+\sigma\varepsilon_t}{\alpha-1}} \left(\alpha e^{\mu_{t+1}+\sigma\varepsilon_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - \delta\right)$$

and

$$\tilde{c}_t + \tilde{k}_t e^{\frac{\mu_t+\sigma\varepsilon_t}{1-\alpha}} = e^{\mu_t+\sigma\varepsilon_t} \tilde{k}_{t-1}^{\alpha} + (1-\delta)\,\tilde{k}_{t-1}.$$

Using the notation in Section 2, $x_{t-1} = \tilde{k}_{t-1}$, $y_t = \tilde{c}_t$, and $\theta_t = \theta_{1t} = \mu_t$, so

$$f\left(y_{t+1}, y_t, x_t, x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t\right) =$$
$$\begin{bmatrix} \frac{1}{\tilde{c}_t} - \beta \frac{1}{\tilde{c}_{t+1}} e^{\frac{\mu_t+\sigma\varepsilon_t}{\alpha-1}} \left(\alpha e^{\mu_{t+1}+\chi\sigma\varepsilon_{t+1}} \tilde{k}_t^{\alpha-1} + 1 - \delta\right) \\ \tilde{c}_t + \tilde{k}_t e^{\frac{\mu_t+\varepsilon_t}{1-\alpha}} - e^{\mu_t+\sigma_t\varepsilon_t} \tilde{k}_{t-1}^{\alpha} - (1-\delta)\,\tilde{k}_{t-1} \end{bmatrix}.$$

Clearly,

$$\tilde{c}_t = g\left(\tilde{k}_{t-1}, \varepsilon_t, \chi, s_t\right),$$
$$\tilde{c}_{t+1} = g\left(\tilde{k}_t, \chi\varepsilon_{t+1}, \chi, s_{t+1}\right),$$
$$\tilde{k}_t = h\left(\tilde{k}_{t-1}, \varepsilon_t, \chi, s_t\right),$$

and the Markov Switching parameter is

$$\mu_{t+1} = \mu\left(\chi, s_{t+1}\right) = \overline{\mu} + \chi\widehat{\mu}\left(s_{t+1}\right).$$

5.2 Solving the RBC Model

This subsection shows how to solve the model using a first order approximation. The first step is to find the steady state, and the second is to define the matrices in expression (16) that are necessary to solve for the policy functions. Finally, after solving the model, simulations demonstrate the decision rules.

5.2.1 Steady State

In order to calculate steady state, set $\chi = 0$. Therefore, $\tilde{c}_t = \tilde{c}_{t+1} = \tilde{c}_{ss}$, $\tilde{k}_{t-1} = \tilde{k}_t = \tilde{k}_{ss}$, and $\mu_{t+1} = \mu_t = \bar{\mu}$. So the equilibrium conditions in steady state are

$$\begin{bmatrix} \frac{1}{c_{ss}} - \beta \frac{1}{c_{ss}} e^{\frac{\bar{\mu}}{\alpha-1}} \left(\alpha e^{\bar{\mu}} \tilde{k}_{ss}^{\alpha-1} + 1 - \delta \right) \\ \tilde{c}_{ss} + \tilde{k}_{ss} e^{\frac{\bar{\mu}}{1-\alpha}} - e^{\bar{\mu}} \tilde{k}_{ss}^{\alpha} - (1-\delta) \tilde{k}_{ss} \end{bmatrix} = 0$$

and solve these produces the steady state values

$$\begin{aligned} \tilde{k}_{ss} &= \left(\frac{1}{\alpha e^{\bar{\mu}}} \left(\frac{1}{\beta e^{\frac{\bar{\mu}}{\alpha-1}}} - 1 + \delta \right) \right)^{\frac{1}{\alpha-1}} \\ \tilde{c}_{ss} &= e^{\bar{\mu}} \tilde{k}_{ss}^{\alpha} + (1-\delta) \tilde{k}_{ss} - \tilde{k}_{ss} e^{\frac{\bar{\mu}}{1-\alpha}}. \end{aligned}$$

5.2.2 The Matrices

The next step is to define the matrices in expression (16), which depend on the derivatives of the function f evaluated at the steady state. Recall in this example that $n_y = 1$, $n_x = 1$, $n_\varepsilon = 1$, and $n_\theta = 2$. The necessary matrices are

$$\mathcal{D}_1 f_{ss}(s', s) = \begin{bmatrix} \frac{1}{c_{ss}^2} \\ 0 \end{bmatrix}, \mathcal{D}_2 f_{ss}(s', s) = \begin{bmatrix} -\frac{1}{c_{ss}^2} \\ 1 \end{bmatrix}, \mathcal{D}_3 f_{ss}(s', s) = \begin{bmatrix} (1-\alpha) \alpha \beta e^{\frac{\alpha \bar{\mu}}{\alpha-1}} \frac{k_{ss}^{\alpha-2}}{c_{ss}} \\ e^{\frac{\bar{\mu}}{1-\alpha}} \end{bmatrix}$$

$$\mathcal{D}_4 f_{ss}(s', s) = \begin{bmatrix} 0 \\ -\frac{e^{\frac{\bar{\mu}}{1-\alpha}}}{\beta} \end{bmatrix}, \mathcal{D}_5 f_{ss}(s', s) = \begin{bmatrix} -\alpha \beta e^{\frac{\alpha \bar{\mu}}{\alpha-1}} \frac{k_{ss}^{\alpha-1}}{c_{ss}} \sigma \\ 0 \end{bmatrix},$$

$$\mathcal{D}_6 f_{ss}(s', s) = \begin{bmatrix} \frac{\sigma}{(1-\alpha) c_{ss}} \\ \left(\frac{e^{\frac{\bar{\mu}}{1-\alpha}} k_{ss}}{1-\alpha} - e^{\bar{\mu}} k_{ss}^{\alpha} \right) \sigma \end{bmatrix},$$

$$\mathcal{D}_{7,8} f_{ss}(s', s) = \begin{bmatrix} -\alpha \beta e^{\frac{\bar{\mu} \alpha}{\alpha-1}} \frac{k_{ss}^{\alpha-1}}{c_{ss}} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{D}_{9,10} f_{ss}(s', s) = \begin{bmatrix} \frac{1}{c_{ss}} & 0 \\ -e^{\bar{\mu}} k_{ss}^{\alpha} + \frac{1}{1-\alpha} e^{\frac{\bar{\mu}}{1-\alpha}} k_{ss} & 0 \end{bmatrix}$$

Given these derivatives, constructing the necessary matrices for the solution is straightforward. The first sets of matrices are for the quadratic system, they are

$$A(1) = \begin{bmatrix} \sum_{s'=1}^{2} p_{1,s'} \mathcal{D}_3 f_{ss}(s', 1) & p_{1,1} \mathcal{D}_1 f_{ss}(1,1) & p_{1,2} \mathcal{D}_1 f_{ss}(2,1) \end{bmatrix},$$

$$A(2) = \left[\begin{array}{ccc} \sum_{s'=1}^{2} p_{2,s'}\mathcal{D}_3 f_{ss}(s',2) & p_{2,1}\mathcal{D}_1 f_{ss}(1,2) & p_{2,2}\mathcal{D}_1 f_{ss}(2,2) \end{array} \right],$$

and

$$\begin{aligned} B(1) &= -\sum_{s'=1}^{2} p_{1,s'} \left[\begin{array}{cc} \mathcal{D}_4 f_{ss}(s',1) & \mathcal{D}_2 f_{ss}(s',1) \end{array} \right], \\ B(2) &= -\sum_{s'=1}^{2} p_{2,s'} \left[\begin{array}{cc} \mathcal{D}_4 f_{ss}(s',2) & \mathcal{D}_2 f_{ss}(s',2) \end{array} \right] \end{aligned}$$

The second set of matrices are used for the derivative with respect to ε_t, and they are

$$\Theta_\varepsilon = \sum_{s'=1}^{2} \left[\begin{array}{cc} p_{1,s'}\mathcal{D}_2 f_{ss}(s',1) & 0 \\ 0 & p_{2,s'}\mathcal{D}_2 f_{ss}(s',2) \end{array} \right],$$

$$\Phi_\varepsilon = \sum_{s'=1}^{2} \left[\begin{array}{cc} p_{1,s'}\mathcal{D}_1 f_{ss}(s',1)\,\mathcal{D}_1 g_{ss}(s') + \mathcal{D}_3 f_{ss}(s',1) & 0 \\ 0 & p_{2,s'}\mathcal{D}_1 f_{ss}(s',2)\,\mathcal{D}_1 g_{ss}(s') + \mathcal{D}_3 f_{ss}(s',2) \end{array} \right]$$

and

$$\Psi_\varepsilon = -\sum_{s'=1}^{2} \left[\begin{array}{c} p_{1,s'}\mathcal{D}_6 f_{ss}(s',1) \\ p_{2,s'}\mathcal{D}_6 f_{ss}(s',2) \end{array} \right].$$

The third set of matrices are used for the derivative with respect to χ, and they are

$$\Theta_\chi = \left[\begin{array}{cc} p_{1,1}\mathcal{D}_1 f_{ss}(1,1) + \mathcal{D}_2 f_{ss}(1,1) & p_{1,2}\mathcal{D}_1 f_{ss}(2,1) \\ p_{2,1}\mathcal{D}_1 f_{ss}(1,2) & p_{2,2}\mathcal{D}_1 f_{ss}(2,2) + \mathcal{D}_2 f_{ss}(2,2) \end{array} \right],$$

$$\Phi_\chi = \sum_{s'=1}^{2} \left[\begin{array}{cc} p_{1,s'}\mathcal{D}_1 f_{ss}(s',1)\,\mathcal{D}_1 g_{ss}(s') + p_{1,s'}\mathcal{D}_3 f_{ss}(s',1) & 0 \\ 0 & p_{2,s'}\mathcal{D}_1 f_{ss}(s',2)\,\mathcal{D}_1 g_{ss}(s') + p_{2,s'}\mathcal{D}_3 f_{ss}(s',2) \end{array} \right],$$

and

$$\Psi_\chi = -\sum_{s'=1}^{2} \left[\begin{array}{c} p_{1,s'}\mathcal{D}_{7,8} f_{ss}(s',1)\,\mathcal{D}\theta(0,s') \\ p_{2,s'}\mathcal{D}_{7,8} f_{ss}(s',2)\,\mathcal{D}\theta(0,s') \end{array} \right].$$

5.3 RBC Solution

Now, to describe the solution, first consider the following parameters.

α	β	δ	σ	$\mu(1)$	$\mu(2)$	$p_{1,1}$	$p_{2,2}$
0.33	0.99	0.025	0.001	0.03	0.01	0.90	0.90

The transition matrix implies that regimes 1 and 2 occur with equal frequency in the ergodic distribution, so the steady state depends upon $\bar{\mu} = 0.02$. The steady state values of capital and consumption are $k_{ss} = 11.4572$ and $c_{ss} = 1.64771$. Consequently the numerical values of the derivatives are

$$\mathcal{D}_1 f_{ss}(s', s) = \begin{bmatrix} 0.3683 \\ 0 \end{bmatrix}, \ \mathcal{D}_2 f_{ss}(s', s) = \begin{bmatrix} -0.3683 \\ 1 \end{bmatrix}, \ \mathcal{D}_3 f_{ss}(s', s) = \begin{bmatrix} 0.0022 \\ 1.0303 \end{bmatrix}$$

$$\mathcal{D}_4 f_{ss}(s', s) = \begin{bmatrix} 0 \\ -1.04071 \end{bmatrix}, \ \mathcal{D}_5 f_{ss}(s', s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathcal{D}_6 f_{ss}(s', s) = \begin{bmatrix} 0.000905828 \\ 0.0153372 \end{bmatrix}$$

$$\mathcal{D}_7 f_{ss}(s', s) = \begin{bmatrix} -0.0383185 \\ 0 \end{bmatrix}, \text{ and } \mathcal{D}_8 f_{ss}(s', s) = \begin{bmatrix} 0.905828 \\ 15.3372 \end{bmatrix}.$$

Using the Gröbner basis with respect to the ordering $\{\mathcal{D}_{1,n_x} h_{ss}(1), \mathcal{D}_{1,n_x} h_{ss}(2), \mathcal{D}_{1,n_x} g_{ss}(1), \mathcal{D}_{1,n_x} g_{ss}(2)\}$ produces the following solutions

	$D_{1,n_x} h_{ss}(1)$	$D_{1,n_x} g_{ss}(1)$	$D_{1,n_x} h_{ss}(2)$	$D_{1,n_x} g_{ss}(2)$
1)	1.08526	−0.0774371	1.08526	−0.0774371
2)	0.930745	0.0817605	0.930745	0.0817605
3)	$1.12 - 0.091i$	$-0.113 + 0.093i$	$1.12 - 0.091i$	$-0.113 + 0.093i$
4)	$1.12 + 0.091i$	$-0.113 - 0.093i$	$1.12 + 0.091i$	$-0.113 - 0.093i$

Now, checking these solutions for MSS, the only stable solution is (2). The full solution is then

$$\begin{aligned} s_t &= 1 : \hat{c}_t = 0.0818\hat{k}_{t-1} + 0.0021\varepsilon_t + 0.0375, \hat{k}_t = 0.9307\hat{k}_{t-1} - 0.0318\varepsilon_t - 0.0182 \\ s_t &= 2 : \hat{c}_t = 0.0818\hat{k}_{t-1} + 0.0021\varepsilon_t - 0.0375, k_t = 0.9307\hat{k}_{t-1} - 0.0318\varepsilon_t + 0.0182 \end{aligned}$$

As an alternative parameterization, consider the same parameters above, but with $p_{1,1} = 0.5$. In the ergodic distribution across regimes for this case, regime 1 occurs with probability $\frac{1}{6}$ and regime 2 occurs with probability $\frac{5}{6}$. Then the steady state has $\bar{\mu} = 0.0133$, $c_{ss} = 1.7967$, and $k_{ss} = 14.6326$, and the first order solution is

$$\begin{aligned} s_t &= 1 : \hat{c}_t = 0.0705\hat{k}_{t-1} + 0.0023\varepsilon_t + 0.0294, \hat{k}_t = 0.9410\hat{k}_{t-1} - 0.0411\varepsilon_t - 0.3526 \\ s_t &= 2 : \hat{c}_t = 0.0705\hat{k}_{t-1} + 0.0023\varepsilon_t - 0.0059, \hat{k}_t = 0.9410\hat{k}_{t-1} - 0.0411\varepsilon_t + 0.0705 \end{aligned}$$

There are two important properties of these first order solutions. First, for both the first case with a symmetric transition matrix and the second case with a non-symmetric transition matrix, the slope coefficients of the solutions are identical across regimes. Second, the additional constant term at the end of the solution is non-zero, which shows the non-certainty equivalence of the first-order solution, and its magnitude depends upon the ergodic probabilities. Since the only regime-switching parameter is the level of growth, the only change in the decision rules is through the constant term, which represent deviations from the steady state due to Markov switching. In the symmetric parameterization, each regime occurs with equal probability in the ergodic distribution, so the steady state is exactly between each regime, and hence the deviations are equally above and below. In the non-symmetric transition matrix parameterization, since regime 2 occurs with a higher probability in the ergodic distribution, the additional constants are much smaller for regime 2, demonstrating that the steady state is closer to regime 2.

Figure 1 shows the policy functions for each regime when the transition matrix is symmetric if $\varepsilon_t = 0$, alongside the fixed regime case, which is no Markov switching but with TFP growth always at $\bar{\mu}$. The plot shows how the policy functions with Markov switching have identical slopes to those without switching, but the constant term associated with Markov switching scales the functions up and down. In the case with a symmetric transition matrix and hence equal ergodic probabilities, the fixed regime case lies exactly between the two lines when there is switching.

Figure 2 shows the policy functions for the non-symmetric transition matrix case. Again, this figure shows that the slopes are the same, but the Markov switching rules are scaled up and down by a constant. Since regime 2 occurs with higher probability in the ergodic distribution, the fixed regime policy function is very close to that for regime 2, while regime 1 is farther away.

5.4 RBC Simulations

To illustrate how Markov switching can play a role in growth dynamics, especially through the non-certainty equivalence of the first-order approximation, Figures 3 and 4 show simulation results of the models discussed above and their ergodic distributions. For both the symmetric and non-symmetric transition matrices, there are 1000 simulations of the economy for a length of 10000 periods, excluding the first 1000 to eliminate the effects of initial conditions.

Figure 3 shows the simulated distributions of output and consumption growth for the symmetric transition matrix economy. Recall that in this specification, both regimes are equally highly persistent, so in the ergodic distribution, both occur with equal probability. While the fixed regime case has a single-peaked distribution, thereby exhibiting growth at close to a constant rate, the switching case has a twin-peaked distribution for both variables, one peak associated with each regime. The parameterization for the fixed regime case suggests that its growth rate peak should be halfway between the growth rates of the two regimes, but simulations show that growth is higher on average in the switching case than the fixed regime case. This result follows from the non-certainty equivalence of the solution; when there is switching between high and low growth regimes, agents understand that they will experience both regimes, and, on average, this decision leads to higher consumption and output growth than if there was only a single regime.

Figure 4 shows the simulated distributions of output and consumption growth for the case of the non-symmetric transition matrix. In this case, regime 2 occurs much more often and is more persistent than regime 1, so the ergodic distribution has higher probability on regime 2. Again, the fixed regime case exhibits almost constant growth: the distribution is single-peaked. The Markov switching case, on the other hand, is no longer twin-peaked. In this case, there is one dominant peak of the distribution, which is associated with regime 2, but there are also several other smaller peaks to the distribution that correspond to different histories of the regimes. For example, the large peak is a result of regime 2 occurring approximately 5/6 of the time, but there will be long periods where only regime 2 occurs, and the small left-most peak is associated with these stretches. The other smaller peaks correspond to various lengths of regime 1 occurring, which happens with lower probability. As in the symmetric case, the ergodic mean of growth in the fixed case is lower than when there is switching, again this is because of the lack of certainty equivalence in the two regimes.

6 Example 2: NK Model

This section presents a second example: a simple New Keynesian model to highlight the issue of determinacy and mean square stability.

6.1 The NK Model

The model is a model with quadratic price adjustment costs where the monetary authority follows a Taylor Rule that changes according to a Markov Process. The reaction coefficient of monetary policy switches with the regime, which Davig & Leeper (2007), Farmer *et al.* (2008b), and Bianchi (2009), among others, have argued captures the changing stance of policy in the United States.

A representative consumer maximizes expected lifetime utility over consumption C_t and hours worked H_t

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - H_t\right)$$

subject to the budget constraint

$$C_t + \frac{B_t}{P_t} = W_t H_t + R_{t-1} \frac{B_{t-1}}{P_t} + T_t + D_t$$

where B_t is next period's nominal bonds, W_t is the real wage, R_{t-1} is the nominal return on bonds, T_t is lump-sum transfers, and D_t is profits from firms.

A competitive final goods producer combines a continuum of intermediate goods $Y_{j,t}$ into a final good Y_t by a CES aggregator

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\eta-1}{\eta}} dj\right)^{\frac{\eta}{\eta-1}}$$

Intermediate goods firms take the wage and their demand function

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\eta} Y_t$$

as given and set their price $P_{j,t}$ demand hours $H_{j,t}$ to produce according to

$$Y_{j,t} = A_t H_{j,t}$$

where total factor productivity follows

$$\log A_t = \mu_t + \log A_{t-1}$$

where, similar to the RBC model in Section 5, the drift can take two values

$$\mu_t = \mu(s_t), \; s_t \in \{1,2\}.$$

These firms face quadratic price adjustment costs according to

$$AC_{j,t} = \frac{\kappa}{2}\left(\frac{P_{j,t}}{P_{j,t-1}} - 1\right)^2.$$

The monetary authority sets prices by a Taylor rule where the coefficient varies over time

$$\frac{R_t}{R_{ss}} = \left(\frac{R_{t-1}}{R_{ss}}\right)^{\rho_r} \Pi_t^{(1-\rho_r)\psi_t} \exp(\sigma_r \varepsilon_{r,t})$$

In a symmetric equilibrium $P_{j,t} = P_t$, $Y_{j,t} = Y_t$, and $H_{j,t} = H_t$ for all j, and market clearing implies

$$Y_t = C_t + \frac{\kappa}{2}(\Pi_t - 1)^2 Y_t.$$

Using the notation in Section 2, $y_t = [\Pi_t, Y_t]'$, $x_{t-1} = R_{t-1}$, $\theta_{1t} = \mu_t$, and $\theta_{2t} = \psi_t$. Then the stationary equilibrium is expressed as

$$f(y_{t+1}, y_t, x_t, x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t) =$$
$$\begin{bmatrix} 1 - \beta \frac{\left(1-\frac{\kappa}{2}(\Pi_t-1)^2\right)\tilde{Y}_t}{\left(1-\frac{\kappa}{2}(\Pi_{t+1}-1)^2\right)\tilde{Y}_{t+1}} \frac{1}{\exp\left(\mu_{t+1}\right)} \frac{R_t}{\Pi_{t+1}} \\ (1-\eta) + \eta\left(1 - \frac{\kappa}{2}(\Pi_t - 1)^2\right)\tilde{Y}_t + \beta\kappa \frac{\left(1-\frac{\kappa}{2}(\Pi_t-1)^2\right)}{\left(1-\frac{\kappa}{2}(\Pi_{t+1}-1)^2\right)}(\Pi_{t+1} - 1)\Pi_{t+1} - \kappa(\Pi_t - 1)\Pi_t \\ \left(\frac{R_{t-1}}{R_{ss}}\right)^{\rho} \Pi_t^{(1-\rho)\psi_t} \exp(\sigma\varepsilon_t) - \frac{R_t}{R_{ss}} \end{bmatrix}$$

6.2 NK Solution

The calibration used is as follows

β	κ	η	ρ	σ	$p_{1,1}$	$p_{2,2}$	$\bar{\mu}$	$\mu(1)$	$\mu(2)$	$\psi(1)$	$\psi(2)$
0.99	161	10	0.8	0.0025	0.90	0.90	0.02	0.03	0.01	3.1	0.9

Using (16) and the described calibration produces a quadratic system to be solved to find $\{\mathcal{D}_{1,n_x}g_{ss}(s), \mathcal{D}_{1,n_x}h_{ss}(s)\}_{s=1}^{n_s}$. Using the Gröbner basis with respect to the ordering

$$\left\{\mathcal{D}_{1,n_x}h_{ss}(1), \mathcal{D}_{1,n_x}h_{ss}(2), \mathcal{D}_{1,n_x}g_{ss}(1)', \mathcal{D}_{1,n_x}g_{ss}(2)'\right\}$$

the solutions are

	$D_{1,n_x}h_{ss}(1)$	$D_{1,n_x}g_{ss}(1)'$		$D_{1,n_x}h_{ss}(2)$	$D_{1,n_x}g_{ss}(2)'$	
1)	0.596	-1.892	-0.318	0.700	-2.892	-0.537
2)	0.777	-3.575	-0.035	1.308	-7.266	2.743
3)	0.799	-1.757	-0.002	1.055	1.332	1.375
4)	1.096-0.438i	-0.791+4.136i	0.463-0.685i	1.337+0.0569i	-9.738-1.895i	2.897+0.307i
5)	1.096+0.438i	-0.791-4.136i	0.463+0.685i	1.337-0.0569i	-9.738+1.895i	2.897-0.307i
6)	1.098-0.208i	-0.963+1.862i	0.467-0.325i	1.026-0.019i	0.962+0.738i	1.217-0.104i
7)	1.098+0.208i	-0.963-1.862i	0.467+0.325i	1.026+0.019i	0.962-0.738i	1.217+0.104i
8)	1.240-0.250i	0.756+2.978i	0.688-0.392i	0.752+0.005i	-2.212+0.615i	-0.261+0.025i
9)	1.240+0.250i	0.756-2.978i	0.688+0.392i	0.752-0.005i	-2.212-0.615i	-0.261-0.025i

Checking these solutions for MSS, the first solution is the only stable one. Constructing the full solution produces:

$$\begin{aligned} s_t &= 1: \begin{bmatrix} \hat{R}_t \\ \widehat{\widetilde{Y}}_t \\ \hat{\Pi}_t \end{bmatrix} = \begin{bmatrix} 0.5965 \\ -1.8919 \\ -0.3184 \end{bmatrix} \hat{R}_{t-1} + \begin{bmatrix} 0.0019 \\ -0.0062 \\ -0.0010 \end{bmatrix} \varepsilon_{r,t} + \begin{bmatrix} -0.0014 \\ 0.0250 \\ -0.0022 \end{bmatrix} \\ s_t &= 2: \begin{bmatrix} \hat{R}_t \\ \widehat{\widetilde{Y}}_t \\ \hat{\Pi}_t \end{bmatrix} = \begin{bmatrix} 0.7004 \\ -2.8919 \\ -0.5366 \end{bmatrix} \hat{R}_{t-1} + \begin{bmatrix} 0.0022 \\ -0.0095 \\ -0.0018 \end{bmatrix} \varepsilon_{r,t} + \begin{bmatrix} -0.0043 \\ -0.0724 \\ -0.0230 \end{bmatrix} \end{aligned}$$

where $\widehat{var}_t = var - var_{ss}$ where var_{ss} is the steady state of var for $var \in \left\{R_t, \widetilde{Y}_t, \Pi_t\right\}$.

As an alternative, suppose now that $\psi(2) = 0.7$. There are still nine total solutions, but now there are two stable solutions:

	$D_{1,n_x}h_{ss}(1)$	$D_{1,n_x}g_{ss}(1)'$		$D_{1,n_x}h_{ss}(2)$	$D_{1,n_x}g_{ss}(2)'$	
1)	0.592109	-1.90874	-0.325381	0.713454	-3.14857	-0.599885
2)	0.858767	-1.70451	0.0919787	1.01631	2.13138	1.49934

which shows that this parameterization does not produce a unique MSS solution.

These two parameterizations demonstrate how MSS as a stability concept determines existence and uniqueness of the solution. In the parameterization with $\psi(2) = 0.9$, solving the system produces 9 total solutions, and only satisfies mean square stability. In the case of $\psi(2) = 0.7$, having two MSS solutions implies non-uniqueness of a stable solution. If, on the other hand, there were no MSS solutions, then a stable solution does not exist.

7 Conclusion

This paper developed a perturbation method for constructing approximations to the solutions to Markov switching DSGE models. The framework allows introducing switching from first principles, including when switching affects the steady state of the economy. While not pursued here, second- or higher-order approximations are straightforward, and follow the single-regime case studied by Schmitt-Grohe & Uribe (2004). Using Gröbner bases to solve the system and mean square stability to check for existence and uniqueness of stable solutions, the method handles a wide variety of models, and shows that switching in parameters that affect the steady state implies that first order approximations are not certainty equivalent.

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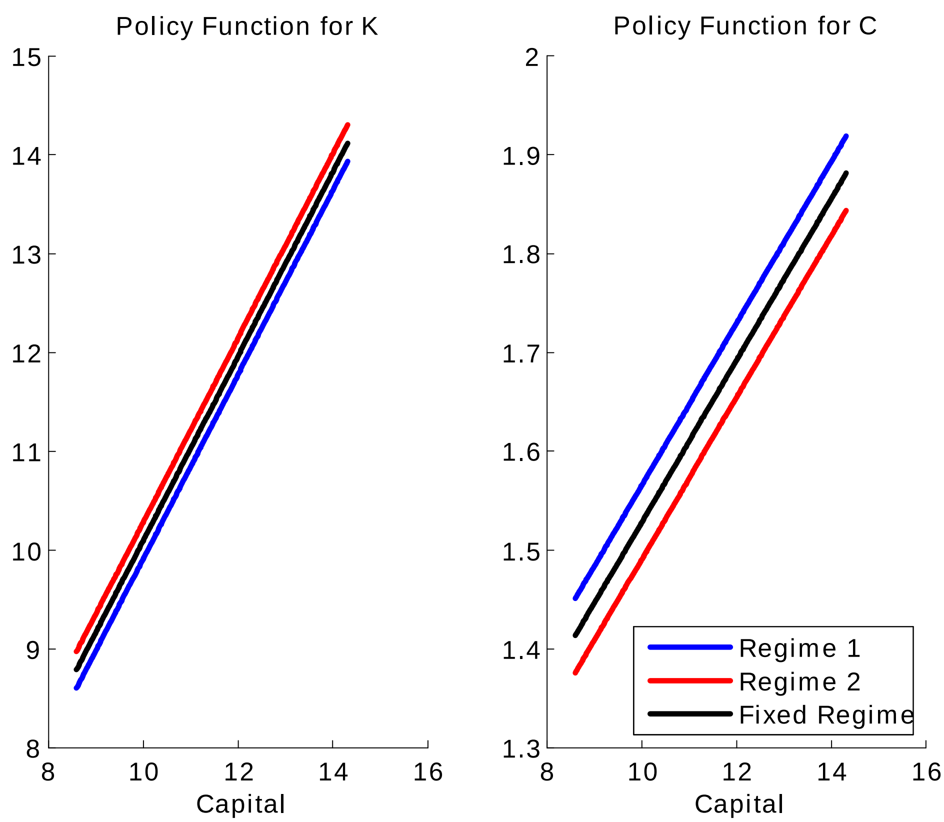

Figure 1: Decision Rules: Symmetric Transition Matrix

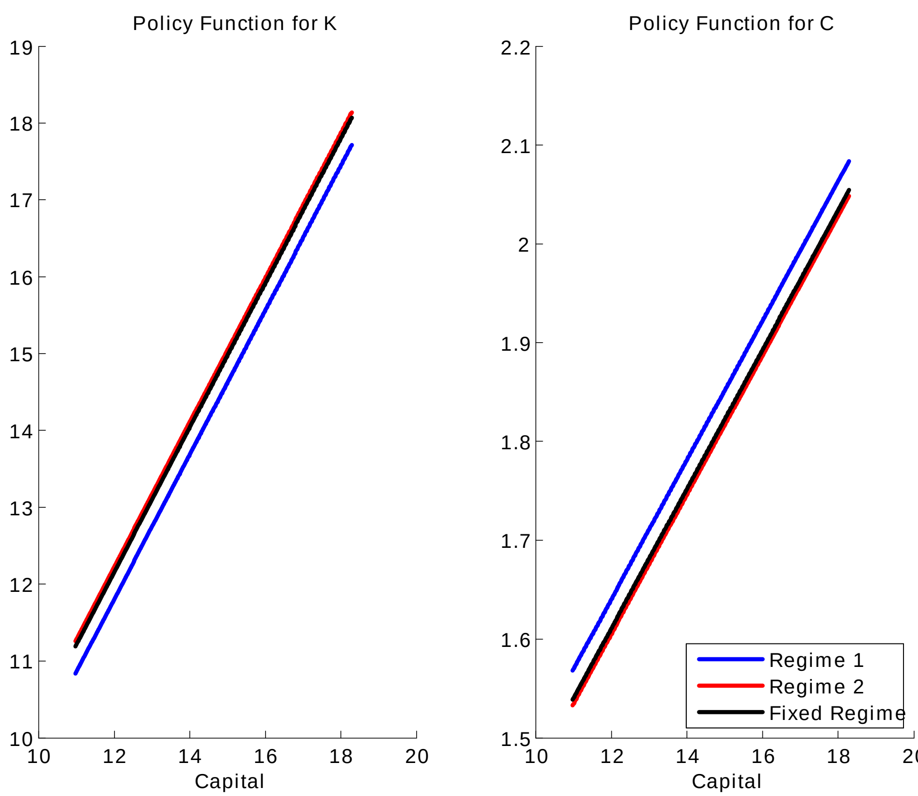

Figure 2: Decision Rules: Non-Symmetric Transition Matrix

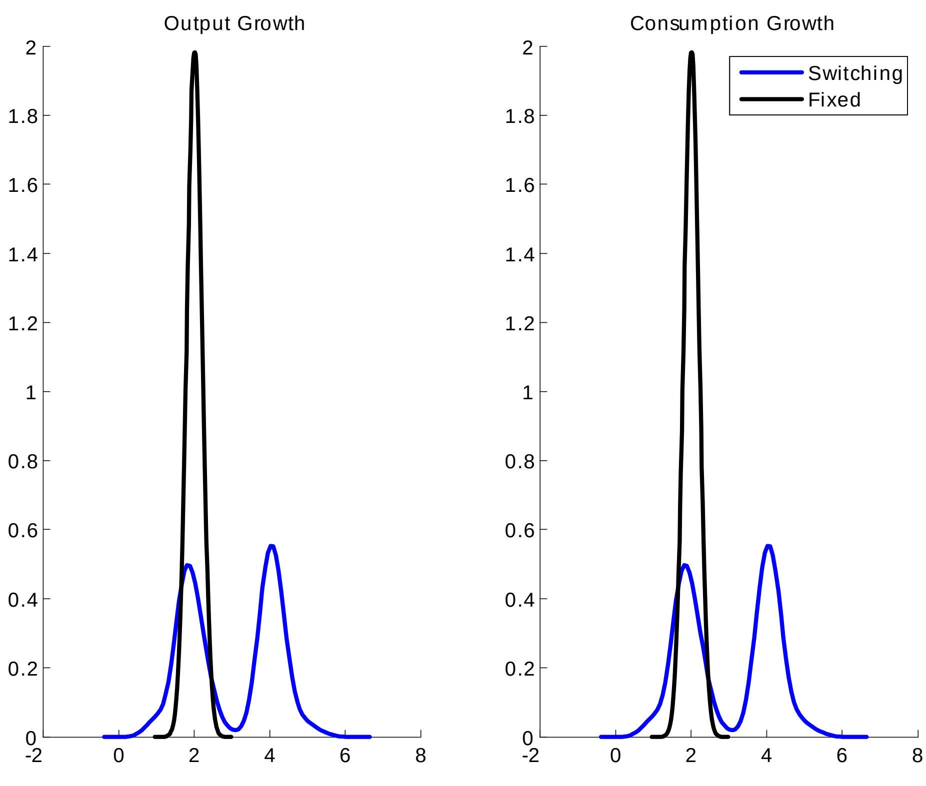

Figure 3: Simulations: Symmetric Transition Matrix

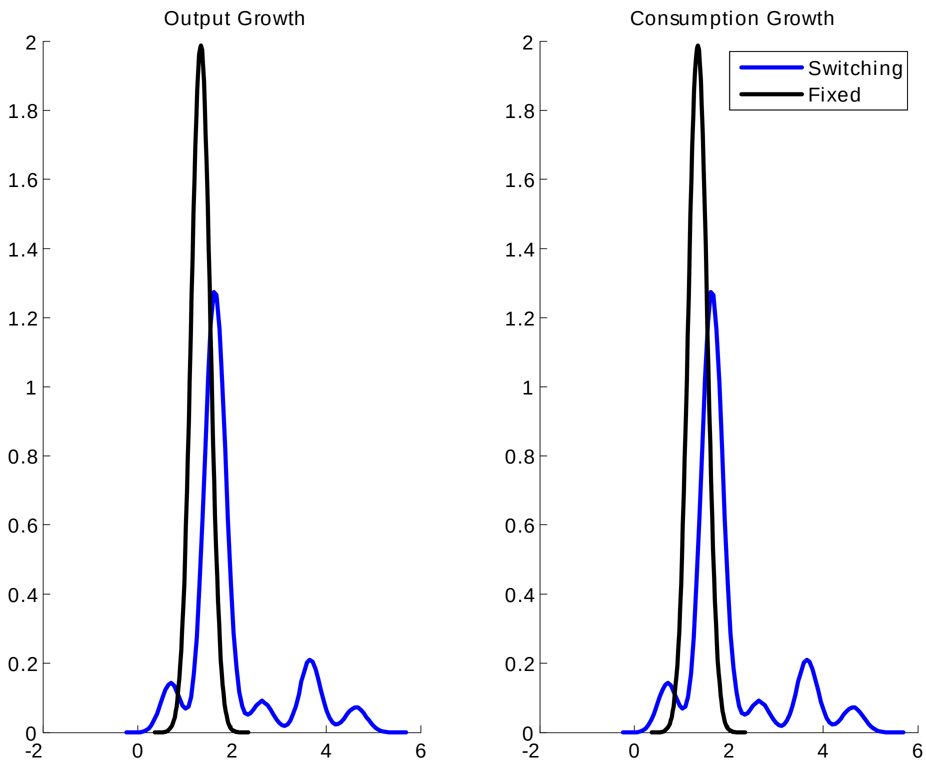

Figure 4: Simulations: Non-Symmetric Transition Matrix