Uncertainty, Productivity and Unemployment in the Great Recession*

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December 19, 2011

Abstract

The 2007-2009 recession has witnessed two phenomena that standard search models of the labor market have difficulty reconciling: a large and persistent increase in unemployment and a sharp rise, above pre-recession levels, in labor productivity following a small initial drop. In addition, these observations were accompanied by a significant increase in the dispersion of firm growth rates. In this paper, I introduce idiosyncratic volatility shocks in search-and-matching models to study if they can account for the patterns observed in US labor markets during the recession. To that end, I develop a tractable dynamic search model of heterogeneous firms with decreasing returns to labor. The model features directed search, allows for endogenous separations, entry and exit of firms, and job-to-job transitions. I show, first, that the model is consistent with a number of observations at the establishment and cross-sectional levels. Second, I find that the introduction of establishment dynamics provides an additional propagation mechanism to the model with productivity shocks only, but cannot account by itself for the total fluctuations in unemployment. Volatility shocks, on the other hand, enhance quite significantly the ability of the model to account for the magnitude of changes observed in output and labor market flows, as well as the coexistence of high unemployment and high productivity. However, the uncertainty induced by the rise in volatility is not sufficient to explain the large persistence in unemployment observed after 2010.

1 Introduction

The recession that followed the 2007-2008 collapse of the financial markets resulted in one of the deepest downturns in post-war U.S labor markets. While GDP contracted by up to 6.8% in the fourth quarter of 2008[1], the unemployment rate grew from 5% in January 2008 to 10.1% in October 2009 according to the Bureau of Labor Statistics. At the same time, labor productivity experienced a small drop, but quicky recovered to a level above its pre-recession trend. While one would expect firms to start hiring again at such productivity levels, the unemployment rate has remained persistently high.

To summarize these facts, I construct detrended time-series of output, unemployment and productivity. The trend is computed with an HP-filter with parameter 1600, as is usually done with

*I am greatly indebted to Nobuhiro Kiyotaki, Esteban Rossi-Hansberg and Oleg Itskhoki for their invaluable advice and support. I also wish to thank Manuel Amador, Yan Bai, Francisco Buera, Pablo Fajgelbaum, Benjamin Moll, Christopher Sims, Mathieu Taschereau-Dumouchel, and participants at the Princeton University Macroeconomics seminar. I gratefully acknowledge financial support from the Center for Economic Policy Studies at Princeton University.

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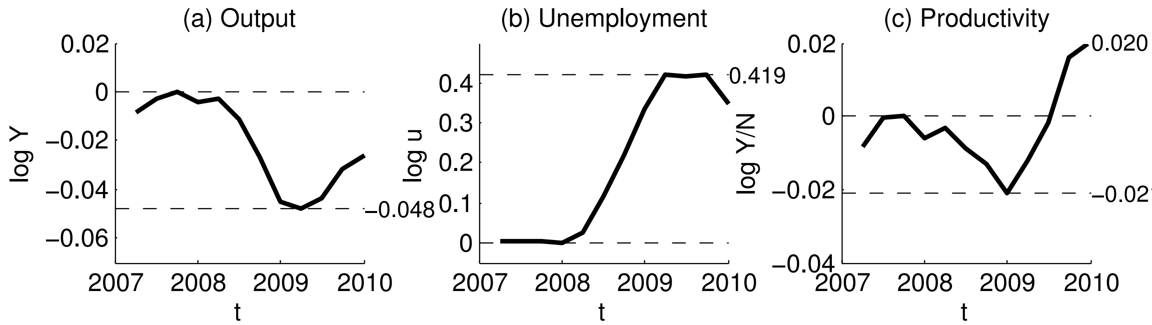
[1] Source: BEA. Percent change in real GDP from preceding period in annual rate.

quarterly data. Figure 1 presents the log of these series, centered at the peak (or trough) preceding the recession. Graph (a) and (b) show the depth of the recession in output and unemployment. Figure (c) presents the productivity of labor measured as output per person. Productivity displays an initial drop of about 2% in the first quarter of 2009, followed by a quick recovery during the rest of the year to a level of 2% above their pre-recession trend in 2010.



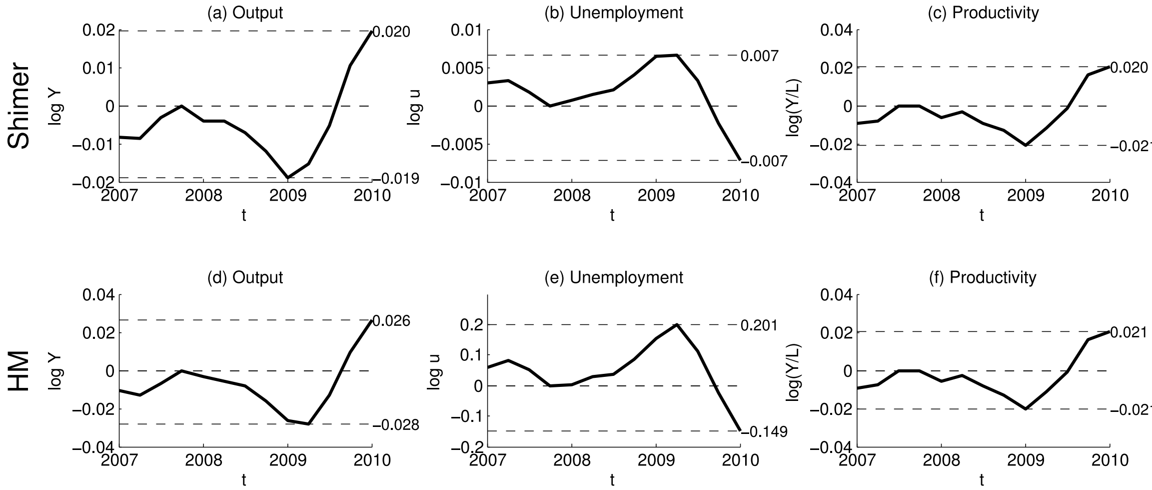
Notes: Data from BEA and BLS. Data is shown in log-deviation from an HP-trend with parameter 1600. Graphs are centered on the peak (trough) preceding the recession.

Figure 1: Output, Unemployment and Productivity in the 2007-2009 Recession

In addition to their welfare implications, these patterns offer a challenge to search-and-matching models of the labor market. First of all, it has been noticed by several authors such as Shimer (2005) and Hall (2005) that standard calibrations of models with matching frictions cannot explain large fluctuations in unemployment. Accounting for the recent increase in the unemployment rate would require a counterfactually large fall of the exogenous component of aggregate productivity, unless one follows the calibration strategy suggested in Hagedorn and Manovskii (2008). But most importantly, a large part of the literature on search and business cycles is based on one underlying shock process in aggregate productivity. In such an environment, recessions are periods of technological regress: it is then difficult to explain the coexistence of high productivity and persistent, high unemployment. To illustrate these results, I present in figure 2 the responses of the Mortensen-Pissarides model in the Shimer (2005) and Hagedorn-Manovskii (2008) calibrations when the aggregate productivity process is calibrated on the empirical time-series of output per person[2]. As mentioned before, the Shimer calibration predicts too little variation in unemployment. The Hagedorn-Manovskii calibration being designed to generate a higher elasticity of unemployment to productivity produces a much larger response (about 50% of the observed increase). However, because of the large recent increase in productivity, the two calibrations predict a counterfactual fall in unemployment from the middle of 2009 to a level much below its pre-recession trend, as well as a counterfactually large recovery in output.

The coexistence of high productivity and high unemployment can suggest that a different kind of shock may be needed to explain the patterns observed in the current recession. A number of commentators have pointed out the fact that volatility and thus uncertainty have largely risen during the recession and could have contributed importantly to the large and persistent rate of unemployment. Bloom et al. (2009) shows evidence that different measures of aggregate and idiosyncratic volatility have sharply increased during the recession. A measure of idiosyncratic volatility at the firm level is the dispersion of firm sales growth rates. I present in figure 3 the distribution of firm sales growth rates taken from Compustat over 2006Q1-2010Q1. The median growth rate dropped significantly between 2008 and 2010, but the situation was quite different across firms. Some firms continued to grow, while some experienced sharp contractions. As a result, the dispersion across firms rose con-

[2] A comparison with Mortensen and Pissarides (1994) with volatility shocks can be found in the appendix.



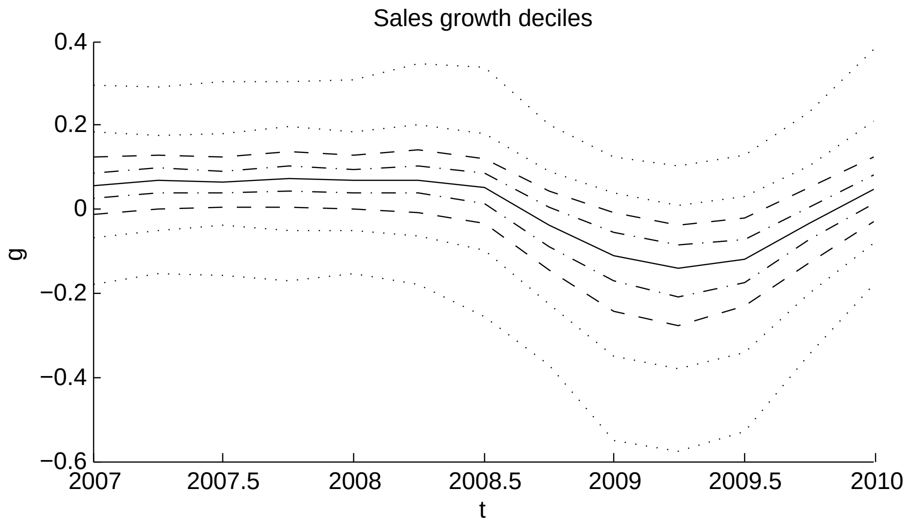
Notes: Log-deviations from initial steady-state. Aggregate productivity shocks calibrated on empirical output per person series.

Figure 2: Predicted paths for output and unemployment in the Shimer (05) and Hagedorn-Manovskii (08) calibrations

siderably over that period. Figure 4 shows two different measures of dispersion, interquartile range and standard deviation. Both display a large increase at the end of 2008. This rise in dispersion can be observed at different levels of aggregation (establishment, firm and industry levels), and in the establishment-level employment growth rates as well. It should also be noted that volatility peaked at the beginning of the 1980s and during the 2001 recession.

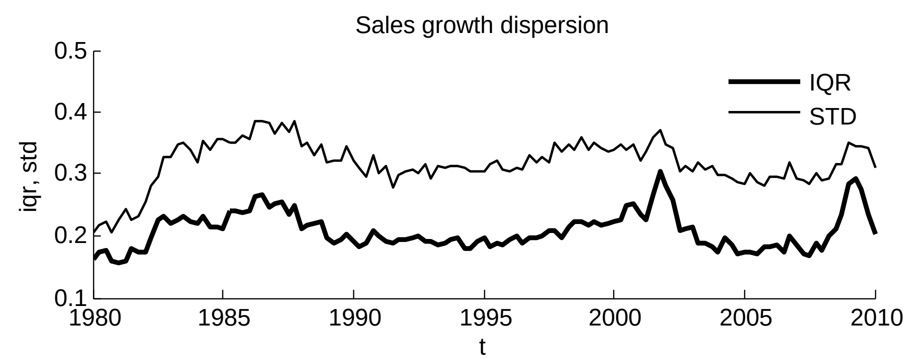
In this paper, I study whether these cross-sectional events can shed some light on the 2007-2009 recession, and, more generally, whether the cross-section of firms matters for business cycles. Volatility shocks, understood as increases in the idiosyncratic productivity of firms, could in principle improve the performance of search-and-matching models along several dimensions. First of all, a greater volatility implies a larger probability for firms to get bad shocks. As a consequence, the number of firm exits should be expected to rise, resulting in more job destructions and a higher inflow of workers into unemployment. Simultaneously, the selection of highly productive firms and the exit of unproductive ones may increase the aggregate measure of productivity. In addition to these effects, volatility also induces higher firm-level uncertainty, which may have a significant impact on the economy. In an economy with labor market frictions, adjustments in a firm's employment are costly. Therefore, in the face of greater uncertainty about future sales prospects, firms may decide to delay or reduce their hirings. Unemployment may rise further as a consequence. At the same time, an additional rise in productivity may be observed because of the decreasing returns to labor as firms reduce their scale of operation. This mechanism depicts how volatility may explain a high level of unemployment along a contraction in output and a high level of productivity.

To investigate the importance of the cross-section of firms and volatility shocks over the business cycle, I introduce the notion of establishment into search models with the assumption of decreasing returns to labor. The model I develop endogenizes a number of dimensions in order to be consistent with some establishment-level observations, and allows for aggregate productivity shocks and time-varying volatility at the firm level. Despite these features, the model retains its tractability, and dynamics can be computed easily. I calibrate the model and show that the introduction of establishment dynamics in the model already provides some additional propagation to the model with productivity shocks only, but cannot account for the total fluctuations in unemployment. Next, I compare the responses of the economy to productivity and volatility shocks. When idiosyncratic volatility increases, the model predicts a substantial rise in unemployment, an increase



Notes: Quarterly sales data from Compustat. Distribution of annual sales growth rates over 1980Q1-2010Q1. The different curves show the deciles of the distribution. The plain curve represents the median growth rate. See data appendix for details.

Figure 3: Distribution of sales growth rates in the 2007-2009 Recession



Notes: Quarterly sales data from Compustat. Interquartile range and standard deviation of annual sales growth rates over 1980Q1-2010Q1. Growth rates are computed with $g_{i,t} = \frac{s_{i,t} - s_{i,t-4}}{1/2(s_{i,t} + s_{i,t-4})}$. Growth rates are detrended by time-industry dummies. See data appendix for details.

Figure 4: Sales growth dispersion over the period 1980-2010

in output per person, and a greater dispersion of firm growth rates, as seen in the data. This does not suffice, however, to account for the contraction in output and initial drop in productivity observed during the 2007-2009 crisis. Motivated by the fact that the distribution of firm growth rates also experienced a downward shift in 2009, I calibrate a combination of productivity and volatility shocks to measure how much of the patterns seen in the data can be accounted by the theory. I find that volatility shocks improve significantly the ability of the model to explain the magnitude of changes in the data. With these shocks, the model can account for a substantial fractions of the variation in output, unemployment and labor market variables. However, firm-level volatility does not seem to contribute to the large persistence of unemployment. I conclude that other sources of uncertainty may be needed to explain this fact. Finally, I turn to previous recessions in which a peak of volatility was observed and find, again, that volatility shocks can improve significantly the explanatory power of search models.

The other main contribution of the paper is to develop a model of firm dynamics and search frictions, that is fully tractable even with aggregate fluctuations. Dynamic models featuring heterogeneous firms usually raise a number of technical issues that make them difficult to solve. The addition of search frictions only complicates the model further. To address this issue, I use the structure of labor markets with directed search developed by Menzio and Shi (2008, 2009) in order to exploit the convenient property of *block recursivity*. I show that, under some conditions, the property continues to hold with the introduction of multiworker firms with decreasing returns. This property allows me to exactly characterize firm and aggregate dynamics. The model features realistic firm dynamics and a rich description of labor markets flows. In the model, heterogeneous firms can endogenously expand/contract, enter/exit over the business cycle. Workers search for new job opportunities both on- and off-the-job, which allows me to distinguish quits from layoffs. I show in section 4, that the model is able to reproduce a range of facts at the establishment and cross-sectional levels. First, it matches a number of features of the micro-level employment policies of establishments in terms of hires, layoffs and quits. It can also replicate the cross-sectional distribution of establishment growth rates as reported in Davis et al. (2006) and Davis et al. (2010). Second, turning to the evolution of the cross-section of firms over the business cycle, I find that the model is able to explain the recent finding that large firms are more cyclically sensitive as reported by Moscarini and Postel-Vinay (2009). Finally, the model has predictions in terms of wages. The presence of on-the-job search leads to a substantial wage dispersion that lies within the range of empirical estimates. The model also produces a realistic size-wage differential.

This paper is related to several strands in the literature. It first relates to the growing literature suggesting uncertainty shocks as a driving force of the business cycles. Bloom (2009) and Bloom et al. (2009) study the effects of time-varying volatility in models of firms with non-convex adjustment costs and show that uncertainty can lead to large drops in economic activity. Arellano et al. (2010) study how financial frictions and uncertainty shocks combine to explain a number of facts related to the current crisis and conclude that uncertainty shocks can explain a substantial fraction of fluctuations in output as well as large movements in the labor wedge. In a slightly different framework, Gilchrist et al. (2010) analyse how time-varying uncertainty and frictions in financial markets in a general equilibrium setting can produce aggregate fluctuations observationally equivalent to TFP-driven cycles. Bachmann and Bayer (2009) shows that realistically calibrated uncertainty shocks do not alter significantly the dynamics of an otherwise standard heterogeneous-firm RBC model. Bachmann et al. (2010) uses business survey data on confidence to analyse the impact of uncertainty on aggregate variables, and finds little evidence of the "wait-and-see" effect highlighted in the literature. From a different perspective, my paper investigates the role of labor market imperfections in explaining the patterns observed in the data, and shows that volatility shocks in a model with search frictions can help us account for a significant fraction of the recent trends in unemployment, produtivity and a range of labor market variables.

This paper also relates to the recent strand in the literature that has seeked to introduce multiworker firms with decreasing returns to scale in search-and-matching models. Acemoglu and Hawkins (2010) and Elsby and Michaels (2010) extend the Mortensen-Pissarides model to firms with decreasing returns and Stole and Zwiebel bargaining. Acemoglu and Hawkins (2010) emphasizes the time-consuming aspect of matching to generate persistence in unemployment. Elsby and Michaels (2010) shows that the gap between average and marginal products of labor resulting from the decreasing returns allows a reasonable calibration of the model to generate large fluctuations in unemployment and vacancies. However, the dynamics are sometimes intractable and their resolution may require the use of approximation methods. My paper explores another approach in which dynamics are easily solvable. This tractability enables me to enrich the model further by adding job-to-job transitions and endogenous firm entry/exit, which play an important role in business cycles. Kaas and Kircher (2010) develops a model which exploits ideas similar to the ones used in this paper, but technically different. Addressing the question of efficiency of search models with large firms, they build a model in which firms also offer long-term contracts and use a device similar to block-recursivity for tractability. Block recursivity usually requires some indifference condition on either side of the labor market. Kaas and Kircher (2010) imposes this indifference condition on the worker side by assuming that workers are homogeneous and cannot search on the job. As a result, firms are not indifferent between contracts, and their model can replicate the empirical fact that growing firms have also higher job-filling rates. They cannot, however, address issues related to job-to-job transitions, which have very specific cyclical properties and account for the largest part of separations. In my model, there is heterogeneity on both sides of the market because workers are allowed to search on the job. Block recursivity still obtains under some conditions with indifference on the firm side. As a consequence, the model cannot explain the differential job filling rate across firms. However, workers are not indifferent between the contracts and the model can replicate some new features of the data, in particular the optimal firm policy in terms of quits and layoffs (figure 10) as evidenced in Davis et al. (2010) and study the dynamics of job-to-job transitions over the business cycle.

2 Model

In order to study the impact of idiosyncratic uncertainty on unemployment and productivity, I build a dynamic search model of heterogeneous firms with decreasing returns. Extending standard search models along this dimension is not trivial and raises a number of difficulties. Allowing firms to differ in size amounts to introduce a new layer of heterogeneity. This significantly complicates the resolution of the model as it raises the dimensionality of the problem and makes the aggregation of the variety of firms' and workers' behaviors more difficult. In this paper, I explore an approach initially suggested by Menzio and Shi (2009). Switching the labor market structure from random search as in Mortensen and Pissarides (1994) to directed search can greatly simplify the resolution of the dynamics. Indeed, the combination of a free-entry condition and a directed search structure delivers the property of *block recursivity*, under which an individual firm's problem can be solved independently from the distribution of employment across other firms. As I explain below, this convenient property keeps the dimensionality of the problem tractable and allows for an exact characterization of the model dynamics. Let me now introduce the model.

2.1 Population and technology

Time is discrete. The economy is populated by a continuum of equally productive workers of mass 1. An unrestricted mass of firms can potentially enter the economy. Firms and workers are risk neutral and share the same discount rate β. Firms all produce an identical homogeneous good. All are subject to the same time-varying aggregate productivity y that takes a finite number of values in $\mathcal{Y} = \{y_{min} < ... < y_{max}\}$ and follows a discrete Markov process with transition matrix $\pi_y(y_{t+1}|y_t)$. In addition, firms differ in their idiosyncratic productivity z, that follows the finite Markov process $\pi_z(z_{t+1}|z_t, s_t)$ in $\mathcal{Z} = \{z_{min} < ... < z_{max}\}$. s_t is a time-varying index of idiosyncratic volatility common to all firms that follows itself a finite Markov process $\pi_s(s_{t+1}|s_t)$ in $\mathcal{S}$. A firm with a mass of n workers operates the production technology

$$e^{y+z}F(n)$$

where F is an increasing concave production function with $F(0) = 0$. Upon entry, firms must pay a sunk entry cost k_e. Also, firms must pay a fixed operating cost k_f every period in order to use the production technology.

2.2 Labor market

Search is directed on the worker and firm sides. Firms post dynamic contracts that guarantee a certain utility x to the workers. Posting is costly and there is a cost c per vacancy. The labor markets are organized in a continuum of submarkets indexed by the utility $x \in [\underline{x}, \overline{x}]$ promised by the firm to the workers. Workers can direct their search and choose in which submarket to look for a job. Each submarket is characterized by its tightness $\theta(x) = v(x)/u(x)$, where $v(x)$ stands for the number of vacancies posted on submarket x and $u(x)$ the corresponding number of searching workers. On a submarket with tightness θ, workers find jobs with probability $p(\theta)$, while firms find candidates with probability $q(\theta) = p(\theta)/\theta$. I allow workers to search on-the-job, but at a reduced efficiency. Denoting λ the relative search efficiency of the employed compared to the unemployed, the job finding probability of employed workers is $\lambda p(\theta)$.

Firms are assumed to post a mass of vacancies so that a law of large number applies and there is no uncertainty on the number of workers they recruit. Figure 5 describes the functioning of the labor market. Notice that the market tightness $\theta(x)$ is the equilibrium variable that adjusts to guarantee that markets clear.

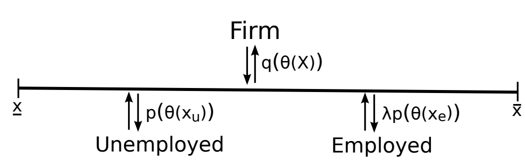


Figure 5: Directed search

2.3 Contracting and timing

The introduction of multiworker firms into search models raises important questions about the wage determination process. Most of the recent literature with large firms has used the multilateral bargaining procedure proposed by Stole and Zwiebel (1996). This wage determination process nicely extends the standard Nash bargaining procedure to firms with multiple workers, but loses its property of efficiency. In this paper, I opt for the less explored alternative of dynamic contracts. On top of lending themselves very naturally to a directed search setting, dynamic contracts lead to an efficient outcome, which considerably simplifies the resolution of the model as I can focus on a planner's allocation.

To simplify the exposition, assume for now that contracts are complete, and that there is full commitment from both worker and firm sides. In particular, this means that a contract can specify any possible variable or action taken by one party in every possible state of the economy. A contract specifies $\{w_{t+s}, \tau_{t+s}, x_{t+s}, d_{t+s}\}_{s=0}^{\infty}$, where w is a wage, τ a firing probability, x the submarket that the worker visits on-the-job, and d an exit dummy for the firm. Each element is contingent on the current state of the firm and the economy. I maintain the assumptions of completeness and full commitment throughout this section, but will however show in section 3 that completeness and commitment from the worker side can be relaxed, as the optimal allocation can be implemented by a contract that satisfies the worker's participation and incentive constraints.

The timing is illustrated in Figure 6. At date t, the shocks to aggregate productivity y and volatility s are revealed. Potentially entering firms decide whether or not to enter. Immediately after, each firm learns its idiosyncratic productivity z. Firms then decide whether to exit ($d_t = 1$) or stay. Separations (layoffs) at probability τ_t then take place. Search and matching between new and incumbent firms on one side and unemployed/employed workers on the other side follows, before production takes place.

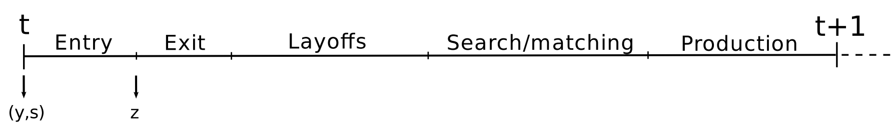


Figure 6: Timing

2.4 Worker's problem

As mentioned earlier, the introduction of firms with decreasing returns in a search model raises the dimensionality of workers' and firms' problems. In principle, the aggregate state variables should include the current aggregate productivity y, idiosyncratic volatility s as well as the distribution of employment across firms $g(z,n)$. Keeping track of g in the state-space is particularly problematic because of its infinite dimension. Fortunately, the structure of the model gives rise to the property of *block recursivity*, in which firms' and workers' problems are independent of distribution g.[3] In that case, the aggregate state space reduces to variables (y,s). I focus the analysis on block-recursive equilibria from now on and will examine in the next section under what conditions such a property arises.

Let me now introduce the worker's problem. In what follows, all value functions are expressed at the time just before production takes place. As a convention, hatted variables denote next period's values.

While unemployed, workers enjoy a utility b from home production or leisure. Unemployed workers choose to visit the submarket $\hat{x}_u$ that offers the best balance between the utility they can get and the probability at which they can find a job. They solve the following problem:

$$\mathbf{U}(y,s) = \max_{\hat{x}_u(\hat{y},\hat{s})} b + \beta \mathrm{E}_{\hat{y},\hat{s}} \Big\{ \big(1 - p(\theta(\hat{x}_u))\big)\mathbf{U}(\hat{y},\hat{s}) + p(\theta(\hat{x}_u))\hat{x}_u \Big\}$$

[3] Another possibility explored in the literature is to use limited-information techniques as introduced by Krusell and Smith (1998) in which distribution g is approximated by a series of moments. Such methods render the analysis tractable, but are still subject to limitations. First, the number of moments needed to approximate the employment distribution may still be large and require intensive computations. Second, it is sometimes difficult to evaluate the accuracy of the approximation.

Abusing notation slightly, denote $p(x) \equiv p(\theta(x,y,s))$. The problem can be rewritten as:

$$\mathbf{U}(y,s) = b + \beta \mathrm{E}_{\hat{y},\hat{s}}\Big\{\mathbf{U}(\hat{y},\hat{s}) + \max_{\hat{x}_u(\hat{y},\hat{s})} p(\hat{x}_u)(\hat{x}_u - \mathbf{U}(\hat{y},\hat{s}))\Big\} \tag{1}$$

Let me turn now to employed workers. The contracts are written in their recursive form. In each period, a contract specifies $\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\}$, where the hatted variables implicitly depend on next period's state $(\hat{y},\hat{s},\hat{z})$. $\hat{W}$ stands for the promised utility that the firm guarantees to the worker in the next period. The utility of a worker employed in a firm with productivity z is:

$$\begin{aligned}\mathbf{W}(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\}) = w + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{\hat{d}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{d})\Big(\hat{\tau}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{\tau})\lambda p(\hat{x})\hat{x} \\ + (1-\hat{\tau})(1-\lambda p(\hat{x}))\hat{W}\Big)\Big\}\end{aligned} \tag{2}$$

The first term in the expectation corresponds to the utility of going back to unemployment if the firm exits. The second is the utility the worker receives when laid-off. The third is the utility he enjoys if he manages to find a job in submarket $\hat{x}$. The last term is the continuation utility offered by the firm.

2.5 Firm's problem

Firms hire a continuum of workers with potentially different contracts. Let $\varphi(W)$ denote the cumulative distribution of promised utilities across workers within a given firm. The total mass of workers is $n = \int \mathrm{d}\varphi$. The contracts offered by the firm

$$\{w(W), \hat{\tau}(\hat{y},\hat{s},\hat{z},W), \hat{x}(\hat{y},\hat{s},\hat{z},W), \hat{d}(\hat{y},\hat{s},\hat{z},W), \hat{W}(\hat{y},\hat{s},\hat{z},W)\}$$

are functions of the promised utility W. To simplify notation, the dependence of contracts on $(\hat{y},\hat{s},\hat{z})$ and W should be considered implicit in what follows.

Firms solve the following problem:

$$\begin{aligned}&\mathbf{J}(y,s,z,n,\varphi) \\ &= \max_{\substack{w,\hat{\tau},\hat{x},\hat{d},\\ \hat{W},\hat{v},\hat{X}}} \; \mathrm{e}^{y+z}F(n) - k_f - \int w\mathrm{d}\varphi + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{(1-\hat{d})(-c\hat{v} + \mathbf{J}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi}))\Big\}\end{aligned} \tag{3}$$

$$\text{s.t. } \forall W, \quad \mathbf{W}\Big(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\}(W)\Big) \geq W \tag{3.a}$$

$$\hat{n} = \int (1-\hat{\tau})(1-\lambda p(\hat{x}))\mathrm{d}\varphi + \hat{v}q(\hat{X}) \tag{3.b}$$

$$\forall W, \hat{\varphi}(W) = \int_{W'|\hat{W}(W') \leq W} (1-\hat{\tau})(1-\lambda p(\hat{x}))\mathrm{d}\varphi(W') + \hat{v}q(\hat{X})\,\mathbb{I}(W \geq \hat{X}) \tag{3.c}$$

where $\hat{v}$ is the number of vacancies posted, $\hat{X}$ is the submarket where the firm searches for new hires[4], and $\mathbb{I}$ is an indicator function. In the current period, the firm earns revenue from production minus the fixed cost k_f and wage bill $\int w\mathrm{d}\varphi$. Next period, the firm stays ($\hat{d} = 0$), or exits ($\hat{d} = 1$), in which case it receives 0. In that case, it pays the vacancy posting cost $c\hat{v}$ for its newly hired workers and goes to the next period. Constraint (3.a) is the *promise-keeping* constraint, ensuring that the firm gives the promised utility to each of its workers. Constraints (3.b) and (3.c) are the laws of motion for the total mass of workers and the corresponding distribution of promised utilities, including newly employed workers on market $\hat{X}$. Notice that for a contract to be optimal, the promise-keeping constraint (3.a) must bind: it is always possible for the firm to reduce the current wage to increase its own profit until the constraint binds.

[4] I implicitly impose here that a firm can only hire in one single market in order to simplify the notation. We will see later that this is no restriction as firms are indifferent between all active submarkets.

2.6 Joint surplus maximization

The firm's problem as stated above is a complicated mathematical problem. It is however possible to greatly simplify it thanks to the contract structure. Indeed, the chosen contracts always maximize the joint surplus of a firm and its workers.

Define the following Bellman equation for the joint surplus of a firm and its current workers by:

$$\begin{aligned}\mathbf{V}(y,s,z,n,\varphi) = \max_{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}} & \mathrm{e}^{y+z}F(n) - k_f + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{ n\hat{d}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{d})\Big(\mathbf{U}(\hat{y},\hat{s})\int \hat{\tau}\mathrm{d}\varphi \\ & + \int (1-\hat{\tau})\lambda p(\hat{x})\hat{x}\mathrm{d}\varphi + \mathbf{V}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi}) - \big(c+q(\hat{X})\hat{X}\big)\hat{v}\Big)\Big\} \\ \text{s.t.} & \begin{cases} \hat{n} = \int (1-\hat{\tau})(1-\lambda p(\hat{x})\mathrm{d}\varphi + \hat{v}q(\hat{X}) \\ \hat{\varphi}(W) = \int_{W'|\hat{W}(W')\leq W} (1-\lambda p(\hat{x}))(1-\hat{\tau})\mathrm{d}\varphi(W') + \hat{v}q(\hat{X})\mathbb{I}(W \geq \hat{X}). \end{cases}\end{aligned} \tag{4}$$

The first term in the joint surplus V corresponds to the present production minus fixed cost. The expectation term includes tomorrow's utility for workers going back to unemployment in case of exit, the utility of laid-off workers, and the utility of workers leaving for another job. The last two terms are next period's surplus net of the utility promised to new hires and vacancy costs. It should be noted that wages cancel out in this expression as they amount to internal transfers between the firm and its employees. In this joint-surplus maximization problem, we also abstract from contracts and promise-keeping constraints because we are looking for the first best allocation. The following proposition states the equivalence between the firm's problem and the joint surplus maximization.

Proposition 1. *The firm's problem and joint surplus maximization are equivalent in the following sense:*

(i) The joint surplus $\mathbf{V}(y,s,z,n,\varphi) \equiv \mathbf{J}(y,s,z,n,\varphi) + \int W\mathrm{d}\varphi(W)$ *solves Bellman equation* (4)*,*

(ii) An optimal policy for the firm $\{w^F,\hat{\tau}^F,\hat{x}^F,\hat{d}^F,\hat{W}^F,\hat{v}^F,\hat{X}^F\}$ *maximizes the joint surplus,*

(iii) Conversely, if $\{\hat{\tau}^S,\hat{x}^S,\hat{d}^S,\hat{W}^S,\hat{v}^S,\hat{X}^S\}$ *maximizes the joint surplus, there exists a unique wage schedule* $w^S(\cdot)$ *such that* $\{w^S,\hat{\tau}^S,\hat{x}^S,\hat{d}^S,\hat{W}^S,\hat{v}^S,\hat{X}^S\}$ *solves the firm's problem.*

Proposition 1 tells us that it is possible to solve first for the real variables ($\hat{\tau}$, $\hat{x}$, $\hat{d}$, $\hat{v}$, $\hat{X}$) by finding the efficient allocation that maximizes the joint surplus of the firm and its workers. As a consequence, there is no need to solve for the particular contracts that implement the allocation if we are only interested in worker and job flows. The promise-keeping constraint can be ignored for that matter. From (iii), it is indeed always possible to adjust the wage so that the promise-keeping constraint is satisfied and the firm's profits maximized.

In its current formulation, the problem can be further simplified. Notice that since the promise-keeping constraint has disappeared, the joint surplus does not depend on distribution φ. In equation (4), φ only appears when we sum over workers. The exact nature of the distribution has no effect on the surplus. In particular, it is possible to find a solution to Bellman equation (4) that does not depend on φ. I will focus on such equilibria from now on. The next section will show that such equilibria are the only solution when a block-recursive competitive equilibrium with positive entry exists. In that case, the firm's profit is equal to

$$\mathbf{J}(y,s,z,n,\varphi) = \mathbf{V}(y,s,z,n) - \int W\mathrm{d}\varphi.$$

2.7 Free entry

Every period after the aggregate shocks (y,s) are realized, new firms are allowed to enter the economy. Firms first decide whether or not to enter. Upon entry, they must pay an entry cost k_e. After entry, an idiosyncratic productivity z is drawn from distribution g_z. Depending of the outcome, firms then decide to exit, in which case they get 0, or to stay and choose a number of vacancies $v(y,s,z)$ and a submarket $X(y,s,z)$ to hire their workers. The ex-post discounted profits of a firm of type z choosing submarket X and a number of workers $n = vq(\theta)$ to hire is

$$\begin{aligned}\mathbf{J}\Big(y,s,z,n,\varphi\Big) - cv &= \mathbf{V}\Big(y,s,z,n\Big) - Xn - cv \\ &= \mathbf{V}\Big(y,s,z,n\Big) - \underbrace{\left(X + \frac{c}{q(\theta(X,y,s))}\right)}_{\text{hiring cost per worker}} n.\end{aligned}$$

Notice that submarket X only enters the firm's entry problem through the hiring cost per worker. Entering firms therefore choose the submarkets that minimize this hiring cost. Define the minimal hiring cost κ as

$$\kappa(y,s) = \min_{\underline{x} \le X \le \bar{x}} X + \frac{c}{q(\theta(X,y,s))}. \tag{5}$$

In equilibrium, active submarkets will have the same hiring cost $\kappa(y,s)$ and firms will be indifferent between them.

The full free-entry condition can be written as:

$$k_e = \max_{n(z)} \mathrm{E}_{g_z} \Bigg(\mathbf{V}(y,s,z,n) - \kappa(y,s)n \Bigg)^+ \tag{6}$$

with the corresponding complementary slackness condition

$$\forall X, \quad \theta(X) \left[\max_{n(z)} \mathrm{E}_{g_z} \left(\mathbf{V}(y,s,z,n) - \left(X + \frac{c}{q(\theta(X,y,s))} \right) n \right)^+ - k_e \right] = 0 \tag{7}$$

where the $(\cdot)^+$ notation stands for $\max(\cdot,0)$.

Condition (6) simply states that new firms enter the economy as long as expected profits exceed the entry cost k_e, driving these profits down to k_e. Condition (7) is essential as it pins down the equilibrium market tightness θ for all submarkets. It imposes that if the submarket is active, then $\theta > 0$ and the expected profits equal k_e. If the submarket is inactive, then $\theta = 0$ and the entry cost can exceed the anticipated profits.

The free-entry condition is crucial to guarantee the existence of a block-recursive equilibrium. Indeed, as was said earlier, the distribution of firms $g_t(n,z)$ with employment n and productivity z could in principle be part of the state-space of agents as they need to forecast the tightness of the different submarkets. This could make the computation of the equilibrium intractable because of high dimensionality. Without free-entry, agents need to know this distribution to forecast how many workers and firms apply to each market. With free-entry, agents understand that entering firms act as a buffer: they do not need to know the exact number of firms on each market, since they understand that there will always be enough entry to bring the tightness to its 'block recursive' level.

In modelling terms, the free-entry condition (7) makes the equilibrium labor market tightness $\theta(x,y,s)$ a function of the joint surplus. If the joint surplus V does not depend on distribution g, then θ does not either. Similarly, since distribution g only enters the joint surplus maximization problem through its potential effect on θ, it is possible to find a *block-recursive* solution in which neither θ, nor V depend on the distribution of employment across firms. This is what makes positive entry a key condition to satisfy for a block-recursive equilibrium to exist. As Proposition 2 will show in the next section, there always exists a *block-recursive* solution to equations (1)-(7) in which the worker-firm joint surplus does not depend on the employment distribution g. For this result to obtain in the presence of heterogeneity on both sides of the market, it is crucial that the free-entry condition can be decomposed into, first, a cost minimization problem and, second, to the entry cost equalization. As long as the hiring cost is the same across firms, though they may differ in productivity, block recursivity may obtain. Key assumptions for this result are the transferability of utility and a law of large numbers for workers at the firm level.

2.8 Unemployment and firm distribution dynamics

Although firm behavior does not depend on the aggregate distribution of employment in a block-recursive equilibrium, the unemployment level and distribution $g_t(z,n)$ follow their own non-trivial dynamics that can easily be computed recursively. Their laws of motion are given by

$$u_{t+1} = u_t(1-p(x_{u,t+1})) + \int n_t \left(d_{t+1} + (1-d_{t+1})\tau_{t+1}\right) g_t(z_t,n_t)\pi_z(z_t,z_{t+1})\mathrm{d}z_t\mathrm{d}z_{t+1}\mathrm{d}n_t \tag{8}$$

and

$$g_{t+1}(z_{t+1},n_{t+1}) = \int_{n_{t+1}=\hat{n}(y_{t+1},s_{t+1},z_{t+1},n_t)} (1-d_{t+1})g_t(z_t,n_t)\pi_z(z_t,z_{t+1})\mathrm{d}z_t\mathrm{d}z_{t+1}\mathrm{d}n_t + \underbrace{h(y_{t+1},s_{t+1},z_{t+1},n_{t+1},g_t)}_{\text{new entrants}} \tag{9}$$

where the number of new entrants h can be computed from period to period as the residual that makes the equilibrium market tightness θ equal to the ratio of vacancies over searching workers.

3 Equilibrium properties

I characterize in this section a few properties of the competitive equilibrium. I first establish the existence of a block-recursive solution to equations (1)-(7) under some weak conditions, which provides us with a well-defined block-recursive competitive equilibrium as long as it implies positive entry in every period. I next show that when they exist, such equilibria coincide with the unique efficient equilibrium of the economy. I then describe some features of firms' optimal behavior such as the existence of a band of inactivity, a characteristic that the model shares with the adjustment cost literature. In the last part, I relax the assumption of contract completeness and commitment from workers to show that the optimal allocation can be decentralized with contracts that satisfy workers' participation and incentive constraints.

3.1 Existence of a block-recursive solution

A block-recursive equilibrium can be defined as follows.

Definition 1. *A block-recursive competitive equilibrium of this economy is a set of value functions* $\mathbf{V}(y,s,z,n)$, $\mathbf{U}(y,s)$, $\mathbf{J}(y,s,z,n,\varphi)$, $\mathbf{W}(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\})$, *optimal policy functions* $\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}$, *where w^* depends on (y,s,z,W) and all the others on $(y,s,z,\hat{y},\hat{s},\hat{z},W)$, and an equilibrium labor market tightness $\theta^*(x,y,s)$ such that equation* (1)-(7) *are satisfied and equations* (8)-(9) *imply a positive number of entrants in every period.*

To prove the existence of a block-recursive solution, I make the following assumptions:

Assumption 1. *F is bi-Lipschitz continuous, i.e there exists $(\underline{\alpha}_F, \overline{\alpha}_F)$ such that*

$$\forall (n_1,n_2), \quad \underline{\alpha}_F |n_2-n_1| \leq |F(n_2)-F(n_1)| \leq \overline{\alpha}_F |n_2-n_1|$$

Assumption 2. *(i) p,q are twice continuously differentiable, (ii) p is strictly increasing, concave; q is strictly decreasing and convex, (iii) $p(0) = 0$, $q(0) = 1$, (iv) $p \circ q^{-1}$ is strictly concave*[5].

To proceed with the proof, I show that there exists a common solution to the joint surplus maximization, free-entry condition and unemployed workers' problem. This establishes the behavior of variables ($\hat{\tau}$, $\hat{x}$, $\hat{d}$, $\hat{v}$, $\hat{X}$) without the need to describe the set of contracts that implement the efficient allocation. I will show in subsections 3.3 and 3.4 how these contracts can be easily recovered from the firm's optimal behavior. Let us first define the set where our optimal surplus $\mathbf{V}$ lies and introduce our last assumption on parameters. Let n be the firm size and $\overline{n}$ an upper-bound chosen sufficiently large so that it does not constrain the equilibrium.

Definition 2. *Let $\mathcal{V}$ be the set of value functions $V : (y,s,z,n) \in \mathcal{Y} \times \mathcal{S} \times \mathcal{Z} \times [0,\overline{n}] \longrightarrow \mathbb{R}$ strictly increasing in n, satisfying $\forall (y,s), \mathrm{E}_{g_z} V(y,s,z,0)^+ \leq \beta k_e$*[6]*, bounded in $[\underline{V},\overline{V}]$, and bi-Lipschitz continuous in n such that*

$$\forall V \in \mathcal{V}, \forall (y,s,z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V (n_2-n_1) \leq V(y,s,z,n_2) - V(y,s,z,n_1) \leq \overline{\alpha}_V (n_2-n_1),$$

with the constants $\underline{\alpha}_V$, $\overline{\alpha}_V$, $\underline{V}$ and $\overline{V}$ defined in the appendix.

Assumption 3. *Assume $\overline{n} > \underline{\alpha}_V^{-1}(k_e+k_f)$.*

Assumption 3 is a sufficient condition on parameters that guarantees that there is always a solution to the free-entry problem. We can now establish the existence of a solution to the free-entry problem.

Lemma 1. *Under Assumptions 1-3, for $V \in \mathcal{V}$, $y \in \mathcal{Y}$, the free-entry problem* (5)-(7) *admits a solution with equality. The hiring cost per worker $\kappa(y,s) = c/q(\theta(X,y,s)) + X$ is equalized across active submarkets and there is and an optimal level of hiring for entering firms $n_e^V(y,s,z)$ such that*

(i) Submarket X is active $\Rightarrow \theta^V(X,y,s) > 0 \Rightarrow c/q(\theta(X,y,s)) + X = \kappa^V(y,s)$,

(ii) $k_e = \max_{n(y,s,z)} \mathrm{E}_{g_z}[V(y,s,z,n) - \kappa^V n]^+ = \mathrm{E}_{g_z}[V(y,s,z,n_e^V) - \kappa^V n_e^V]^+$,

(iii) $\theta^V(X,y,s) = \begin{cases} q^{-1}\left(\frac{c}{\kappa^V(y,s)-X}\right), & \text{for } \underline{x} \leq X \leq \kappa^V(y,s) - c, \\ 0, & \text{for } X \geq \kappa^V(y,s) - c. \end{cases}$

This lemma tells us that there is a continuum of active submarkets for $x \in [\underline{x}, \kappa^V(y,s) - c]$ in which firms can enter. It is important to notice that the free-entry problem amounts to an indifference between active submarkets from the point of view of firms. This results from the fact that the firm's decision to apply in one submarket rather than another is equivalent to the minimization of a hiring cost per worker. As a consequence, this hiring cost $\kappa(y,s)$ must be identical across active

[5] (iv) is a regularity condition ensuring that workers' problem is well defined and concave.

[6] This condition guarantees the existence of a solution to the free-entry problem.

submarkets. It can be decomposed between the expected utility promised to the worker X and the total vacancy cost to attract one worker $c/q(\theta(X,y,s))$. Figure 7 displays how the market tightness evolves with X. We see that markets that offer a low utility to the workers have a higher tightness: as it is less costly for them in terms of utility-wage, firms keep entering these markets until the probability to find a worker becomes so low that they would actually prefer to offer higher wages. Another point worth noticing is that the hiring cost is the same for both incumbent and entering firms: the indifference result applies to all.

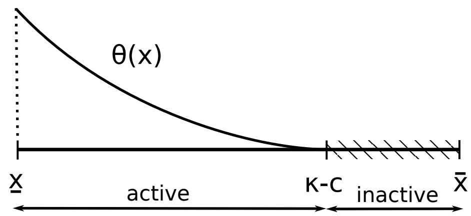


Figure 7: Equilibrium market tightness

We can now turn to the main proposition of this section that establishes the existence of a block-recursive solution.

Proposition 2. *Under Assumptions 1-3, there exists a block-recursive solution to equations* (1)-(7)*, i.e the mapping $T : \mathcal{V} \longrightarrow \mathcal{V}$ such that*

$$TV(y,s,z,n) = \max_{\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{n}_i,\hat{X}} e^{y+z}F(n) - k_f + \beta E_{\hat{y},\hat{s},\hat{z}} \Bigg\{ n\hat{d}\mathbf{U}^V(\hat{y},\hat{s}) + (1-\hat{d})\Big(\mathbf{U}^V(\hat{y},\hat{s})\int \hat{\tau}\mathrm{d}j$$
$$+ \int (1-\hat{\tau})\lambda p^V(\hat{x})\hat{x}\mathrm{d}j - \kappa^V(\hat{y},\hat{s})\hat{n}_i + V(\hat{y},\hat{s},\hat{z},\hat{n})\Big)\Bigg\}$$

with $\hat{n} = \int(1-\hat{\tau})(1-\lambda p^V(\hat{x}))\mathrm{d}j + \hat{n}_i$, (θ^V,κ^V) solution to the free-entry problem (5)-(7) *and $\mathbf{U}^V$ solution to* (1)*, admits a fixed point. The existence of valuations $\mathbf{J}^V$, $\mathbf{W}^V$ and optimal policies $\{\hat{\tau}^V,\hat{x}^V,\hat{d}^V,\hat{W}^V,\hat{n}_i^V,\hat{X}^V\}$ follows.*

Notice that I have substituted without loss of generality the arbitrary distribution φ with a summation on a uniform distribution j for $j \in [0,n]$ as it does not affect the equilibrium.

Proposition 2 shows the existence of a joint solution to the surplus maximization, free-entry and unemployed workers' problems. To be a well-defined competitive block-recursive equilibrium, one must still check one additional feasibility condition: since we have imposed a free-entry condition, we must check that the required number of entrants is always non-negative. Unfortunately, this condition cannot be put into a block-recursive form as the number of entrants depends on the distribution of employment across firms. We can, however, easily check in the simulations whether this condition is satisfied. This provides us with a constructive way to find candidates for block-recursive equilibria.

3.2 Uniqueness and efficiency

To study efficiency, I now introduce the planning problem of this economy. In particular, I show what conditions guarantee the uniqueness and efficiency of a block-recursive equilibrium. Let $h_t(y^t,s^t)$ be the total number of new entrants every period. The planner's objective is to maximize the total welfare in the economy

$$\mathrm{E}_{y,s}\sum_{t}\beta^{t}\Big[\sum_{z_t,n_t} g_t(z_t,n_t)(1-d_t)\big(F(n_t)-k_f-cv_t\big)-k_e h_t+u_t b\Big] \tag{10}$$

which is the discounted sum of production net of operating cost k_f and vacancy posting cost c over all existing firms, minus total entry costs for new firms h_t every period, plus home production b of unemployed agents.

The following proposition establishes the existence and unicity of the planning solution and characterizes when a competitive equilibrium is efficient.

Proposition 3. *(i) There exists a unique solution to the social planner's problem. (ii) If a block-recursive competitive equilibrium with positive entry exists, it coincides with the unique efficient allocation.*

Although a solution to the planning problem always exists, there is no easy way to characterize or compute the optimal allocation in general because of the high dimensionality of the problem. This is where the block-recursivity of the competitive equilibrium is useful, as it provides us with a constructive and tractable way to solve for it. Proposition 3 highlights the importance of the feasibility condition requiring that there is non-negative gross entry in every period, under which a block-recursive solution to equations (1)-(7) is a well-defined competitive equilibrium. It shows, in addition, that when such an equilibrium exists, it coincides with the unique efficient allocation. In contrast, when the positive entry condition is not satisfied, a solution to the planning problem still exists, but no longer satisfies block-recursivity: the analysis loses its tractability. This raises the natural question of how strong and restrictive such a condition is. Data from the U.S and other developped economies shows that this is true empirically: available datasets usually display a positive number of firm entry even during the deepest recessions. The condition of positive entry is therefore no restriction for any empirical application. It would however start to matter if one were to test counterfactual hypotheses or policies faraway from the calibrated economy. Theoretically, one may also wonder whether a block-recursive equilibrium with positive entry always exists. The answer depends on the parameters, but it is always possible to introduce a minimum amount of exogenous exits in the model to make sure that a positive number of firms always enter.

3.3 Optimal firm behavior

Now that the results of existence and efficiency of the equilibrium have been established, we can characterize a few features of the optimal policy for firms in terms of hirings, layoffs, quits and exit. A few analytical results can be established about symmetry and uniqueness of the layoff probability τ, the market for job-to-job transitions x, and wages, but we rapidly face the problem that the model is too complex to be solved analytically. We can however go farther by doing numerical simulations.

In what follows, I simulate the model with some parameters and show in figure 8 how the optimal decision of firms varies in the (z,n)-space in terms of expansion (hirings), contraction (layoffs and quits[7]) and exit.

We see that exits are concentrated at small unproductive firms. Indeed, small firms with low productivity are those for which the current production and expected future surpluses are not enough to cover the operating cost k_f. This is a feature that has also been observed empirically, as evidenced in Evans (1987). Hirings tend to occur at small productive firms, since they have a high marginal product of labor. These firms expand until the marginal product of workers is equalized to the hiring cost $\kappa(y,s)$. In contrast, separations occur mostly at large unproductive firms, because

[7] In the model, all quits are job-to-job transitions.

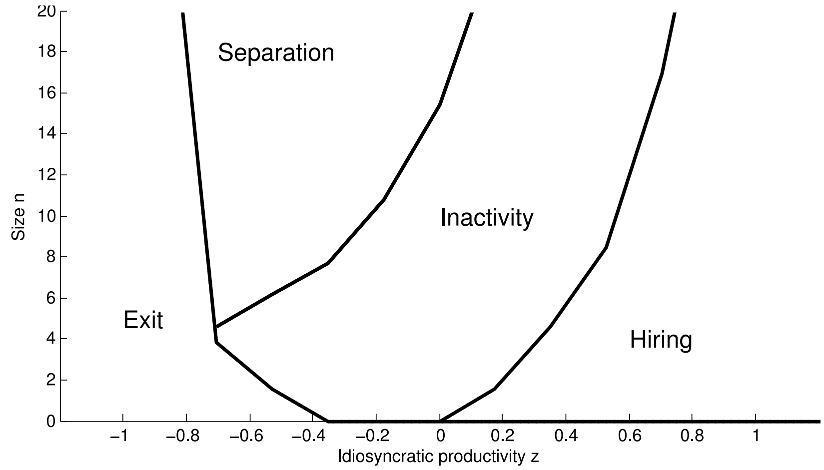


Figure 8: Optimal hirings, quits, layoffs and exit as a function of (z,n)

their marginal product of labor is low. These firms dismiss workers up to the point where their marginal productivity is equal to the utility they could get unemployed, $\mathbf{U}(y,s)$. To proceed with these separations, firms use quits and layoffs. Quits are in general more profitable than layoffs since workers can immediately get a higher utility by avoiding a spell of unemployment. Firms therefore let their workers quit more easily if they do not have to contract too much. They start using layoffs when the job finding probability for quits becomes too low and it is too costly to retain these extra workers. Finally, we see that there is a large band of inactivity, in which firms experience no change in employment. This result is due to the existence of a gap between the utility of unemployment, $\mathbf{U}(y,s)$, and the marginal productivity of workers at hiring firms, which is also equal to the hiring cost $\kappa(y,s)$. As a consequence, there is a non-negligible mass of firms with a marginal product of labor comprised between these two values that do not adjust their employment in every period. This fact has been often documented in the data as in Davis et al. (1996).

After this short discussion of the qualitative features of the optimal firm behavior, I now establish a few analytical results about the contracts offered by firms. In the model, firms are allowed to discriminate among workers by offering them different contracts. I examine in proposition 4 how different elements of the contracts (layoff probability τ, market for on-the-job search x, etc.) vary between workers in a given firm.

Proposition 4. *If an equilibrium joint surplus* $\mathbf{V} \in \mathcal{V}$ *exists with its corresponding optimal contract* $\{w_t, \hat{\tau}_{t+1}, \hat{x}_{t+1}, \hat{d}_{t+1}, \hat{W}_{t+1}\}_{t\geq 0}$*, then:*

(i) If workers can commit, wages are not uniquely determined. In particular, the transformation $\{w+\Delta, \hat{\tau}, \hat{x}, \hat{d}, \hat{W} - a\Delta\}$ *leaves the worker and the firm indifferent, with* $a = [\beta(1-\hat{d})(1-\hat{\tau})(1-\lambda p(\hat{x}))]^{-1}$ *and* $\Delta \in \mathbb{R}$.

(ii) $\hat{x}$ *is identical for all workers in the same firm.*

(iii) only the total number of layoffs $\int \hat{\tau} \mathrm{d}\varphi$ *is determined; the distribution of layoffs* $\hat{\tau}$ *over workers is not.*

Proposition 4 shows that there is a variety of contracts that can implement the optimal allocation. First, if workers can commit, as I have assumed so far, then an infinite number of wage profiles can be used by the firm. This result is due to the risk-neutrality of firms and workers and the fact that utility is transferable. The submarket chosen for on-the-job search x is the same for across workers in a given firm. This result may seem surprising if we think that firms prefer to separate early from their workers with high promised utility. This does not happen here because firms are committed to their contracts: even if they were to dismiss one of these highly paid workers, the firms would have to fulfill their promise and compensate the worker upon dismissal, instead of later, in a way that leaves the firms indifferent. The promise-keeping constraint therefore does not distort the firm's decision in favor of some group of workers. Finally, the individual layoff probability τ is not uniquely determined, but only the total number of layoffs at the firm level ($\int \hat{\tau} d\varphi$) is. Indeed, since all agents value utility in the same way, it is always possible to design compensation schemes to sustain any distribution of layoff probabilities among workers to yield the same utility profile to each of them. I assume a uniform distribution of τ in the quantitative applications.

3.4 Decentralization with incentive compatible contracts

I have assumed so far that contracts are complete and that firms and workers are able to commit. These assumptions may seem somewhat restrictive. I show in this subsection that they can be relaxed along two dimensions. First, the feature that contracts specify $\hat{x}$, the submarket for on-the-job search, seems unrealistic. I show that it is possible for the firm to write contracts specifying only $\{w, \hat{\tau}, \hat{d}, \hat{W}\}$ that induce the worker to choose the right submarket $\hat{x}$. Second, commitment from the worker side can be relaxed. It is indeed possible for the firm to write incentive compatible contracts that satisfy workers participation constraint.

Relaxing these two assumptions reduce to the following additional constraints on the equilibrium contract $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}$. To guarantee that the worker chooses the right submarket, we have the incentive constraint:

$$\begin{aligned} \hat{x}^* &= \underset{\hat{x}}{\text{argmax}}\ \lambda p(\hat{x})\hat{x} + \big(1-\lambda p(\hat{x})\big)\hat{W}^* \\ \Leftrightarrow \hat{x}^* &= \underset{\hat{x}}{\text{argmax}}\ p(\hat{x})(\hat{x}-\hat{W}^*). \end{aligned} \tag{11}$$

Now, to guarantee that the worker never wants to break the contract at the time of separation and leave the company to go back to unemployment, we have the following participation constraint:

$$\lambda p(\hat{x}^*)\hat{x}^* + (1-\lambda p(\hat{x}^*))\hat{W}(\hat{x}^*) \geq \mathbf{U}(\hat{y}, \hat{s}). \tag{12}$$

I now show that for any given contract with $\{\hat{\tau}^*, \hat{x}^*, \hat{d}^*\}$, there is a unique equivalent contract with wage w^{IC} and future utility $\hat{W}^{IC}$ that satisfies the above incentive and participation constraints and delivers the same promised utility to the worker.

Proposition 5. *Given an optimal contract $\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}$, there exists a unique equivalent incentive-compatible contract $\{w^{IC}, \hat{\tau}^{IC}, \hat{d}^{IC}, \hat{W}^{IC}\}$ such that:*

(i) $\hat{\tau}^{IC} = \hat{\tau}^*$ *and* $\hat{d}^{IC} = \hat{d}^*$,

(ii) $\lambda p(\hat{x}^*)\hat{x}^* + (1-\lambda p(\hat{x}^*))\hat{W}^{IC} \geq \mathbf{U}(\hat{y}, \hat{s})$,

(iii) $\hat{x}^* = \underset{\hat{x}}{\text{argmax}}\ p(\hat{x})(\hat{x}-\hat{W}^{IC})$,

(iv) $\mathbf{W}(y,s,z;\{w^*, \hat{\tau}^*, \hat{x}^*, \hat{d}^*, \hat{W}^*\}) = \mathbf{W}(y,s,z;\{w^{IC}, \hat{\tau}^{IC}, \hat{x}^*, \hat{d}^{IC}, \hat{W}^{IC}\})$.

Proposition 5 tells us that the allocation that maximizes the worker-firm joint surplus can be implemented by an incentive-compatible contract. In particular, the layoff and exit probabilities are the same: $\hat{\tau}^{IC} = \hat{\tau}^*$, $\hat{d}^{IC} = \hat{d}^*$, and the submarket $\hat{x}^*$ chosen by the worker coincides with the efficient one. The wage and future utility $(w^{IC}, \hat{W}^{IC})$ are the only elements that adjust to insure that the two additional constraints (11) and (12) are satisfied. In addition to being more realistic than complete contracts with full commitment, these contracts offer the advantage of pinning down wages uniquely. They thus offer an alternative to other bargaining procedures. Moreover, we will see in part 4 that the wages they imply match a number of empirical facts, such as a realistic wage dispersion and size-wage differential.

4 Standard business cycles and cross-sectional properties

In this section, I calibrate the model and evaluate its predictions at the establishment and aggregate levels. I first discuss some establishment-level and cross-sectional properties of the model in terms of growth and wages and show that it can capture a number of stylized facts. Turning to the aggregate level, I present some standard business cycle statistics from the model and compare them to a Mortensen-Pissarides model calibrated as in Shimer (2005). I show that if the introduction of firm dynamics seems to add an additional propagation mechanism to search-and-matching models with productivity shocks only, it does not substantially affect their business cycle predictions. Volatility shocks, on the other hand, enable the model to generate more realistic fluctuations in unemployment and other variables.

4.1 Calibration

4.1.1 Functional forms and stochastic processes

Let me first introduce some functional forms. The production function is the concave function $F(n) = An^{\alpha}$ where α governs the amount of diminishing returns in the economy. I normalize A to 1. Since time is discrete, I must choose a job finding probability function bounded between 0 and 1, which rules out Cobb-Douglas matching functions. Following Menzio and Shi (2009), I pick the CES contact rate functions

$$p(\theta) \equiv \theta(1+\theta^{\gamma})^{-1/\gamma}, \quad q(\theta) \equiv p(\theta)/\theta = (1+\theta^{\gamma})^{-1/\gamma}.$$

In addition to providing a good fit to the data, these functions satisfy all the regularity conditions stated in Assumption 2. To parameterize them, I estimate function p by non-linear least squares using the job finding rate series constructed by Shimer (2007)[8] and a measure of market tightness θ. I construct the latter using data on vacancies from the Job Openings and Labor Turnover Survey (JOLTS) and the Conference Board's Help Wanted Index with unemployment data from the Bureau of Labor Statistics (BLS)[9]. The regression yields $\gamma = 1.60$ with a good fit ($R^2 = 0.90$).

The aggregate and idiosyncratic productivity processes follow AR(1) processes:

$$\begin{cases} y_t = \rho_y y_{t-1} + \sigma_y \sqrt{1-\rho_y^2}\varepsilon_{y,t}, & \varepsilon_{y,t} \sim \mathcal{N}(0,1) \\ z_t = \rho_z z_{t-1} + s_{t-1}\sqrt{1-\rho_z^2}\varepsilon_{z,t}, & \varepsilon_{z,t} \sim \mathcal{N}(0,1). \end{cases}$$

The idiosyncratic volatility process s_t follows

$$\log s_t = (1-\rho_s)\log\overline{\sigma_z} + \rho_s \log s_{t-1} + \sigma_s\sqrt{1-\rho_s^2}\varepsilon_{s,t}, \quad \varepsilon_{s,t} \sim \mathcal{N}(0,1)$$

where $\overline{\sigma_z}$ is the mean of the time-varying volatility process as well as the average variance of the idiosyncratic productivity process. In the data, idiosyncratic volatility is countercyclical. I therefore allow the innovations $\varepsilon_{y,t}$ and $\varepsilon_{s,t}$ to be negatively correlated. Innovations to z_t are independent across agents. In the simulations, I approximate these processes on finite grids using the method described in Tauchen (1986).

[8]This data was constructed by Robert Shimer. For additional details, please see Shimer (2007) and his webpage http://sites.google.com/site/robertshimer/research/flows.

[9]Because the direct vacancy measure by JOLTS is only available since 2001, I use the Conference Board's Help Wanted Index to complete the measure from 1951Q1 to 2000Q4 and adjust its level to match the JOLTS stock of vacancies in 2001Q1. Unemployment comes from the monthly seasonally-adjusted unemployment rate constructed by the BLS. Data is averaged over 3-month periods and detrended using an HP filter with parameter 1600.

4.1.2 Calibration strategy

The calibration uses a method of simulated moments. To simplify the comparison with existing literature in search and business cycles, I follow the calibration strategy in Menzio and Shi (2008) (MS08 hereafter) as closely as possible. Given that the model has a number of additional dimensions, I target other moments that have been widely used in the search-and-matching literature. This calibration strategy targets mostly average aggregate labor market flows, as in Shimer (2005). In particular, such calibrations are known to lead to the Shimer puzzle. It is conservative in that sense.

The time period is set to one month. I set the discount rate β to 0.996 so that the annual interest rate is about 5%. To calibrate the decreasing returns to scale parameter α, I choose to match literature estimates of total diminishing returns at the establishment level. From a brief survey of Basu and Fernald (1995), Basu (1996) and Basu and Kimball (1997), I set $\alpha = 0.85$ in the middle of the range of estimates[10]. The parameters left to calibrate are the following: the productivity parameters (ρ_y, σ_y), $(\rho_z, \overline{\sigma_z})$ and (ρ_s, σ_s), home production b, vacancy posting cost c, entry cost k_e, fixed operating cost k_f and the relative search efficiency of employed workers compared to unemployed ones λ. The calibration targets are set to the historical averages of the following monthly transition rates: an Unemployment-Employment (UE) rate of 45%, an Employment-Unemployment (EU) rate of 2.6% according to Shimer (2005), and an Employment-Employment (EE) rate of 2.9% following estimates by Nagypál (2007). The ratio of home production b over average marginal productivity is set to match a ratio of 71% in accordance with Hall and Milgrom (2008).

Turning to establishment-related moments, I must now depart from the MS08 calibration slightly. I fit an average establishment size of 15.6 computed from the 2002 Economic Census. Regarding firm entry, I target an average fraction of hires at opening establishments of 21% according to quarterly data from the Business Employment Dynamics (BED) over the period 1992Q3-2010Q1. To calibrate the average variance of idiosyncratic productivity shocks, I target an average dispersion of quarterly growth rates in employment (g_n) of 0.22 taken from Davis et al. (2010) using BED data. The time-varying volativility parameters (ρ_s, σ_s) are calibrated with the help of my measure of dispersion of sales quarterly growth rates (g_s) constructed from Compustat (I use the standard deviation measure, labeled $STD(g_s)$). I target the autocorrelation and standard deviation of the logarithm of this volatility measure. The correlation between the innovations to the productivity and volatility processes $cor(\varepsilon_{y,t}, \varepsilon_{s,t})$ is calibrated to match the correlation between output and my volatility measure $cor(Y, STD(g_s))$.

Finally, the aggregate productivity parameters (ρ_y, σ_y) are set to match the autocorrelation and standard deviation of log-detrended output. To do so, I use seasonally-adjusted quaterly real GDP data from the Bureau of Economic Analysis from 1947Q1 to 2010Q1 and detrend it using an HP filter with parameter 1600. The targets are set to 0.841 for the autocorrelation and 0.017 for the standard deviation.

The parameters are calibrated simultaneously using a search algorithm in the parameter space that minimizes the distance between the empirical and simulated moments, with weights chosen to yield relative errors of the same amplitude for each moment. Table 1 summarizes the parameter values that result from the calibration. Table 2 shows the fit of the model with the targeted moments.[11]

[10] I choose to match the total decreasing returns at the firm level because I am interested in explaining firm dynamics, despite the absence of capital in the model. A previous version of this paper targeted a wage share of 0.66, with little incidence on the final results.

[11] Notice that the fit is close but not perfect given the complexity of the parameter search and the time required for each simulation.

Parameter	Value	Description
Pre-calibrated:		
A	1	Technology parameter
β	0.996	Monthly discount factor
γ	1.60	Job-finding probability parameter
α	0.85	Decreasing returns to scale coefficient
Calibrated:		
b	0.45	Home production
c	0.8	Vacancy posting cost
λ	0.253	Relative search efficiency of employees
k_e	4.1	Entry cost
k_f	1.95	Fixed operating cost
ρ_y	0.95	Persistence of aggregate productivity y
σ_y	0.03	Standard deviation of aggregate productivity y
ρ_z	0.6	Persistence of idiosyncratic productivity z
$\overline{\sigma_z}$	0.41	Standard deviation of idiosyncratic productivity z
ρ_s	0.74	Persistence of volatility process s
σ_s	0.055	Standard deviation of volatility process s
ρ_{ys}	-1	Correlation between $\varepsilon_{y,t}$ and $\varepsilon_{s,t}$

Notes: Autocorrelations and standard deviations are monthly.

Table 1: Calibrated parameters

Moment	Empirical value	Simulated
UE rate	0.450	0.467
EU rate	0.026	0.026
EE rate	0.029	0.029
b / productivity	0.710	0.690
Average size of firms	15.6	15.4
Entry / Hirings	21%	21.2%
$\rho[log(output)]$	0.841	0.828
$\sigma[log(output)]$	0.017	0.017
$STD(g_n)$	0.220	0.241
$\rho[log(STD(g_s))]$	0.313	0.377
$\sigma[log(STD(g_s))]$	0.039	0.039
$cor(Y, STD(g_s))$	-0.165	-0.122

Notes: UE, EU and EE are monthly transition rates. Autocorrelation, standard deviations of log-detrended output and volatility are quarterly.

Table 2: Calibrated moments

4.2 Establishment growth

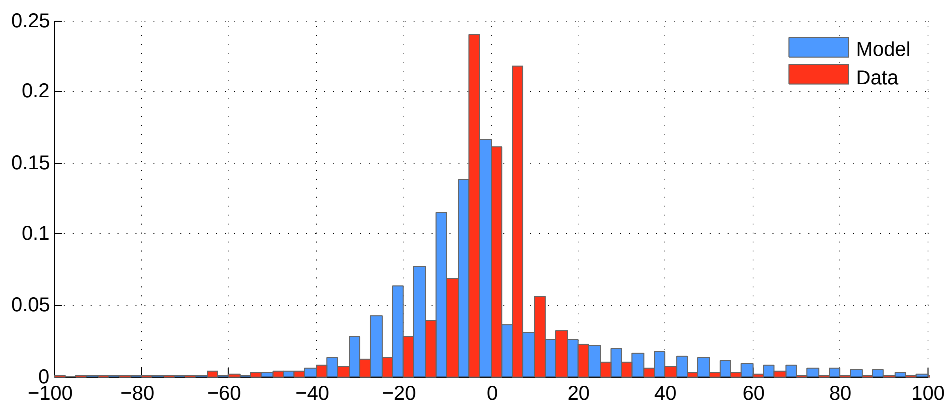
As a validation exercise, I now examine some predictions of the model in terms of establishment growth. It is convenient to introduce the following measure of establishment growth rates as initially used by Davis et al. (1996). Denoting $n_{i,t}$ the total employment of establishment i at date t, define growth rate $g_{i,t}$ as:

$$g_{i,t} = \frac{n_{i,t} - n_{i,t-1}}{\frac{1}{2}(n_{i,t} + n_{i,t-1})}.$$

This measure takes the ratio of net employment growth to the average size of the firm between periods $t-1$ and t. This measure is convenient in that it can account for entry and exit of firms and treats them in a symmetric fashion. A growth rate of 2 therefore means entry, while -2 stands for exit.

4.2.1 Growth rate distribution

Davis et al. (2010)[12] reports the quarterly growth rate distribution of establishments using data from the Business Employment Dynamics dataset in 2008. I compare it with the same distribution from my model calibrated as in part 5. I simulate the model at the aggregate steady state over a 3-month period and compute the growth rate distribution.



Notes: Quarterly data from 2008 tabulated from the BED dataset by Davis, Faberman and Haltiwanger (DFH). Simulated distribution computed over a three-month interval at the aggregate steady state using the same methodology as DFH.

Figure 9: Distribution of quarterly establishment growth rates

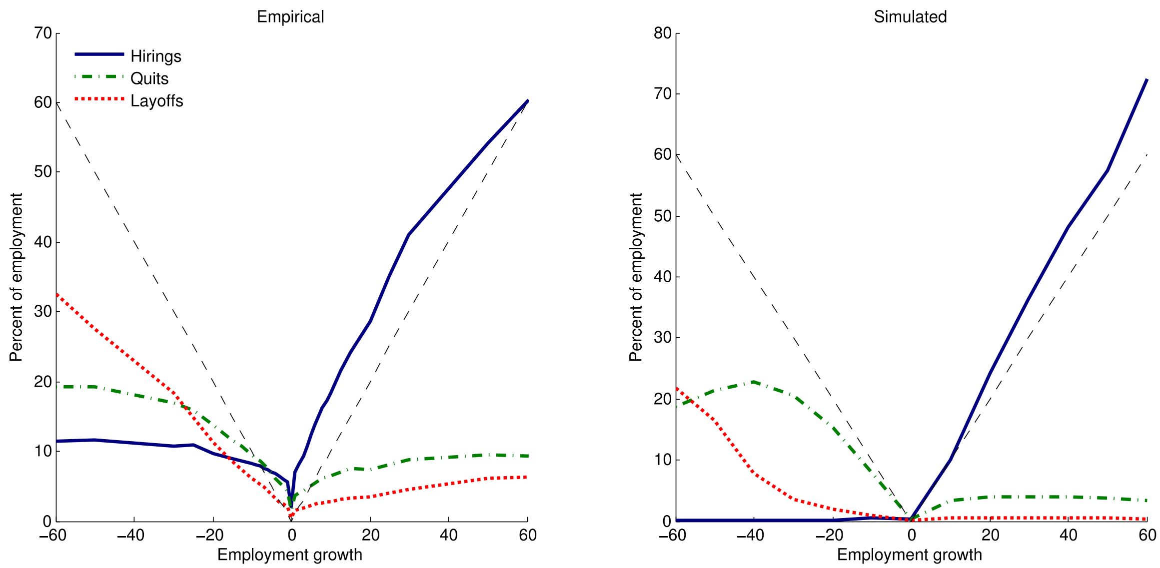
Figure 9 compares the two distributions. The two have similar shapes, but differ on several dimensions since the model is not calibrated to match this distribution. The empirical dispersion is 0.22 for a simulated dispersion of 0.24. Both present a large peak at 0 (16.1% in the data, 16.8% in the model): a substantial number of establishments do not adjust their employment at all in a quarter. On the negative side, the distribution generated by the model is more skewed to the left. This results from the fact that search frictions manifest themselves as linear adjustment costs. Hiring is costly, while separations are not. Therefore, there is a larger tendency for firms to contract by a small

[12] I would like to thank Steven Davis, Jason Faberman and John Haltiwanger for allowing me to use their tabulations from the Business Employment Dynamics dataset.

amount, while expansions are usually larger. The model-generated distribution has indeed larger tails. This also explains why the model cannot reproduce the large peak for growth rates between 0 and 5%. A possible way to improve this dimension would be to introduce firing costs and other types of adjustment costs in the model in order to replicate the whole shape of the distribution.

4.2.2 Employment policy

Empirical evidence shows that firms with different growth rates have different hiring, layoff and quit rates. The composition of hirings against separations and the balance between layoffs and quits present some important non-linearities at the establishment-level. Davis et al. (2010) show with some empirical exercises that these establishment-level regularities are important features to replicate in order to improve the time-series predictions of search models.



Notes: Tabulations from the BED dataset by Davis, Faberman and Haltiwanger. Simulated policy computed by aggregation over a three-month period. The averages are employment weighted. The dashed lines are the $-45°$ and $+45°$ lines to show the minimal level of separations and hirings needed.

Figure 10: Empirical and simulated employment policies as a function of growth

Figure 10 displays the empirical and simulated employment-weighted levels of hirings, quits and layoffs as a function of establishment growth. To produce this graph, I simulate the model for a large number of periods and compute the corresponding series by aggregating over 3-month periods. The model can reproduce a number of qualitative and quantitative features of the average level of quits and layoffs. In particular, it is able to match the change in their composition for expanding and contracting firms. Establishments that contract by a small amount tend to favor quits over layoffs, as workers can be directly employed without experiencing unemployment. However, the more the firm contracts, the more it will lay off workers. The reason is that in order to dismiss a large number of workers, firms have to direct them to labor markets where the probability to find a job is higher and where wages are lower. At some point, the utility promised to workers on these markets becomes so low that the firm starts using layoffs. Turning to hirings, the model performs reasonably well for expanding firms, but fails for contracting ones. The empirical data shows a non-negligible amount of churning: establishments that grow also separate from some workers, and vice versa. The model is able to generate churning to a certain extent for growing establishments, but very little for contracting ones. The reason lies in the fact that hiring is costly, which breaks the symmetry between expansion versus contraction. For that reason, very little hiring is observed at establishments that contract over a 3-month period.

4.2.3 Differential growth rates over the business cycle

Empirical regularities in the evolution of the cross-sectional distribution of establishments have recently drawn attention from a number of authors. Moscarini and Postel-Vinay (2009) document that large firms are more cyclically sensitive than small ones. They show that large firms shrink faster during recessions, but create more jobs in the later stages of the following expansion. As a result, the differential growth rate between large and small establishments $g_{large,t} - g_{small,t}$ is strongly procyclical. Using different datasets and controls, they report correlations between unemployment and the differential growth rate that range from -0.61 to -0.38 in the U.S. To investigate whether the model can reproduce this different cyclical sensitivity, I compute the differential growths rate between the 50% largest and smallest establishments in the model with aggregate disturbances in productivity. The correlation between the differential and unemployment level is -0.42. This negative correlation remains robust to different definitions of large and small firms. Such a strong correlation can be surprising as one would expect it to be 0 in a frictionless model. The explanation for this differential cyclical sensitivity in the model lies in a subtle interaction between the distribution of firms and their optimal policy. Figure 17 depicts the change in firms' policy after a negative productivity shock. In the benchmark calibration, the separation threshold is more cyclically sensitive than the firing threshold. In particular, as I will explain in the next section, the separation area shrinks more than the hiring area expands. Now, small firms are mostly firms that received a bad shock and laid some workers off: most of them lie close to the firing threshold. Conversely, large firms tend to lie close to the hiring threshold. As a result, when a negative productivity shock hits, the growth rate of small firms is less affected because the firing threshold moves farther away (their growth rate is less negative). To further examine this effect, I simulate the model with volatility shocks only. As figure 19 shows, the optimal policy shifts in the other direction when volatility increases and the hiring threshold is much more reactive. The correlation between unemployment and the differential growth rate becomes positive in this case, confirming in some way the above explanation. The mechanism at play in the model sheds some light on the origin of this differential cyclical sensitivity. Search frictions and adjustment costs can deliver such effects if policy thresholds respond differently to the business cycle.

4.3 Wage predictions

The use of optimal dynamic contracts in search models provides an alternative to the standard assumptions of Nash or Stole & Zwiebel bargaining. In particular, since workers can search for jobs while employed, employers have to design contracts that give the right incentives for workers to stay in the firm or apply to the right labor market. Using the incentive-compatible contracts described in part 3.4 yields a unique characterization of wages. Under this specification, wages can vary substantially from one worker to the other, even if they belong to the same firm. Because of this rich incentive structure, the model is able to predict an important wage dispersion for observationally equivalent workers, and accounts for a significant fraction of the empirical variation. It also predicts that wages tend to grow with firm size, as seen in the data.

4.3.1 Wage dispersion and elasticity

Hornstein et al. (2007) report that standard calibrations of search-and-matching models without on-the-job search cannot generate much dispersion in wages. In their basic calibration of a standard random search model, they obtain a mean-min ratio for wages of 1.036, while their preferred empirical estimate is about 1.70, and a corresponding coefficient of variation of only 1/12th of the variation in the data. Using wage data from the 1990 Census with different sets of controls, their estimates for the empirical coefficient of variation of residual wages range from 0.35 to 0.49. I estimate the same dispersion measure in my model by simulating over a large number of periods and obtain an average coefficient of variation of 0.23, which explains between 47% and 66% of the observed dispersion in wages.

Regarding the evolution of wages over business cycles, both the average wage and total wage dispersion appear to be procyclical. Given the recent attention that wage stickiness has received in the search-and-matching literature, I measure the elasticity of wages with respect to productivity (output per person). My estimated elasticity is of 0.91, above the empirical estimates of about 0.5 reported in the literature. An interesting future development could be to introduce risk-aversion for workers. Combined with the dynamic contracting framework of the model, this extension would connect search theory to the implicit contract literature and provide us with a theory of endogenous stickiness, in which this dimension could be significantly improved.

4.3.2 Size-wage differential

It is a common finding that firm size can explain part of the variation in wages. Brown and Medoff (1989) report that with a variety of data sets and a range of controls chosen to capture much of the differences in labor quality, a substantial size-wage differential remains: an employee working at a firm with log(employment) one standard deviation above average may expect to earn a wage between 6-15% above the one at a firm with log(employment) one standard deviation below average. To investigate whether the model can reproduce this finding, I compute the wages in every establishment at the aggregate steady-state. I then run the following regression

$$\log(wage) = \alpha + \beta \log(employment) + \varepsilon$$

and evaluate by how much the wage of a worker employed at a firm with log(employment) one standard deviation above average increases compared to the one with one standard deviation below: I obtain an increase of 10.3% in line with the empirical estimates. Interestingly, this size-wage differential can be explained by a mechanism due to search frictions quite different from standard explanations based on labor quality or institutions. The mechanism at work in the model is due to the way firms deal with worker incentives. In this economy, firms that want to expand prefer to retain their current workers in order to save on hirings costs. To do so, they backload incentives to future periods by promising them a high future utility. Therefore, all other things being equal, firms that grow tend to offer lower wages today than firms that shrink, since workers get a higher future compensation. Turning back to firm size, large firms are those that received a high idiosyncratic shock in the recent past, and vice versa for small firms. The idiosyncratic shock being transitory, there is regression towards the mean. Large firms tend to shrink and offer high wages, while small ones tend to grow and offer low wages. This mechanism underlines the role of search frictions and job-to-job transitions to generate dispersion in wages. It also emphasizes establishment growth as a key determinant for wages.

To go a little further, I simulate the economy after a negative productivity shock and study how the coefficient of dependance of wage with respect to firm size responds. The coefficient β appears to be procyclical, decreasing during a recession, in line with the previous finding that wage dispersion is procyclical.

4.4 Business cycle statistics

It has been argued by a number of recent studies that the introduction of establishment dynamics into search-and-matching models can provide an additional propagation mechanism that standard models lack. In this section, I simulate the model under the benchmark calibration with aggregate productivity and volatility shocks, along with a version of the model with productivity shocks only, calibrated using the same targets, except those related to the autocorrelation and standard deviation of firm growth dispersion. I compute a selection of moments and compare them to a Mortensen-Pissarides model (MP hereafter) calibrated as in Shimer (2005).[13] Table 3 compares the standard deviations and correlations with output of a number of empirical time series to the ones generated by the two models.

It should be noticed first that the model developed in this paper produces larger standard deviations in unemployment, vacancies and labor market flows than the MP model, even with productivity shocks only. This tells us that the introduction of establishments and the presence of a slow-moving distribution of employment across them does increase the propagation of the model. The version with productivity shocks only, however, still produces standard deviations that are significantly below the ones observed in the data, with the exception of entry. Volatility shocks, on the other hand, do generate large fluctuations in unemployment: the model can reproduce about 87% of the total unemployment volatility. Turning to the comovements with output, the MP model usually generates very high contemporaneous correlations because of the fact that aggregate productivity shocks are the only source of fluctuations and the economy adjusts very quickly. The version of the model with productivity shocks only generates some correlations closer to the empirical ones, most likely because of the slow adjustment of the distribution of establishments, which adds some sluggishness to the model. When volatility shocks are introduced, these contemporaneous correlations become weaker and slightly too low compared to the empirical ones. This results from the fact that there are now two shocks driving the fluctuations. The model matches the pro- and countercyclicality of all the variables. Entry and exits are, however, too volatile compared to the data. One reason is that block recursivity makes entry a residual variable. Entry must therefore absorb all the residual shocks in the economy. Exits seem to respond too much to volatility shocks, but we will return to this issue in the next section when we turn to the 2007-2009 recession.

Overall, the introduction of an establishment dimension enables the model to have more realistic predictions than a standard MP model. In addition, volatility shocks allow us to generate much larger fluctuations in unemployment and to a lesser extent in vacancies.

5 Current and Past Recessions

I now study in details the effects of idiosyncratic volatility shocks and examine whether these shocks can improve the ability of search-and-matching models to explain empirical time-series. To highlight the mechanism at play in the model, I first present the response of the model to changes in aggregate productivity, before introducing idiosyncratic volatility shocks. Such shocks allow the model to explain the coexistence of high unemployment and high productivity, as observed since 2009, but fail to capture alone certain aspects of the 2007-2009 crisis. I therefore calibrate a combination of productivity and volatility shocks on their empirical counterparts to measure how much of the patterns in the data can be accounted by these shocks. In a final exercise, I proceed to the same accounting for the 1981-1982 and 2001 recessions in which a peak in volatility was observed.

5.1 Productivity shocks

In this subsection, I present the response of the economy to a permanent negative productivity shock, and discuss why such a shock fails to explain some characteristic features of the 2007-2009 recession.

[13] The calibration is identical to the one in the original article except that I target the autocorrelation (0.841) and standard deviation of output (0.017) instead of output-per-person to harmonize it with my own calibration strategy.

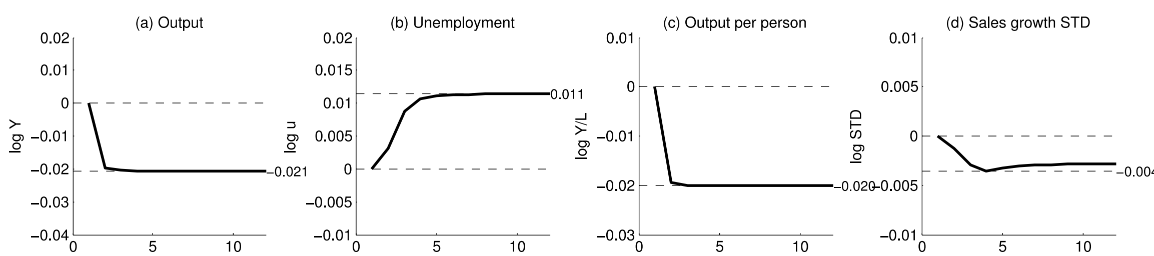
	Data		Shimer (05)	
	Std Dev.	cor(Y,x)	Std Dev.	cor(Y,x)
Y	0.017	1	0.017	1
Y/L	0.014	0.628	0.017	1
U	0.143	-0.840	0.007	-0.982
V	0.137	0.640	0.021	0.993
Hirings	0.057	0.617	0.003	0.448
Quits	0.100	0.631	-	-
Layoffs	0.064	-0.403	0.001	0.931
Entry	0.045	0.319	-	-
Exit	0.046	-0.004	-	-
	Model (y+s)		**Model (y only)**	
	Std Dev.	cor(Y,x)	Std Dev.	cor(Y,x)
Y	0.017	1	0.018	1
Y/L	0.016	0.950	0.018	0.999
U	0.124	-0.355	0.030	-0.831
V	0.052	0.166	0.030	0.887
Hirings	0.052	0.166	0.030	0.887
Quits	0.064	0.573	0.053	0.960
Layoffs	0.139	-0.162	0.013	-0.804
Entry	0.256	0.009	0.138	0.331
Exit	0.171	-0.082	0.003	-0.617

Notes: Time series are presented in logs. Quarterly time series detrended using an HP filter with parameter 1600. Y is output, Y/L output per person, U unemployment, V vacancies. Entry and exit are number of workers in entering/exiting firms. See data appendix for data sources.

Table 3: Business cycle statistics

When an aggregate negative productivity shock hits the economy, firms and workers see their joint surplus decrease. The triggered drop in profits discourages firms to enter the economy as they can no longer recover the entry cost. As a result, the market tightness goes down, and it becomes increasingly difficult for workers to find jobs. The total number of new hires and job-to-job transitions (quits) in the economy decreases. The effect on layoffs and exits is ambiguous. As the productivity of their workers drops, firms become more willing to separate from some workers. However, general equilibrium effects may invert this trend. As market tightness dips, the value of unemployment goes down due to a lower job finding rate. As a result, firms may find it more profitable to keep their workers if the value of unemployment drops more than their marginal productivity. Turning to exits, a negative productivity shock causes a number of firms at the bottom of the productivity distribution to exit as they can no longer pay the fixed operating cost. However, as the hiring costs go down with the market tightness, firms expected profits do not drop as much, so that the exit threshold may not change substantially. Also, as the mass of firms present in the economy goes down, the total number of exits may actually decrease. As time goes on, the cleansing effect of the recession fades out after the exit of the least productive firms, and a recovery in entry can be observed as firms start entering again and create new jobs.

To illustrate these effects, figure 17 in the appendix describes how the optimal decision of firms evolves in the (z,n)-space after a negative aggregate productivity shock in the calibrated model. In this example, general equilibrium effects seem to dominate: firms dismiss less workers after the shock, while the exit threshold is unchanged. Interestingly, incumbent firms increase slightly their sizes because of the lower hiring cost per worker, but not enough to compensate for the drop in job creation at entering firms. Figure 11 shows the aggregate response of the model to a permanent negative aggregate productivity shock calibrated to produce a 2% drop in productivity (output per person) as observed during the 2007-2009 recession. The response of other labor market variables can be found in figure 18 in the appendix. The shock triggers a drop in output and productivity with a small increase in unemployment. The dispersion of sales growth decrease by less than 1%, instead of the 11.2% increase observed in Compustat's quarterly growth data. We conclude that aggregate productivity shocks are not sufficient to account for the patterns observed during the recession. To generate the dramatic increase in unemployment (42%) observed between 2008 and 2009, one would need a counterfactually large drop in aggregate productivity, incompatible with its strong subsequent recovery observed in the data.



Notes: Responses shown in log deviation from steady-state. The aggregate productivity shock is matched to produce a -2.2% drop in output per person as observed between 2007Q4-2009Q1.

Figure 11: Response to a permanent decrease in aggregate productivity

5.2 Volatility shocks

Motivated by the observation that firm-level dispersion in growth rates has increased substantially since 2008, I now study the effect of volatility shocks to see whether they can improve the ability of the model to explain the patterns observed in the data. In this subsection, I discuss the general effect of positive shocks to idiosyncratic volatility s.

The effect of volatility shocks can be decomposed mainly into two categories. First, when idiosyncratic volatility increases, firms face higher uncertainty about their future prospects, leading to the "wait-and-see" effect put forward in the uncertainty-driven business cycle literature. Because of the search frictions, hiring is costly in the model. Therefore, when volatility increases, firms may decide to cut on hiring in order to avoid paying hiring costs repeatedly for workers that may end up laid-off in the next period. To put it differently, the option value of waiting rises in the face of higher uncertainty. However, such "wait-and-see" effects are counterbalanced by the opposite force of "realized volatility". Higher volatility, understood as a increase in idiosyncratic volatility s, corresponds to a higher probability of getting both good and bad shocks. However, since firms have the option to exit when their situation worsens, a volatility shock can actually translate on average into higher expectation of future surpluses. If risk aversion is low, as is the case in this model, such shocks can be positive on average.

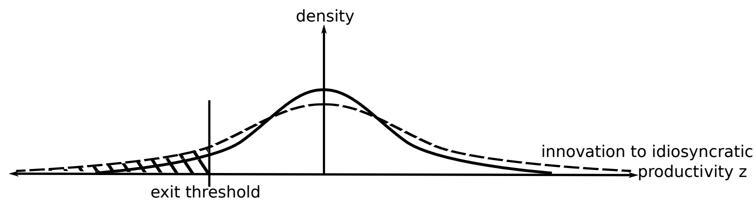


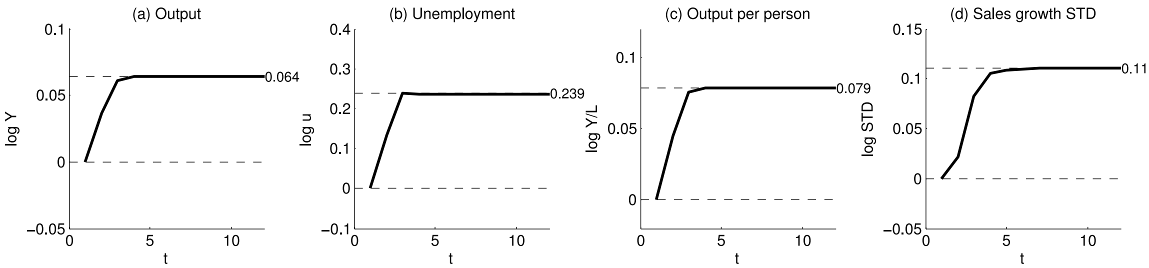
Figure 12: Volatility shock

Figure 12 shows how a mean-preserving spread of the distribution of the innovation to idiosyncratic productivity z can lead to higher expected surplus. Lower values of productivity are cut-off since firms can endogenously decide to exit. A mean-preserving spread thus loads proportionaly more on the right-hand side of the graph if the exit threshold does not change too much. Of course, the threshold also adjusts when the shock hits. But since it is endogenously chosen by the firm, the firm can do at least as good as if it kept the threshold constant. Thus, the average expected surplus increases when volatility rises.[14] Given that agents are risk neutral in this model, this effect will in general dominate the wait-and-see effect explained above. As a result, when volatility rises, more firms enter the economy as they perceive a higher probability to make large profits in the future. This drives the market tightness up, so that workers can find jobs more easily. The total number of hirings and quits therefore goes up. On the general equilibrium side, hiring costs rise, causing incumbent firms to shrink slightly, reinforcing the wait-and-see effect on hiring. The value of unemployment goes up with market tightness, leading to more layoffs. Simultaneously, the mass of firms getting a low shock increases, resulting in a jump in exits and further layoffs.

To illustrate this, figure 19 in the appendix displays the optimal firm decision in the (z,n)-space before and after a positive volatility shock. As explained above, the hiring area shrinks because of the combined forces of wait-and-see and general equilibrium effects. Layoffs go up, while the exit area decrease slightly because of the realized volatility effect. Figure 13 presents the aggregate response of the model when volatility s rises to match an increase of about 11% as observed over the 2007-2009 period. The responses of other labor market variables can be found in figure 20 in the appendix. It should be noted that the realized volatility effect largely dominates. Output and productivity jump up because of the asymmetric distributional impact of volatility. Unemployment rises by 24% (42% in the data), as do the rest of labor market flows, because of the large reallocation taking place on the labor market: the intensified reallocation of jobs between firms triggered by the

[14]Note that this effect is similar to the result that the price of an option increases with volatility.

rise in volatility leads larger flows of workers to experience a spell of unemployment, as it takes time to create new matches. A direct consequence of the volatility shock is a substantial increase in the dispersion of sales growth rates.



Notes: Responses shown in log deviation from steady-state. The aggregate volatility shock is matched to produce a 11.2% increase in sales growth dispersion as observed between 2007Q4-2009Q1.

Figure 13: Response to a permanent increase in idiosyncratic volatility s

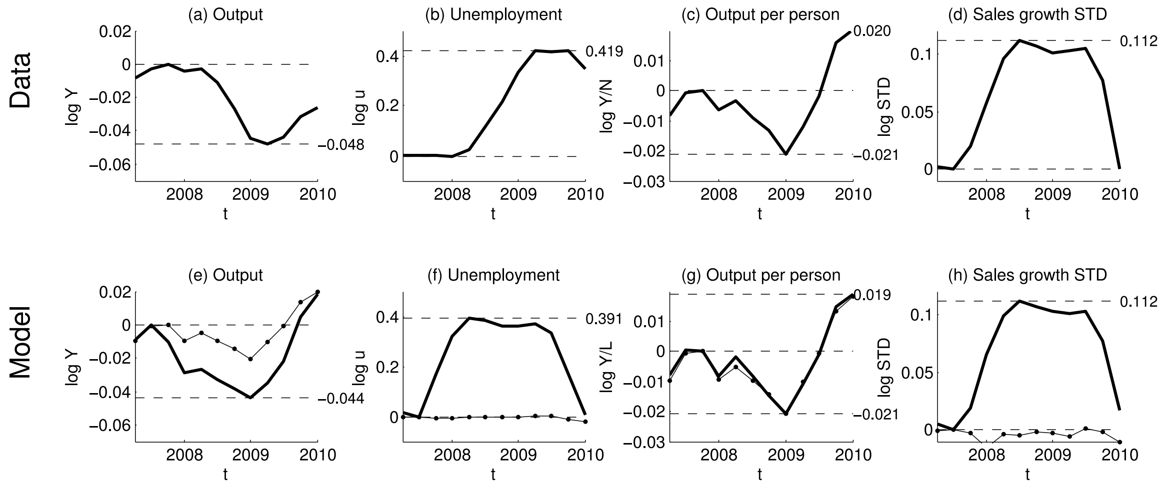
If volatility shocks can explain a large rise in unemployment similar to the one seen in the data, along with a sharp rise in productivity and in the dispersion of firms, they cannot explain why output fell so much during the 2007-2009 recession. An obvious reason is, in particular, the difficulty to isolate the wait-and-see from the realized volatility effect, which seems to be too strong and should be counterbalanced by some opposite force.

5.3 The 2007-2009 Recession

Figure 3 showed in the introduction that the distribution of firm sales growth rates underwent both a downward shift in its mean, as well as an increase in its dispersion. It makes sense therefore to combine first moment shocks in productivity y to second moment volatility shocks in s to examine what the model would predict for the 2007-2009 recession. The objective is to see whether the first moment shocks to productivity are large enough to reproduce the magnitude of the initial drops observed in output and output per person, and whether volatility can explain the sharp rise in unemployment. Another question is to answer whether firm-level uncertainty can explain why unemployment has remained so persistent since the beginning of the crisis despite the recovery in many other economic indicators.

I calibrate two series of shocks to aggregate productivity y and idiosyncratic volatility s to match the data series in output per person (BLS) and the standard deviation of firms' quarterly growth rates (Compustat). Since these variables are endogenous in the model and I am using grids to simulate the economy, I cannot calibrate these series perfectly, but I choose those that offer the closest fit to their empirical counterparts. Figure 14 presents the aggregate response of the model to these shocks, along with a version of the model with productivity shocks only. Graphs (g) and (h) present how the calibrated series fit the data. Graphs (e) and (f) show what the model would predict for output and unemployment. Figure 21 in the appendix displays the prediction of the model for other labor market flows.

As figure 14 shows, the model predicts a drop in output (-4.4%) somewhat smaller than in the data (-4.8%), so about 92% of the total. It explains an increase in unemployment of 39.1%, explaining about 93% of the change in the data. The introduction of volatility shocks seems therefore to improve quite substantially the ability of the model to account for the magnitudes observed in the data. However, even though volatility can explain the coexistence of high unemployment and high productivity, the unemployment does not display much persistence and goes back to its original level in 2010. The main reason for these particular dynamics is that volatility decreases quickly in the data by the end of 2009. For the same reason, output recovers very quickly in 2010 because of



Notes: Responses shown in log deviation from initial steady-state. Productivity shocks are calibrated to match the output per person series and volatility shocks are calibrated to match Compustat's quarterly sales growth standard deviation series. The dotted line presents the same counterfactual exercise with productivity shocks only.

Figure 14: Empirical and simulated time-series with aggregate productivity and volatility shocks

	Y/L	STD(g)	Y	U	Hiring	Quit	Layoff	Entry	Exit
Data	**-0.021**	**0.112**	-0.048	0.419	-0.189	-0.325	0.255	-0.144	0.108
Model	**-0.020**	**0.114**	-0.044	0.391	0.161	-0.116	0.413	-0.164	0.417

Notes: The peak-trough variations are computed with the log-deviation from an HP-trend with parameter 1600. Output per person and standard deviation of quarterlysales growth rates (STD) are the targeted moments.

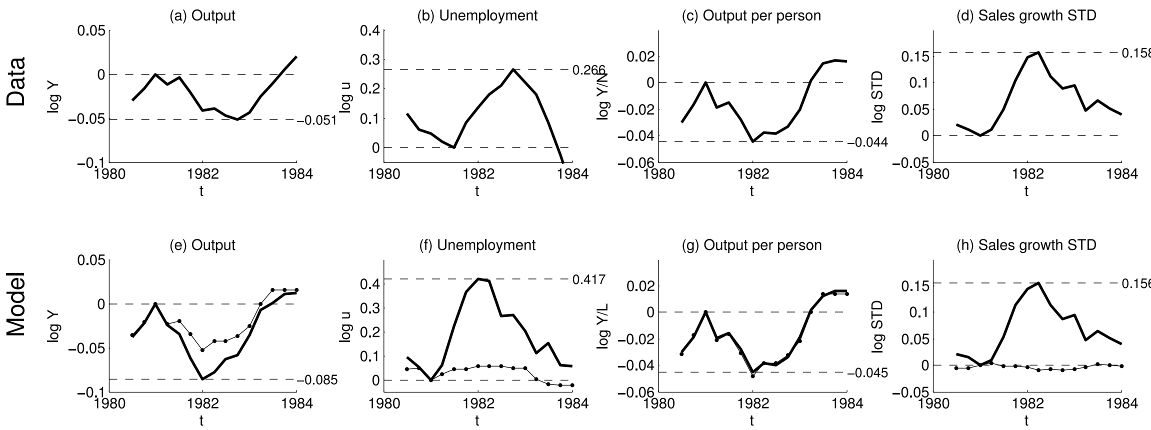
Table 4: Peak-trough variations for the 2007-2009 recession and calibrated model with productivity and uncertainty shocks

the decrease in uncertainty and the contemporaneous rise in productivity. Turning to labor market flows, as reported in table 4, the model can explain most signs of the changes observed in the data, except for hirings. Hirings drop substantially in the data, but the model cannot account for this fact. The model is, however, broadly consistent with the empirical magnitudes of changes in the other variables, such as layoffs and entry. Exits, however, increase slightly too much in the model compared to the data. In other words, the model compensates the insufficient drop in hirings by additional separations. Even though the rise in unemployment is very close to the data, the balance between inflow and outflow to unemployment is different.

The above exercise shows that the introduction of volatility shocks can significantly improve the ability of the model to account quantitatively for the patterns observed during the 2007-2009 recession, especially if compared to a model with productivity shocks only. However, these shocks, when calibrated on Compustat data, do not seem to explain the large persistence in unemployment. If uncertainty is to explain some of its persistence, it may be necessary to consider other types of uncertainty: aggregate, policy, stock market uncertainty, and possibly others; or use different measures of uncertainty. An interesting extension could be to use business survey data as suggested in Bachmann et al. (2010) to measure subjective uncertainty as perceived by businessmen, which may not be reflected in volatility data.

5.4 Past recessions

To further examine how time-varying volatility can help search-and-matching models to explain aggregate data series, I turn to past recessions in which a peak in volatility was observed. The 1981-1982 and 2001 NBER dated recessions were both accompanied by a large increase in idiosyncratic volatility. I proceed to the same accounting exercise as in the previous section and calibrate productivity and volatility shocks to match the empirical output-per-person and volatility series. Figure 15 presents the model's prediction for aggregate variables in the 1981-82 recession. The model does quite a good job to match the joint dynamics of output and unemployment. The peak-trough measures are somewhat larger in the model than in the data, but the empirical and simulated time series behave in a similar fashion.



Notes: Responses shown in log deviation from initial steady-state. Productivity shocks are calibrated to match the output per person series and volatility shocks are calibrated to match Compustat's quarterly sales growth standard deviation series. The dotted line presents the same counterfactual exercise with productivity shocks only.

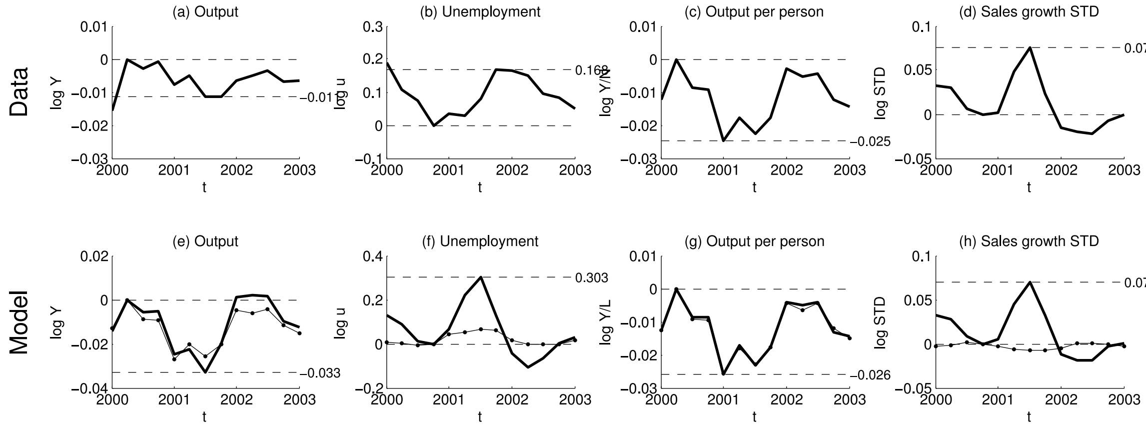
Figure 15: Empirical and simulated time-series with aggregate productivity and volatility shocks for 1981-1982 recession

Figure 16 presents the model's prediction for the 2001 recession. As in the previous case, the overall dynamics are quite similar, but the response of the model to the shocks are larger than in the data. The rise in unemployment and drop in output predicted by the model exceed their empirical counterparts. One reason for this may be that I attribute the aggregate rise in volatility to all sectors, while it originated mostly from the IT sector. As a result, the impact on unemployment may be exaggerated. Another issue remains in the exit dimension. Exits are too flexible and contribute too much to unemployment. Fixing this dimension should be an important step towards a better understanding of labor market dynamics.

With these two examples, even though the model tends to overpredict the magnitude of fluctuations in the data series, I find further evidence that the model with volatility shocks outperforms the one with productivity shocks only. It seems therefore that taking into account movements in the cross-section of firms matters for the business cycle, in particular if one desires to explain the magnitude of labor market reallocation and unemployment.

6 Conclusion

In this paper, I have developed a dynamically tractable search model of firms with decreasing returns. I use this framework to study the effects of productivity and volatility shocks. The introduction of establishment dynamics improves the propagation properties of the model even with



Notes: Responses shown in log deviation from initial steady-state. Productivity shocks are calibrated to match the output per person series and volatility shocks are calibrated to match Compustat's quarterly sales growth standard deviation series. The dotted line presents the same counterfactual exercise with productivity shocks only.

Figure 16: Empirical and simulated time-series with aggregate productivity and volatility shocks for the 2001 recession

productivity shocks only, but volatility shocks remain crucial to generate large fluctuations in unemployment. In particular, these shocks significantly improve our ability to explain the patterns observed in the data during the 2007-2009 recession, as well as in the 1981-82 and 2001 recession. With productivity and volatility shocks calibrated to match output-per-person and the dispersion of firm growth rates, the model can explain an important fraction of the changes in unemployment, output and other labor market flows. On the negative side, I find, however, that the high uncertainty induced by the rise in volatility observed in 2009 is not sufficient to explain the large persistence of unemployment in the data.

The model has a range of implications at the establishment and cross-sectional levels. The growth rate distribution and employment behavior of establishments produced by the model replicate a number of features that have been observed in establishment-level data. Concerning wages, the presence of search frictions and on-the-job search help generate a large wage dispersion, as well as a realistic size-wage differential. It would be interesting for future research to examine these dimensions further in interaction with the business cycle.

Because of the high tractability of its dynamics, the model presented in this paper could be used as a framework in a variety of applications in which multiworker firms and search frictions play a role. Concerning the 2007-2009 recession, the entire policy dimension remains to be studied. Could firing costs have attenuated the depth of the recession as suggested by the experience of some European countries? What role did the unemployment benefits extension play in the crisis for the persistence of unemployment? Could it explain the recently observed deviation of unemployment and vacancies from the historical Beveridge curve? Would hiring subsidies or other policies encourage firms to start hiring again? Another interesting extension would be to introduce financial frictions in the model to study the effect of credit tightening on firm employment decisions and see if the negative productivity shocks could arise endogenously from the misallocation of capital across firms. I leave these topics for future research.

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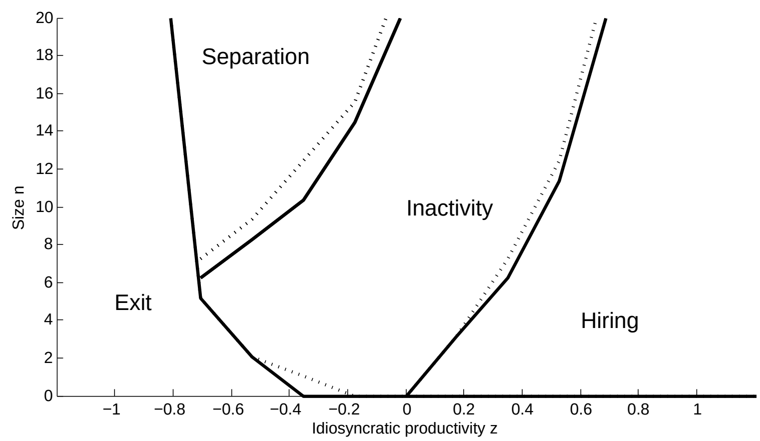
Appendices

A Additional graphs



Notes: The plain line corresponds to the firm's optimal policy before the shock, the dashed line is after the shock.

Figure 17: Firm's optimal policy after a negative productivity shock

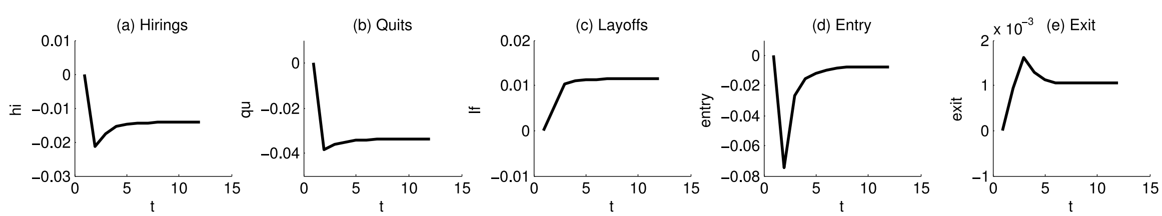
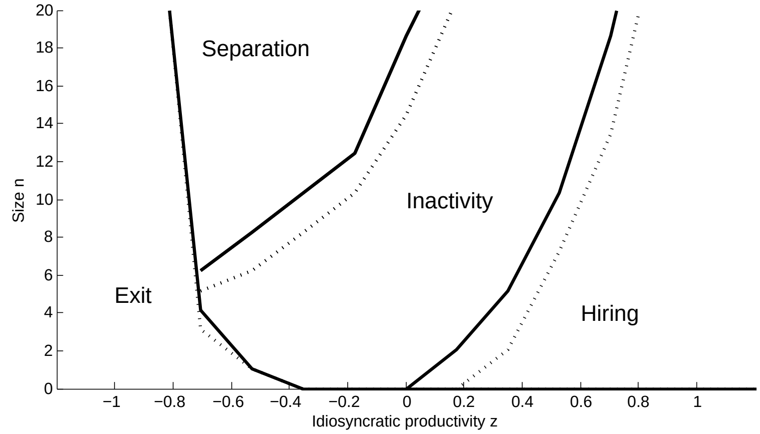


Figure 18: Response of labor market flows to a permanent decrease in aggregate productivity (-2% in output per person)



Notes: The plain line corresponds to the firm's optimal policy before the shock, the dashed line is after the shock.

Figure 19: Firm's optimal policy after an uncertainty shock

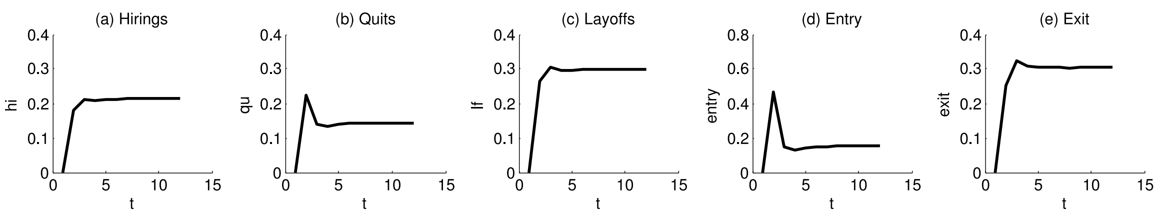
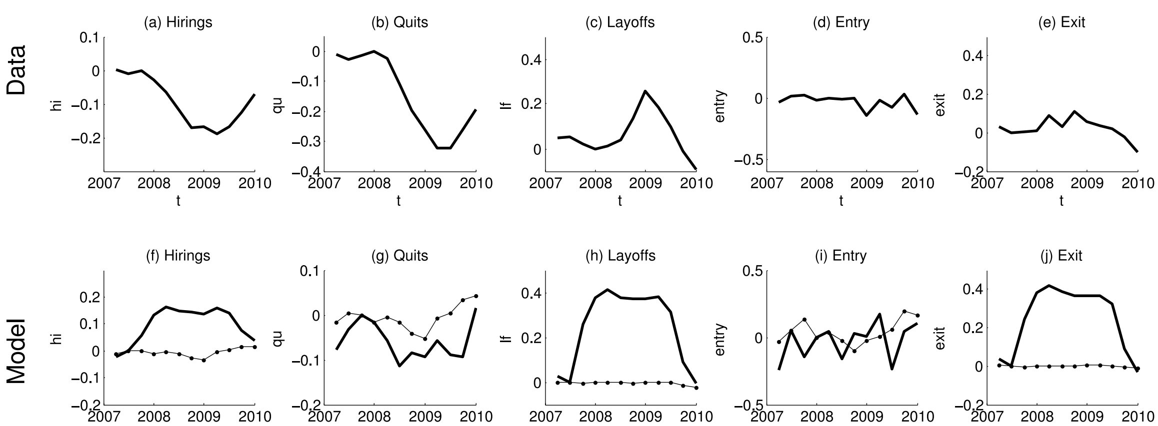


Figure 20: Response to a permanent increase in aggregate volatility s (+11.9% in sales growth dispersion)



Notes: Responses shown in log deviation from initial steady-state. Productivity shocks are calibrated to match the output per person series and volatility shocks are calibrated to match Compustat's quarterly sales growth standard deviation series. The dotted line presents the same counterfactual exercise with productivity shocks only.

Figure 21: Empirical and simulated time-series for labor market flows with aggregate productivity and volatility shocks in the 2007-2009 recession

B Data appendix

This section details the construction and sources of the empirical time series used throughout the paper.

- Output is taken from the NIPA tables constructed by the Bureau of Economic Analysis. I use quarterly GDP in 2005 dollars from 1947Q1 to 2010Q1.
- Productivity Y/L is seasonally adjusted real average output per person in the non-farm sector over the period 1947Q1-2010Q1 from the Bureau of Labor Statistics.
- Unemployment is the seasonally adjusted monthly unemployment rate constructed by the BLS from the Current Population Survey over the period January 1948-March 2010 (for people aged 16 and over). Similarly, I use the total civilian labor force for people aged at least 16 from the BLS over the same period. Data is averaged over quarters.
- Vacancy is quarterly average of the monthly vacancy measure from the Job Openings and Labor Turnover Survey. The measure being only available since 2001, I use the Conference Board's Help Wanted Index to complete the measure from 1951Q1 to 2000Q4 and adjust its level to match the JOLTS stock of vacancies in 2001Q1.
- Historical UE and EU monthly transition rates are taken from Shimer (2007) over the period 1948Q1-2007Q1. For later periods, I use the monthly series on labor force status flows from the Current Population Survey constructed by the BLS over February 1990 to March 2010.
- EE is constructed by taking the ratio of quits from JOLTS over employment $(1-U)$ from January 2001 to March 2010.
- Entry and exit are quarterly openings and closures in employment terms for total private industries taken from the Business Employment Dynamics over the period 1992Q3-2010Q1.
- Labor market flows for hiring, quits and layoffs are quarterly sums of the JOLTS measures from January 2001 to March 2010. Data is normalized by total labor force.
- Firm-level data on sales is taken from Compustat. I use quarterly sales (SALEQ) in dollars for active US firms over the period 1961-2010Q1. I keep only quarters with more than 500 observations (1970Q4 to 2010Q1). I only keep firms that have 100+ observations. Sales growth is computed with $g_{i,t} = \frac{s_{i,t}-s_{i,t-4}}{1/2(s_{i,t}+s_{i,t-4})}$ for the annual rates shown in figure 4. The calibration and simulations use the quarterly growth rate $g_{i,t} = \frac{s_{i,t}-s_{i,t-1}}{1/2(s_{i,t}+s_{i,t-1})}$ seasonally adjusted using the X-12-Arima procedure of the US Census Bureau. The dispersion measures are detrended with time-industry dummies (2-digit NAICS).

C Comparison with Mortensen-Pissarides (1994)

I briefly compare in this subsection how the Mortensen-Pissarides (1994) model responds to uncertainty shocks. This version of the model has an endogenous separation margin, so uncertainty shocks are not neutral as in the exogenous separation case. Eventhough the model is not designed to address issues related to the cross-section of firms, it is still possible to study the effect of fluctuations in the variance of match specific productivity. I calibrate a discrete time version of the model using the same targets as in part 5 when possible and evaluate the response of the model to a series of aggregate productivity shocks and an unexpected uncertainty shock that hits the economy between 2008 and 2009Q3 (best fit for my measure of uncertainty).

C.1 Model

Let us first introduce the model's equations. Unless stated otherwise, I keep the same notation as in my model. As in the original article, aggregate productivity y follows an AR(1) process with parameters (ρ_y, σ_y). To allow comparison with my model, I let the match-specific productivity z follow an AR(1) process with constant volatility:

$$z_t = \rho_z z_{t-1} + \sigma_z \sqrt{1-\rho_z^2}\varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0,1).$$

I model the uncertainty shock to be an unexpected increase in σ_z so as to keep the analysis simple. By this, the effect of the uncertainty shock should be magnified, so the response of unemployment may be considered an upper bound to what would obtain with anticipated shocks. The production function is now linear: a firm-worker pair produces Ae^{y+z}. There is a unique labor market with tightness θ where all firms and workers meet. Firms are free to enter in any period. The timing is identical as in the rest of the paper. At the beginning of the period, the aggregate shock y is realized. Entering firms pay a vacancy cost c to open a vacancy. At that time, new firms draw their productivity from distribution g_z, while incumbent firms learn their new productivity according to the AR(1) process. Firms are then allowed to exit if their value of z is too low. Workers and firms are finally matched and production takes place.

The value of unemployment is:

$$\mathbf{U}(y) = b + \beta \mathrm{E}_{\hat{y}} \left\{ \mathbf{U}(\hat{y}) + p(\hat{\theta}) \left(\int_{\underline{z}(\hat{y})} \mathbf{W}(\hat{y},\hat{z}) \frac{g_z(\hat{z})}{1-G_z(\underline{z}(\hat{y}))} d\hat{z} - \mathbf{U}(\hat{y}) \right) \right\}$$

where $\underline{z}(y)$ is the productivity threshold below which firms exit, and G_z the cumulative distribution function of g_z. The value of employment is:

$$\mathbf{W}(y,z) = w(y,z) + \beta \mathrm{E}_{\hat{y},\hat{z}} \left\{ \hat{\tau}\mathbf{U}(\hat{y}) + (1-\hat{\tau})\mathbf{W}(\hat{y},\hat{z}) \right\}$$

where $w(y,z)$ is the equilibrium wage schedule determined by Nash bargaining, as described below. The firm's problem is given by:

$$\begin{aligned}
\mathbf{J}(y,z) &= \max_{\hat{\tau}} Ae^{y+z} - w(y,z) + \beta \mathrm{E}_{\hat{y},\hat{z}} \left\{ (1-\hat{\tau})\mathbf{J}(\hat{y},\hat{z}) \right\} \\
&= Ae^{y+z} - w(y,z) + \beta \mathrm{E}_{\hat{y},\hat{z}} \left\{ \max\ (\mathbf{J}(\hat{y},\hat{z}),0) \right\}
\end{aligned}$$

The free-entry condition tells us that firms post vacancies until expected profits exactly equal the vacancy cost c:

$$\forall y, \quad c = q(\theta) \int \max(\mathbf{J}(y,z),0) g_z(z) dz = q(\theta) \int_{\underline{z}(y)} \mathbf{J}(y,z) g_z(z) dz.$$

Wages are determined by Nash bargaining. Denote $S(y,z)$ the joint surplus of the firm and worker. If μ is the worker's bargaining power, we can write:

$$\begin{cases} \mathbf{J}(y,z) = (1-\mu)S(y,z) \\ \mathbf{W}(y,z) - \mathbf{U}(y) = \mu S(y,z) \end{cases}$$

and

$$S(y,z) = \mathbf{J}(y,z) + \mathbf{W}(y,z) - \mathbf{U}(y).$$

Combining the above equations, the joint surplus solves the following Bellman equation:

$$S(y,z) = Ae^{y+z} - b + \beta \mathrm{E}_{\hat{y},\hat{z}} \left\{ \max\ (\mathbf{S}(\hat{y},\hat{z}),0) - \mu p(\hat{\theta}) \int_{\underline{z}(\hat{y})} S(\hat{y},\tilde{z}) \frac{g_z(\tilde{z})}{1-G_z(\underline{z}(\hat{y}))} d\tilde{z} \right\}.$$

and the free-entry condition is now:

$$\forall y, \quad c = q(\theta(y))(1-\mu) \int \max(\mathbf{S}(y,z),0) g_z(z) dz$$

C.2 Calibration

Let me now calibrate the model. To allow comparison with my model, I use the same targets as in part 5 when they apply (i.e unrelated to multiple-worker firms/decreasing returns): the UE and EU rates, the ratio of home production b over productivity, the autocorrelation and variance of output. To calibrate the match specific productivity process, I cannot target the dispersion of growth rates in employment anymore, but choose to match the similar cross-sectional dispersion of growth rates in sales. I set the bargaining power μ to 0.5 as in the original article. Table 5 and 6 summarize the calibration.

Parameter	Value	Description
Pre-calibrated:		
A	1	Technology parameter
β	0.996	Monthly discount factor
γ	1.60	Job-finding probability parameter
μ	0.5	Bargaining power of workers
Calibrated:		
b	0.88	Home production
c	1.85	Vacancy posting cost
ρ_z	0.88	Persistence of idiosyncratic productivity z
σ_z	0.46	Standard deviation of idiosyncratic productivity z
ρ_y	0.98	Persistence of aggregate productivity y
σ_y	0.04	Standard deviation of aggregate productivity y

Table 5: Calibrated parameters of the MP94 model

Moment	Empirical value	Simulated
UE rate	0.45	0.450
EU rate	0.026	0.025
b / productivity	71%	71%
$\rho[log(output)]$	0.84	0.80
$\sigma[log(output)]$	0.017	0.018
$STD(g_s)$	0.27	0.27

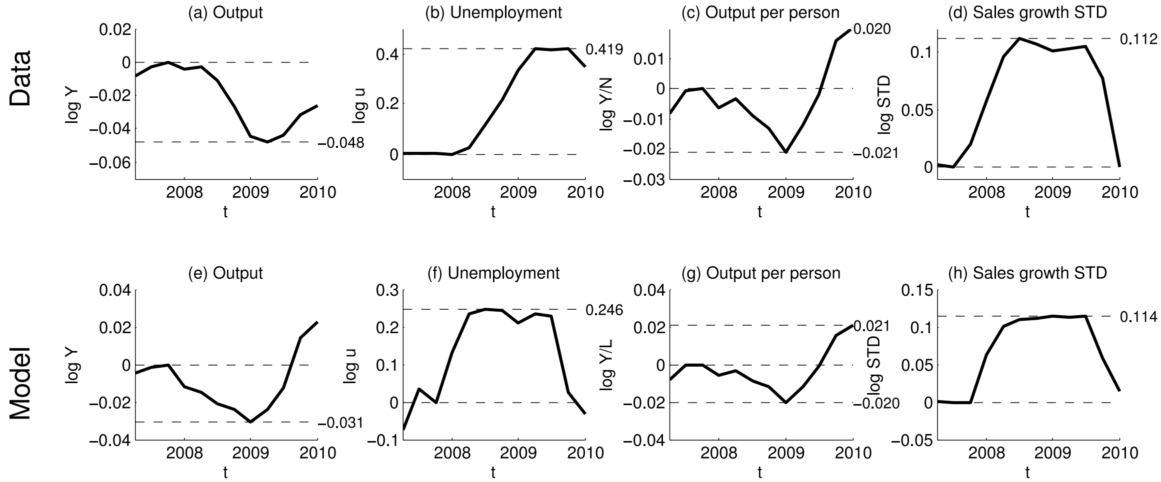
Notes: UE and EU are monthly transition rates. Autocorrelation, standard deviation of log-detrended output and sales growth rate are quarterly.

Table 6: Calibrated moments of the MP94 model

C.3 Response to aggregate productivity and uncertainty shocks

I now proceed to the same exercise as with my model. I examine the response of the model to a series of aggregate productivity shocks calibrated to reproduce the empirical series of output per person between 2007 and 2010. An additional uncertainty shock hits the economy from 2008Q1 to 2009Q3. The shock is calibrated to match an increase in the standard deviation of sales growth rates of 11.2% as in the data. An increase of 14% in σ_z offers a nice fit. Figure 22 displays the series produced by the model.

The response of the economy is qualitatively similar to that of my model. The basic intuition explaining the impact of uncertainty shock is still valid under a Mortensen-Pissarides, except that all



Notes: Log-deviations from steady-state.

Figure 22: Response of MP94 to a series of aggregate productivity and uncertainty shocks

the action takes place on the extensive margin (whether the job is created or destroyed). Unemployment still rises as a result of the larger number of jobs receiving bad shocks, while productivity increases because of the selection of firms. The quantitative results are a little less successful. The model predicts an initial drop in output of 3.1%, smaller than the 4.8% in the data. Output then experiences a counterfactually large recovery, stronger than the one predicted by my model and much larger than the one observed in the data. Unemployment peaks at about 25% (42% in the data) and goes back to a level below trend when uncertainty vanishes. We can conclude from this exercise that uncertainty shocks can still help the model generate much larger fluctuations in unemployment, but that it fails on some other dimensions quantitatively. An important channel is indeed absent from that model: without an intensive margin of adjustment, uncertainty shocks do not induce "wait-and-see" effects. There is no option value of waiting as firms have no incentive to wait before creating jobs. The recession is thus less pronounced and the recovery stronger.

D Proofs of part 2

D.1 Part 2.6

Proof of proposition 1. Denote some firm's policy by $\{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}\}$ and write the sum of the surpluses:

$$
\begin{aligned}
&\tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}\}) \\
&\equiv \mathbf{J}(y,s,z,n,\varphi,\{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}\}) + \int \mathbf{W}(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\})\mathrm{d}\varphi \\
&= \mathrm{e}^{y+z}F(n) - k_f + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Bigg\{ n\hat{d}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{d})\Big(\mathbf{U}(\hat{y},\hat{s})\int \hat{\tau}\mathrm{d}\varphi + \int(1-\hat{\tau})\lambda p(\hat{x})\hat{x}\mathrm{d}\varphi \\
&\qquad - c\hat{v} + \underbrace{\mathbf{J}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi}) + \int (1-\hat{\tau})(1-\lambda p(\hat{x}))\hat{W}\mathrm{d}\varphi}_{\mathbf{J}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi})+\int \hat{W}\mathrm{d}\hat{\varphi}-\hat{v}q(\hat{X})\hat{X}}\Big)\Bigg\}.
\end{aligned}
$$

Rewriting the last expression, we obtain:

$$\begin{aligned}
&\tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}\}) \\
&= e^{y+z}F(n) - k_f + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Bigg\{ n\hat{d}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{d})\Big(\mathbf{U}(\hat{y},\hat{s})\int \hat{\tau}\mathrm{d}\varphi + \int (1-\hat{\tau})\lambda p(\hat{x})\hat{x}\mathrm{d}\varphi \\
&\quad - (c+q(\hat{X})\hat{X})\hat{v} + \underbrace{\mathbf{J}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi}) + \int \hat{W}\mathrm{d}\hat{\varphi}(\hat{W})}_{\equiv \mathbf{V}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi})}\Big)\Bigg\}. \qquad (13)
\end{aligned}$$

Function $\tilde{\mathbf{V}}$ is the joint surplus for some arbitrary contract (not necessarily optimal). To show the result, I will first show that if $\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}$ solves the firm's problem, it must maximize function $\tilde{\mathbf{V}}$. By contradiction, assume that $\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}$ does not, and assume that there exists another policy $\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}$ such that:

$$\tilde{\mathbf{V}}(y,s,z,n,\varphi,\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}) > \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}).$$

If the firm wanted to use the new contract policy $\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}$, only one problem may arise: it may not satisfy the promise-keeping constraint. However, it is possible for the firm to increase or reduce the wage of its workers so that the promise keeping constraint is exactly satisfied. This transformation does not affect the joint surplus, as it only requires internal transfers between the firm and the workers. Without loss of generality, we can therefore choose wages $\tilde{w}$ such that:

$$\forall W, \quad \mathbf{W}(y,s,z;\{\tilde{w}(W),\tilde{\tau}(W),\tilde{x}(W),\tilde{d}(W),\tilde{W}(W)\}) = W.$$

The firm's profit under this contract is:

$$\begin{aligned}
&\mathbf{J}(y,s,z,n,\varphi,\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}) = \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}) - \int W\mathrm{d}\varphi \\
&> \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}) - \int W\mathrm{d}\varphi \qquad (14)
\end{aligned}$$

Since $\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}$ solves the firm's problem by assumption, the promise keeping constraint is satisfied:

$$\forall W, \quad \mathbf{W}(y,s,z;\{w^*(W),\hat{\tau}^*(W),\hat{x}^*(W),\hat{d}^*(W),\hat{W}^*(W)\}) \geq W.$$

Therefore, $\int \mathbf{W}(y,s,z;\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*\})\mathrm{d}\varphi \geq \int W\mathrm{d}\varphi$ and substituting back in (14), we can conclude:

$$\begin{aligned}
&\mathbf{J}(y,s,z,n,\varphi,\{\tilde{w},\tilde{\tau},\tilde{x},\tilde{d},\tilde{W},\tilde{v},\tilde{X}\}) \\
&> \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}) - \int \mathbf{W}(y,s,z;\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*\})\mathrm{d}\varphi \\
&> \mathbf{J}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\})
\end{aligned}$$

This is the contradiction we were looking for. We have thus proved result (ii). We can now conclude for (i):

$$\begin{aligned}
\mathbf{V}(y,s,z,n,\varphi) &\equiv \mathbf{J}(y,s,z,n,\varphi) + \int W\mathrm{d}\varphi(W) \\
&= \mathbf{J}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}) + \int \mathbf{W}(y,s,z;\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*\})\mathrm{d}\varphi \\
&= \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w^*,\hat{\tau}^*,\hat{x}^*,\hat{d}^*,\hat{W}^*,\hat{v}^*,\hat{X}^*\}) \\
&= \max_{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}} \tilde{\mathbf{V}}(y,s,z,n,\varphi,\{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{v},\hat{X}\})
\end{aligned}$$

In other words, we have proved (i), i.e $\mathbf{V}$ solves Bellman equation (4).

(iii) This part follows closely the previous proof. Start with a surplus maximizing policy $\{\hat{\tau}^S, \hat{x}^S, \hat{d}^S, \hat{W}^S, \hat{v}^S, \hat{X}^S\}$. Choose the unique wage schedule $w^S(\cdot)$ such that the promise-keeping constraint binds for all workers. Then, the firm's profit is maximized:

$$\begin{aligned}
&\mathbf{J}(y,z,n,\varphi,\{w^S,\hat{\tau}^S,\hat{x}^S,\hat{d}^S,\hat{W}^S,\hat{v}^S,\hat{X}^S\}) \\
&= \tilde{\mathbf{V}}(y,z,n,\varphi,\{w^S,\hat{\tau}^S,\hat{x}^S,\hat{d}^S,\hat{W}^S,\hat{v}^S,\hat{X}^S\}) - \int W \mathrm{d}\varphi(W) \\
&= \mathbf{V}(y,z,n,\varphi) - \int W \mathrm{d}\varphi(W) = \mathbf{J}(y,z,n,\varphi).
\end{aligned}$$

□

E Proofs of part 3

E.1 Part 3.1

Definition 2 (full) Let $\mathcal{V}$ be the set of value functions $V : (y,s,z,n) \in \mathcal{Y} \times \mathcal{S} \times \mathcal{Z} \times [0,\overline{n}] \longrightarrow \mathbb{R}$ (i) strictly increasing in n, (ii) satisfying $\forall (y,s), \mathrm{E}_{g_z} V(y,s,z,0)^+ \leq \beta k_e$, (iii) bounded in $[\underline{V}, \overline{V}]$, (iv) bi-lipschitz continuous in n such that

$$\forall V \in \mathcal{V}, \forall (y,s,z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V (n_2 - n_1) \leq V(y,s,z,n_2) - V(y,s,z,n_1) \leq \overline{\alpha}_V (n_2 - n_1),$$

with

$$\begin{aligned}
&\underline{\alpha}_V = \mathrm{e}^{\underline{y}+\underline{z}} \underline{\alpha}_F + \beta(1-\beta)^{-1} b > 0, \\
&\overline{\alpha}_V = (1-\beta)^{-1} (\mathrm{e}^{\overline{y}+\overline{z}} \overline{\alpha}_F + \beta \Big((1-\beta)^{-1} (b + \beta \overline{x}) + \lambda \overline{x}) \Big), \\
&\underline{V} = -k_f, \\
&\overline{V} = (1-\beta)^{-1} [\mathrm{e}^{\overline{y}+\overline{z}} F(\overline{n}) - k_f + \beta \overline{n} (\lambda \overline{x} + (1-\beta)^{-1} (b + \beta \overline{x}))].
\end{aligned}$$

Proof of lemma 1. For $V \in \mathcal{V}$, $y \in \mathcal{Y}$, $s \in \mathcal{S}$ and $\kappa \in \mathbb{R}$, define

$$\psi^V(\kappa) = \max_{0 \leq n_e(z) \leq \overline{n}} \mathrm{E}_{g_z} (V(y,s,z,n_e) - \kappa n_e)^+$$

where $\overline{n}$ is an upper bound on the firm sizes, chosen sufficiently large so that it does not constrain the equilibrium. Since V is continuous in n and z has a finite support, ψ^V is a well-defined function for $\kappa \in \mathbb{R}$. The Theorem of the Maximum tells us that ψ^V is a continuous function of κ. Notice that V being increasing in n, $\psi^V(0) = \mathrm{E}_{g_z} V(y,s,z,\overline{n})^+$. Also, since V is bi-lipschitz continuous with parameters $(\underline{\alpha}_V, \overline{\alpha}_V)$, for $\kappa \geq \overline{\alpha}_V$, $\mathrm{E}_{g_z} V(y,s,z,n_e) - \kappa n_e$ is maximized at $n_e = 0$ and $\psi^V(\kappa) = \mathrm{E}_{g_z} V(y,s,z,0)^+$. Let us show that ψ^V is a decreasing function of κ. Take $\kappa_1 < \kappa_2$ and the corresponding $n_{i,i=1,2}$ that solve the maximization problem. Denote $\mathcal{Z}_i = \{z \in \mathcal{Z} | V(y,s,z,n_i) - \kappa_i n_i \geq 0\}$. Then, we have:

$$\begin{aligned}
\psi^V(\kappa_1) - \psi^V(\kappa_2) &= \mathrm{E}_{g_z} (V(y,s,z,n_1) - \kappa_1 n_1)^+ - \mathrm{E}_{g_z} (V(y,s,z,n_2) - \kappa_2 n_2)^+ \\
&\geq \mathrm{E}_{g_z} [(V(y,s,z,n_2) - \kappa_1 n_2) \mathbb{I}_{\mathcal{Z}_2}] - \mathrm{E}_{g_z} [(V(y,s,z,n_2) - \kappa_2 n_2) \mathbb{I}_{\mathcal{Z}_2}] \\
&\geq (\kappa_2 - \kappa_1) \mathrm{E}_{g_z} [n_2 \mathbb{I}_{\mathcal{Z}_2}]
\end{aligned}$$

symmetrically, we can establish that $\psi^V(\kappa_1) - \psi^V(\kappa_2) \leq (\kappa_2 - \kappa_1) \mathrm{E}_{g_z} [n_1 \mathbb{I}_{\mathcal{Z}_1}]$. ψ^V is thus decreasing. But this also tells us that if we denote $\overline{\kappa}$ the smallest κ such that $\psi^V(\kappa) = \mathrm{E}_{g_z} V(y,s,z,0)^+$ (i.e for which $n_e = 0$ is optimal for all z), then we have that ψ^V is strictly decreases on $[0, \overline{\kappa}]$ from $\mathrm{E}_{g_z} V(y,s,z,\overline{n})^+$ to $\mathrm{E}_{g_z} V(y,s,z,0)^+$ and remains constant thereafter.

If $\mathrm{E}_{g_z}V(y,s,z,0)^+ < k_e < \mathrm{E}_{g_z}V(y,s,z,\overline{n})^+$, the Intermediate Value Theorem tells us that there exists a unique $\kappa^V(y,s)$ such that $\psi^V\left(\kappa^V(y,s)\right) = k_e$. This establishes (i):

$$\theta^V(X,y,s) > 0 \Leftrightarrow c/q(\theta(X,y,s)) + X = \kappa^V(y,s).$$

Also, we have (ii): there exists a $n_e^V(y,s,z) \geq 0$ chosen by entering firms so that

$$k_e = \mathrm{E}_{g_z}V(y,s,z,n_e^V(y,s,z)) - \kappa^V(y,s)n_e^V(y,s,z).$$

For firms that decide not to enter, set $n_e^V(y,s,z) = 0$.

To conclude, we only need to check that $\mathrm{E}_{g_z}V(y,s,z,0)^+ < k_e < \mathrm{E}_{g_z}V(y,s,z,\overline{n})^+$. The left-hand side is guaranteed by the fact that $V \in \mathcal{V}$. The right-hand side is guaranteed by assumption 3 as we have $\mathrm{E}_{g_z}V(y,s,z,\overline{n})^+ \geq \mathrm{E}_{g_z}V(y,s,z,0) + \underline{\alpha}_V\overline{n} > -k_f + \underline{\alpha}_V\overline{n} \geq k_e$.

(iii) The complementary slackness condition (7) implies that either:

$$\theta(X,y,s) = 0 \quad \text{or} \quad c/q(\theta(X,y,s)) + X = \kappa^V(y,s).$$

For $X > \kappa^V(y,s) - c$, the second expression admits no solution as the probability q must remain below 1. So θ must be 0 in this region. For $X \leq \kappa^V(y,s) - c$, it admits the unique solution $q^{-1}\left(\frac{c}{\kappa^V(y,s)-X}\right)$. In this region: $c/q(0) + X < \kappa^V$, so $\psi^V(c/q(0)+X) > k_e$. $\theta(X,y,s)$ cannot be 0 otherwise it would violate the free-entry condition (6). To summarize our results:

$$\theta^V(X,y,s) = \begin{cases} q^{-1}\left(\frac{c}{\kappa^V(y,s)-X}\right), & \text{for } \underline{x} \leq X \leq \kappa^V(y,s) - c, \\ 0, & \text{for } X \geq \kappa^V(y,s) - c. \end{cases}$$

□

Proof of proposition 2. To prove the existence, I will proceed in 4 steps: (1) establish existence, uniqueness and boundedness of $U^V(y,s)$ given some $V \in \mathcal{V}$, (2) show that T is a well-defined mapping from $\mathcal{V}$ to $\mathcal{V}$, (3) T is a continuous mapping, (4) $T(\mathcal{V})$ is an equicontinuous family. Since $\mathcal{V}$ is closed, bounded and convex, using Schauder's Fixed Point Theorem as stated in Stokey & Lucas, Theorem 17.4 p.520, this will establish the existence of a solution $\mathbf{V}$ in $\mathcal{V}$ to Bellman equation (4).

Step 1. For $V \in \mathcal{V}$, lemma 1 gives the existence and uniqueness of functions κ^V, n_e^V, θ^V and therefore p^V and q^V. We are going to show that the following mapping M_V that defines U^V is a contraction from the space of functions $U : \mathcal{Y} \times \mathcal{S} \longrightarrow \mathbb{R}$, bounded between some $\underline{U}$ and $\overline{U}$, to be defined later.

$$M^V U(y,s) = b + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\left\{U(\hat{y},\hat{s}) + \max_{\hat{x}_u(\hat{y},\hat{s})} p^V(\hat{x}_u)(\hat{x}_u - U(\hat{y},\hat{s}))\right\}$$

Applying Blackwell's sufficient conditions for a contraction mapping, check *discounting*: for $a \geq 0$,

$$\begin{aligned} M^V(U+a) &= b + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\left\{U(\hat{y},\hat{s}) + a + \max_{\hat{x}_u(\hat{y},\hat{s})} p^V(\hat{x}_u)(\hat{x}_u - U(\hat{y},\hat{s}) - a)\right\} \\ &\leq M^V U + \beta a. \end{aligned}$$

Check *monotonicity*: for $U_1 \leq U_2$, and corresponding optimal choices $\hat{x}_i$, for $i = 1,2$,

$$\begin{aligned} & M^V(U_2) - M^V(U_1) \\ &= \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\left\{U_2(\hat{y},\hat{s}) - U_1(\hat{y},\hat{s}) + p^V(\hat{x}_2)(\hat{x}_2 - U_2(\hat{y},\hat{s})) - p^V(\hat{x}_1)(\hat{x}_1 - U_1(\hat{y},\hat{s}))\right\} \\ &\geq \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}}\left\{U_2(\hat{y},\hat{s}) - U_1(\hat{y},\hat{s}) + p^V(\hat{x}_1)(U_1(\hat{y},\hat{s}) - U_2(\hat{y},\hat{s}))\right\} \geq 0. \end{aligned}$$

It is easy to show now that if $\underline{U} \le U \le \overline{U}$, then

$$b + \beta \underline{U} \le M^V U \le b + \beta(\bar{x} + \overline{U}).$$

The unique fixed point of M^V is therefore bounded between $\underline{U} = (1-\beta)^{-1} b$ and $\overline{U} = (1-\beta)^{-1}(b + \beta\bar{x})$.

Step 2. Let us now check that T is a well defined mapping from $\mathcal{V}$ to $\mathcal{V}$. For what follows, it is useful to denote some policy $\gamma = \{\hat{\tau}, \hat{x}, \hat{d}, \hat{n}_i, \hat{X}\}$, and define

$$\Phi^V(y,s,z,n,\gamma) = e^{y+z} F(n) - k_f + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}} \Bigg\{ n\hat{d}\mathbf{U}^V(\hat{y},\hat{s}) + (1-\hat{d})\Big(\mathbf{U}^V(\hat{y},\hat{s}) \int \hat{\tau} \mathrm{d}j + \int (1-\hat{\tau})\lambda p^V(\hat{x})\hat{x}\mathrm{d}j - \kappa^V(\hat{y},\hat{s})\hat{n}_i + V(\hat{y},\hat{s},\hat{z},\hat{n})\Big) \Bigg\}.$$

Φ denotes the current joint surplus evaluated at some arbitrary policy γ.

(i) If $V \in \mathcal{V}$, then TV is strictly increasing in n. Take $n_1 < n_2$ and the corresponding optimal policies γ_1 and γ_2.

$$\begin{aligned} TV(y,s,z,n_2) - TV(y,s,z,n_1) &= \Phi(y,s,z,n_2,\gamma_2) - \Phi(y,s,z,n_1,\gamma_1) \\ &\ge \Phi(y,s,z,n_2,\tilde{\gamma}) - \Phi(y,s,z,n_1,\gamma_1) \end{aligned}$$

with a suboptimal policy $\tilde{\gamma} = \{\tilde{\tau}, \tilde{x}, \tilde{d}, \tilde{n}_i, \tilde{X}\}$ such that $\tilde{x} = \hat{x}_1$, $\tilde{d} = \hat{d}_1$, $\tilde{n}_i = \hat{n_{i1}}$, $\tilde{X} = \hat{X_{i1}}$, and $\tilde{\tau} = \hat{\tau}_1$ for $j \in [0, n_1]$ and 1 for $j \in [n_1, n_2]$. In that case, we have $\tilde{\hat{n}} = \hat{n}_1$, and many terms cancel to yield the desired result that TV is strictly increasing in n.

$$\begin{aligned} &TV(y,s,z,n_2) - TV(y,s,z,n_1) \ge \Phi(y,s,z,n_2,\tilde{\gamma}) - \Phi(y,s,z,n_1,\gamma_1) \\ &\ge e^{y+z}(F(n_2) - F(n_1)) + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}} \Bigg\{ (n_2 - n_1)\hat{d}_1 \mathbf{U}^V(\hat{y},\hat{s}) + (1-\hat{d}_1)\mathbf{U}^V(\hat{y},\hat{s})(n_2 - n_1) \Bigg\} > 0. \end{aligned}$$

(ii) If $V \in \mathcal{V}$, then $\forall (y,s), \mathrm{E}_{g_z} TV(y,s,z,0)^+ < k_e$. Recall that

$$TV(y,s,z,0) = \max_{w,\hat{\tau},\hat{x},\hat{d},\hat{W},\hat{n}_i,\hat{X}} - k_f + \beta \mathrm{E}_{\hat{y},\hat{s},\hat{z}} \Bigg\{ (1-\hat{d})\Big(-\kappa^V(\hat{y},\hat{s})\hat{n}_i + V(\hat{y},\hat{s},\hat{z},\hat{n}_i)\Big) \Bigg\} \ge -k_f.$$

Thus, integrating over the stationary distribution g_z and using Fubini's theorem:

$$\begin{aligned} \mathrm{E}_{g_z} TV(y,s,z,0) &= \max_{\hat{d}(\hat{y},\hat{s},\hat{z}),\hat{n}_i(\hat{y},\hat{s},\hat{z})} - k_f + \beta \mathrm{E}_{g_z} \mathrm{E}_{\hat{y},\hat{s},\hat{z}} \Bigg\{ (1-\hat{d})\Big(-\kappa^V(\hat{y},\hat{s})\hat{n}_i + V(\hat{y},\hat{s},\hat{z},\hat{n}_i)\Big) \Bigg\} \\ &= \max_{\hat{n}_i(\hat{y},\hat{s},\hat{z})} - k_f + \beta \mathrm{E}_{\hat{y},\hat{s}} \mathrm{E}_{g_z} [V(\hat{y},\hat{s},\hat{z},\hat{n}_i) - \kappa^V(\hat{y},\hat{s})\hat{n}_i]^+ \\ &= -k_f + \beta k_e \end{aligned}$$

where we have recognized the free-entry problem in the next period. Since $TV(y,s,z,0) \ge -k_f$,

$$\begin{aligned} \mathrm{E}_{g_z} TV(y,s,z,0)^+ &= \mathrm{E}_{g_z} \max(TV(y,s,z,0),0) \le \mathrm{E}_{g_z} \max(TV(y,s,z,0) + k_f, 0) \\ &\le \mathrm{E}_{g_z} TV(y,s,z,0) + k_f \le \beta k_e \end{aligned}$$

(iii) If $V \in \mathcal{V}$, then TV is bounded in $[\underline{V}, \overline{V}]$ with $\underline{V} = -k_f$ and $\overline{V} = (1-\beta)^{-1}[e^{\bar{y}+\bar{z}} F(\bar{n}) - k_f + \beta \bar{n}(\lambda \bar{x} + (1-\beta)^{-1}(b + \beta\bar{x}))]$.

$$TV(y,s,z,n) \le e^{\bar{y}+\bar{z}} F(\bar{n}) - k_f + \beta[\bar{n}\overline{U} + \bar{n}\lambda\bar{x} + \overline{V}] \le \overline{V}.$$

Now, for the lower bound:

$$\begin{aligned} TV(y,s,z,n) &\geq \Phi(y,s,z,n_2,\tilde{\gamma}) \\ &\geq e^{\underline{y}+\underline{z}} F(n) - k_f + \beta n \underline{U} \geq -k_f = \underline{V} \end{aligned}$$

with suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = 1$.

(iv) If $V \in \mathcal{V}$, then

$$\forall (y,s,z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V(n_2 - n_1) \leq TV(y,s,z,n_2) - TV(y,s,z,n_1) \leq \overline{\alpha}_V(n_2 - n_1).$$

Take $n_2 \geq n_1$, and corresponding optimal policies $\gamma_i, i = 1,2$. Choose a suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = \hat{d}_2$, $\tilde{x} = \hat{\tau}_2$, $\tilde{n}_i = \hat{n_{i2}}$, $\tilde{X}_i = \hat{X}_{i2}$, $\tilde{\tau}(j) = \hat{\tau}_2(j)$ for $j \in [0,n_1]$ and 1 for $j \in [n_1,n_2]$.

$$\begin{aligned} & TV(y,s,z,n_2) - TV(y,s,z,n_1) = \Phi(y,s,z,n_2,\gamma_2) - \Phi(y,s,z,n_1,\gamma_1) \\ & \leq \Phi(y,s,z,n_2,\gamma_2) - \Phi(y,s,z,n_1,\tilde{\gamma}) \\ & \leq e^{y+z}(F(n_2) - F(n_1)) + \beta E_{\hat{y},\hat{z}} \Bigg\{ (n_2 - n_1)\hat{d}_2 \mathbf{U}^V(\hat{y},\hat{s}) + (1 - \hat{d}_2)\Big(\mathbf{U}^V(\hat{y}) \int_{n_1}^{n_2} \hat{\tau}_2 \mathrm{d}j \\ & + \int_{n_1}^{n_2} (1 - \hat{\tau}_2)\lambda p^V(\hat{x}_2)\hat{x}_2 \mathrm{d}j + V(\hat{y},\hat{s},\hat{z},\hat{n}_2) - V(\hat{y},\hat{s},\hat{z},\hat{n}_1)\Big) \Bigg\} \\ & \leq \Big[e^{\overline{y}+\overline{z}} \overline{\alpha}_F + \beta(\overline{U} + \lambda \overline{x} + \overline{\alpha}_V) \Big](n_2 - n_1) = \overline{\alpha}_V(n_2 - n_1) \end{aligned}$$

Proceed similarly for the other side and choose a policy $\tilde{\gamma}$ such that $\tilde{d} = \hat{d}_1$, $\tilde{x} = \hat{\tau}_1$, $\tilde{n}_i = \hat{n}_{i1}$, $\tilde{X} = \hat{X}_1$, $\tilde{\tau}(j) = \hat{\tau}_1(j)$ for $j \in [0,n_1]$ and 1 for $j \in [n_1,n_2]$.

$$\begin{aligned} & TV(y,s,z,n_2) - TV(y,s,z,n_1) = \Phi(y,s,z,n_2,\gamma_2) - \Phi(y,s,z,n_1,\gamma_1) \\ & \geq \Phi(y,s,z,n_2,\tilde{\gamma}) - \Phi(y,s,z,n_1,\gamma_1) \\ & \geq e^{y+z}(F(n_2) - F(n_1)) + \beta E_{\hat{y},\hat{s},\hat{z}} \Bigg\{ (n_2 - n_1)\hat{d}_2 \mathbf{U}^V(\hat{y},\hat{s}) + (1 - \hat{d}_2)(n_2 - n_1)\mathbf{U}^V(\hat{y},\hat{s}) \Bigg\} \\ & \geq \Big[e^{\underline{y}+\underline{z}} \underline{\alpha}_F + \beta \underline{U} \Big](n_2 - n_1) = \underline{\alpha}_V(n_2 - n_1) \end{aligned}$$

Therefore, TV is bi-lipschitz continuous with the desired coefficients.

Step 3. We are now going to show that $T : \mathcal{V} \longrightarrow \mathcal{V}$ is a continuous mapping. Denote by $||\cdot||$ the infinite norm, i.e $||V|| = \sup_{(y,s,z,n) \in \mathcal{Y} \times \mathcal{S} \times \mathcal{Z} \times [0,\overline{n}]} V(y,s,z,n)$. Take $V_1, V_2 \in \mathcal{V}$. For (y,s,z,n) fixed, denote γ_i, $i = 1,2$, the corresponding optimal policies.

$$\begin{aligned} & TV_1(y,s,z,n) - TV_2(y,s,z,n) = \Phi^{V_1}(y,s,z,n,\gamma_1) - \Phi^{V_2}(y,s,z,n,\gamma_2) \\ & \leq \Phi^{V_1}(y,s,z,n,\gamma_1) - \Phi^{V_2}(y,s,z,n,\gamma_1) \\ & \leq \beta E_{\hat{y},\hat{s},\hat{z}} \Bigg\{ n\hat{d}_1(\mathbf{U}^{V_1}(\hat{y},\hat{s}) - \mathbf{U}^{V_2}(\hat{y},\hat{s})) + (1 - \hat{d}_1)\Big((\mathbf{U}^{V_1}(\hat{y},\hat{s}) - \mathbf{U}^{V_2}(\hat{y},\hat{s})) \int \hat{\tau}_1 \mathrm{d}j + \\ & \int (1 - \hat{\tau}_1)\lambda(p^{V_1}(\hat{x}_1) - p^{V_2}(\hat{x}_1))\hat{x}_1 \mathrm{d}j - (\kappa^{V_1}(\hat{y},\hat{s}) - \kappa^{V_2}(\hat{y},\hat{s}))\hat{n}_{i1} + V_1(\hat{y},\hat{s},\hat{z},\hat{n}_1) - V_2(\hat{y},\hat{s},\hat{z},\hat{n}_1)\Big) \Bigg\} \\ & \leq \beta \Big[\overline{n}||\mathbf{U}^{V_1} - \mathbf{U}^{V_2}|| + \overline{n}\lambda\overline{x}||p^{V_1} - p^{V_2}|| + \overline{n}||\kappa^{V_1} - \kappa^{V_2}|| + ||V_1 - V_2|| \Big] \end{aligned}$$

According to lemma 2 below, we can control each term:

$$TV_1(y,s,z,n) - TV_2(y,s,z,n) \leq \beta(\overline{n}\alpha_U + \overline{n}\lambda\overline{x}\alpha_p + \overline{n}\alpha_\kappa + 1)||V_1 - V_2||$$

which can be made arbitrarily small as $||V_1 - V_2||$ gets smaller. Therefore, T is a continuous mapping.

Lemma 2. *If $V_1, V_2 \in \mathcal{V}$, then*

(i) $||\theta^{V_1} - \theta^{V_2}|| \leq \alpha_\theta ||V_1 - V_2||$, *with* $\alpha_\theta = \frac{1}{c n_{min} |q'(\theta_{max}|}$,

(ii) $||p^{V_1} - p^{V_2}|| \leq \alpha_p ||V_1 - V_2||$, *with* $\alpha_p = p'(0)\alpha_\theta$,

(iii) $||\kappa^{V_1} - \kappa^{V_2}|| \leq \alpha_\kappa ||V_1 - V_2||$, *with* $\alpha_\kappa = c\frac{|q'(0)|}{q^2(\theta_{max})}\alpha_\theta$,

(iv) $||\mathbf{U}^{V_1} - \mathbf{U}^{V_2}|| \leq \alpha_U ||V_1 - V_2||$, *with* $\alpha_U = \beta(1-\beta)^{-1}(\bar{x} + \overline{U})\alpha_p$.

Proof. To prove the lemma, we first need to establish the two following results. Let us prove that there exists $\theta_{max} > 0$ such that

$$\forall V \in \mathcal{V}, \theta^V(\cdot) \leq \theta_{max},$$

and there exists $n_{min} > 0$ such that

$$\forall V \in \mathcal{V}, \mathrm{E}_{g_z} n_e^V(y,s,z) \geq n_{min}.$$

The first result can be established by the fact that $\kappa_V \leq \overline{\alpha}_V$. Then for some $X \in [\underline{x}, \bar{x}]$:

$$c/q(\theta^V(y,s,X)) + X \leq \overline{\alpha}_V \Rightarrow q(\theta^V(y,s,X)) \geq c(\overline{\alpha}_V - X)^{-1} \Rightarrow \theta^V(y,s,X) \leq q^{-1}[c(\overline{\alpha}_V - \underline{x})^{-1}].$$

Setting $\theta_{max} = q^{-1}[c(\overline{\alpha}_V - \underline{x})^{-1}]$ yields the desired result.
Now, for the second result, remember the free-entry condition with positive entry:

$$k_e = \mathrm{E}_{g_z}[V(y,s,z,n_e^V) - \kappa^V n_e^V]^+.$$

Then, using the fact that V is bi-lipschitz:

$$k_e \leq \mathrm{E}_{g_z} V(y,s,z,n_e^V)^+ \leq \mathrm{E}_{g_z}[\overline{\alpha}_V n_e^V + V(y,s,z,0)]^+ \leq \overline{\alpha}_V \mathrm{E}_{g_z}[n_e^V] + \mathrm{E}_{g_z} V(y,s,z,0)^+$$

Since $\mathrm{E}_{g_z} V(y,s,z,0)^+ \leq \beta k_e$, we have

$$\mathrm{E}_{g_z} n_e^V \geq \overline{\alpha}_V^{-1}(1-\beta)k_e \equiv n_{min}.$$

(i) For $y \in \mathcal{Y}, s \in \mathcal{S}$, $X \in [\underline{x}, \bar{x}]$, and $n : z \in \mathcal{Z} \longrightarrow [0, \bar{n}]$. Define:

$$\Psi^V(y,s,X,\theta,n) = \mathrm{E}_{g_z}[V(y,s,z,n) - (c/q(\theta) + X)n(z)]^+.$$

Recognize that Ψ represents the expected value of firms considering to enter on submarket X with tightness θ. Fix (y,s,X), and denote $(\theta_i, n_i) \equiv (\theta^{V_i}(y,s,X), n_e^{V_i}(y,s,z,X))$ the free-entry solutions corresponding the two V's. By definition we have

$$k_e \geq \Psi^{V_i}(y,s,X,\theta_i,n_i)$$

with equality if submarket X is open, $\theta_i = 0$ otherwise. Denote $\mathcal{Z}_i = \{z \in \mathcal{Z} | V_i(y,s,z,n_i) - \kappa^{V_i} n_i \geq 0\}, i = 1,2$. Notice that $\mathbb{P}(z \in \mathcal{Z}_i) > 0$ since there is always strictly positive free-entry. If X is closed for V_1 and V_2, there is nothing to prove as $\theta_1 = \theta_2 = 0$. Now, let us look at the case where submarket X is open for both V_1 and V_2. We have:

$$\begin{aligned} 0 &= \Psi^{V_1}(y,s,X,\theta_1,n_1) - \Psi^{V_2}(y,s,X,\theta_2,n_2) \geq \Psi^{V_1}(y,s,X,\theta_1,n_2) - \Psi^{V_2}(y,s,X,\theta_2,n_2) \\ &\geq \mathrm{E}_{g_z}\left[\left(V_1(y,s,z,n_2) - V_2(y,s,z,n_2) + n_2\left(\frac{c}{q(\theta_2)} - \frac{c}{q(\theta_1)}\right)\right)\mathbb{I}_{z \in \mathcal{Z}_2}\right]. \end{aligned}$$

We get

$$\mathrm{E}_{g_z}[n_2] c \frac{q(\theta_1) - q(\theta_2)}{q(\theta_1)q(\theta_2)} \leq ||V_1 - V_2||.$$

Symmetrically, we can establish that

$$\mathrm{E}_{g_z}[n_1]c\frac{q(\theta_1)-q(\theta_2)}{q(\theta_1)q(\theta_2)} \geq -||V_1-V_2||.$$

By assumption, q is convex. Therefore:

$$q'(\min(\theta_1,\theta_2))(\theta_1-\theta_2) \leq q(\theta_1)-q(\theta_1) \leq q'(\max(\theta_1,\theta_2))(\theta_1-\theta_2).$$

Combining with the previous inequalities, we obtain the desired inequality:

$$|\theta_1-\theta_2| \leq \frac{||V_1-V_2||}{cn_{min}|q'(\theta_{max})|}.$$

We are left with the case where market X is closed for one of the V's but not the other. Assume without loss of generality that X is closed for V_1, i.e $\theta_1=0$. In that case:

$$\forall n_1, \Psi(y,s,X,0,n_1) < k_e.$$

We can still derive the above inequality that said

$$\forall z \in I_2, \quad cn_2\frac{q(\theta_1)-q(\theta_2)}{q(\theta_1)q(\theta_2)} \leq ||V_1-V_2||.$$

In which case we have:

$$0 \leq \theta_2-\theta_1 \leq \frac{||V_1-V_2||}{cn_2|q'(\theta_2)|}.$$

To summarize, in all cases we have:

$$||\theta^{V_1}-\theta^{V_2}|| \leq \alpha_\theta||V_1-V_2||$$

with $\alpha_\theta = \frac{1}{cn_{min}|q'(\theta_{max}|}$.
(ii) Fix (y,s,X). By definition:

$$p^{V_1}(y,s,X)-p^{V_2}(y,s,X) = p(\theta^{V_1}(y,s,X))-p(\theta^{V_2}(y,s,X)).$$

Using concavity of p:

$$|p(\theta^{V_1}(y,s,X))-p(\theta^{V_2}(y,s,X))| \leq p'(0)|\theta^{V_1}(y,s,X)-\theta^{V_2}(y,s,X)| \leq p'(0)\alpha_\theta||V_1-V_2||.$$

(iii) Fix y and take some submarket X open for both V's. Using the fact that q is decreasing convex:

$$\begin{aligned}
|\kappa^{V_1}(y,s)-\kappa^{V_2}(y,s)| &= \left|\frac{c}{q(\theta^{V_1}(y,s,X))}-\frac{c}{q(\theta^{V_2}(y,s,X))}\right| \\
&\leq c\frac{|q'(0)|}{q^2(\theta_{max})}|\theta^{V_1}(y,s,X)-\theta^{V_2}(y,s,X)| \leq c\frac{|q'(0)|}{q^2(\theta_{max})}\alpha_\theta||V_1-V_2||.
\end{aligned}$$

(iv) Fix (y,s). Denote $\hat{x}_{u,i}, i=1,2$ the corresponding optimal choices for unemployed workers.

$$\begin{aligned}
U^{V_1}(y,s)-U^{V_2}(y,s) &= \beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{U^{V_1}(\hat{y},\hat{s})+p^{V_1}(\hat{x}_{u,1})(\hat{x}_{u,1}-U^{V_1}(\hat{y},\hat{s})) \\
&\quad -U^{V_2}(\hat{y},\hat{s})-p^{V_2}(\hat{x}_{u,2})(\hat{x}_{u,2}-U^{V_2}(\hat{y},\hat{s}))\Big\} \\
&\leq \beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{U^{V_1}(\hat{y},\hat{s})-U^{V_2}(\hat{y},\hat{s})+p^{V_1}(\hat{x}_{u,1})(\hat{x}_{u,1}-U^{V_1}(\hat{y},\hat{s}))-p^{V_2}(\hat{x}_{u,1})(\hat{x}_{u,1}-U^{V_2}(\hat{y},\hat{s}))\Big\} \\
&\leq \beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{U^{V_1}(\hat{y},\hat{s})-U^{V_2}(\hat{y},\hat{s})+\hat{x}_{u,1}(p^{V_1}(\hat{x}_{u,1})-p^{V_2}(\hat{x}_{u,1})) \\
&\quad -p^{V_1}(\hat{x}_{u,1})U^{V_1}(\hat{y},\hat{s})+p^{V_2}(\hat{x}_{u,1})U^{V_1}(\hat{y},\hat{s})-p^{V_2}(\hat{x}_{u,1})U^{V_1}(\hat{y},\hat{s})+p^{V_2}(\hat{x}_{u,1})U^{V_2}(\hat{y},\hat{s})\Big\} \\
&\leq \beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{(1-p^{V_2}(\hat{x}_{u,1}))(U^{V_1}(\hat{y},\hat{s})-U^{V_2}(\hat{y},\hat{s}))+(\hat{x}_{u,1}-U^{V_1}(\hat{y},\hat{s}))(p^{V_1}(\hat{x}_{u,1})-p^{V_2}(\hat{x}_{u,1}))\Big\} \\
&\leq \beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{(1-p^{V_2}(\hat{x}_{u,1}))||U^{V_1}-U^{V_2}||+(\hat{x}_{u,1}-U^{V_1}(\hat{y},\hat{s}))||p^{V_1}-p^{V_2}||\Big\}.
\end{aligned}$$

We can now conclude:

$$||U^{V_1} - U^{V_2}|| \leq \beta(1-\beta)^{-1}(\bar{x}+\overline{U})\alpha_p||V_1 - V_2||. \quad \square$$

Step 4. We can now proceed to the last step of the proof of proposition 2. We must show that the family $T(\mathcal{V})$ is equicontinuous, i.e $\forall \varepsilon > 0$, there exists $\delta > 0$ such that for $\xi_i = (y_i, s_i, z_i, n_i), i = 1, 2$,

$$||\xi_1 - \xi_2|| < \delta \Rightarrow |TV(\xi_1) - TV(\xi_2)| < \varepsilon, \forall V \in \mathcal{V}.$$

Fix $\varepsilon > 0$ and denote

$$\begin{cases} \eta_y = \min\limits_{y_1 \neq y_2 \in \mathcal{Y}} |y_1 - y_2| \\ \eta_s = \min\limits_{s_1 \neq s_2 \in \mathcal{S}} |s_1 - s_2| \\ \eta_z = \min\limits_{z_1 \neq z_2 \in \mathcal{Z}} |z_1 - z_2|. \end{cases}$$

Choose $\delta < \min(\eta_y, \eta_s, \eta_z, \varepsilon/\overline{\alpha}_V)$. Take (ξ_1, ξ_2) such that $||\xi_1 - \xi_2|| < \delta$. Therefore, $y_1 = y_2$, $s_1 = s_2$ and $z_1 = z_2$. Take $V \in \mathcal{V}$. Using the fact that V is bi-lipschitz:

$$|TV(\xi_1) - TV(\xi_2)| \leq \overline{\alpha}_V |n_1 - n_2| \leq \overline{\alpha}_V ||\xi_1 - \xi_2|| < \varepsilon.$$

Conclusion: $T(\mathcal{V})$ is equicontinuous. Schauder's Fixed Point Theorem applies and tells us that there exists a fixed point **V** to the mapping T. All other equilibrium objects **U**, **W**, **J**, θ, κ and optimal policy functions are then well defined. $\square$

E.2 Part 3.2

Proof of prop. 3.
We are now going to define the planner's problem in this economy. There is a first difficulty arising in the way we are going to describe the different labor markets. Since the planner can freely allocate workers between firms without respect to any promised utility, the only relevant information concerning each submarket is its tightness. Let us therefore label each submarket by its tightness θ instead of x. Denote by $(\theta_x, \theta_X, \theta_u)$ the markets chosen respectively by firms for on-the-job search, for hirings, and the one chosen by unemployed workers to search. Second, every workers are identical in the eyes of the planner. Given the strict concavity of the problem and therefore the uniqueness of the maximum, I restrict immediately the allocation to be symmetrical between workers of the same firm for x_t. Similarly, as proposition 4 will make it apparent, τ_t should be understood as the total fraction of layoffs, which is uniquely determined in equilibrium, whereas the exact distribution of layoffs across workers in the same firm is not. To simplify notation, I will drop the explicit dependence of each variables below on the aggregate state variables $y^t = (y_0, ..., y_t)$ and $s^t = (s_0, ..., s_t)$ as well as the firm's characteristics at the beginning of the period (z_t, n_{t-1}) after the aggregate and idiosyncratic shocks are realized. The planning problem is to maximize

$$\mathrm{E}_{y,s} \sum_t \beta^t \Big[\sum_{z_t, n_t} g_t(z_t, n_t)(1-d_t)\big(F(n_t) - k_f - cv_t\big) - k_e h_t + u_t b \Big]$$

subject to: $\forall (t, y^t, s^t)$,

$$u_t = u_{t-1}(1 - p(\theta_{u,t})) + \sum_{z_{t-1}, z_t, n_{t-1}} g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) n_{t-1}[d_t + (1-d_t)\tau_t] \quad (15)$$

$$\forall (z_t, n_{t-1}), \quad n_t = n_{t-1}(1-\tau_t)(1 - \lambda p(\theta_{xt})) + q(\theta_{Xt}) v_t \quad (16)$$

$$\forall (z, n), g_t(z, n) = \sum_{n_{t-1} | n_t = n} (1-d_t) g_{t-1}(z_{t-1}, n_{t-1}) \pi_z(z_{t-1}, z_t) + h_t g_z(z) \mathbb{I}(n_{et}(z) = n). \quad (17)$$

The objective function is the discounted sum of production net of operating cost k_f and vacancy posting cost c over all existing firms, minus total entry costs for new firms h_t every period and home production b of unemployed agents. The constraints are the law of motions for the unemployment level u_t, the employment in every firm of type (z_t, n_{t-1}), and distribution g_t. In addition, the planner is subject to constraints for each labor market specifying that the number of workers finding a job is equal to the number of successful job openings. More precisely, for each labor market θ in every period:

$$\sum_{(z_{t-1},z_t,n_{t-1})|\theta_{xt}=\theta} \underbrace{g_{t-1}(z_{t-1},n_{t-1})\pi_z(z_{t-1},z_t)(1-d_t)n_{t-1}(1-\tau_t)\lambda p(\theta_{xt})}_{\text{successful job-to-job transitions}} + \underbrace{\mathbb{I}(\theta_{ut}=\theta)u_{t-1}p(\theta_{ut})}_{\text{unemployed finding a job}}$$
$$= \sum_{(z_{t-1},z_t,n_{t-1})|\theta_{Xt}=\theta} \underbrace{g_{t-1}(z_{t-1},n_{t-1})\pi_z(z_{t-1},z_t)(1-d_t)v_t q(\theta_{Xt})}_{\text{successful job openings by incumbent firms}} + h_t \sum_{z|\theta_{Xt}=\theta} \underbrace{g_z(z)(1-d_t)v_t q(\theta_{Xt})}_{\text{by entering firms}} \tag{18}$$

Let us now write the Lagrangian corresponding to the planner's problem. Write μ_t the Lagrange multiplier on constraint (15) and $\eta_t(\theta)$ the one for each market equilibrium (18). The formal planning problem can be written as follows:

$$\max_{\substack{\tau_t,\theta_{xt},d_t,v_t,\\ \theta_{Xt},h_t,\theta_{ut}\\ u_t,n_t,g_t}} \mathrm{E}\sum_t \beta^t \Big\{ \sum_{z_{t-1},z_t,n_{t-1}} g_{t-1}(z_{t-1},n_{t-1})\pi_z(z_{t-1},z_t)\Big[(1-d_t)\big(F(n_t)-k_f-cv_t$$
$$-\eta_t(\theta_{Xt})q(\theta_{Xt})v_t+\eta_t(\theta_{xt})n_{t-1}(1-\tau_t)\lambda p(\theta_{xt})\big)+\mu_t n_{t-1}(d_t+(1-d_t)\tau_t)\Big]$$
$$+h_t\sum_z g_z(z)\Big[(1-d_t)\big(F(n_t)-k_f-cv_t-\eta_t(\theta_{Xt})q(\theta_{Xt})v_t\big)-k_e\Big]$$
$$+u_t b-\mu_t\big(u_t-u_{t-1}(1-p(\theta_{ut}))\big)+\eta_t(\theta_{ut})u_{t-1}p(\theta_{ut})\Big\}$$

subject to equations (16) and (17). As it is written, the objective function is not concave in all its variables, but if we proceed to the changes of variables $\xi_{xt}=p(\theta_{xt})$, $\xi_{ut}=p(\theta_{ut})$ and $\zeta_{Xt}=q(\theta_{xt})$, then the problem becomes a well-defined strictly concave maximization problem subject to a convex constraint set. The optimum therefore exists and is unique, and the first-order conditions are sufficient to guarantee optimality.

To complete our proof, we are now going to show that our block-recursive competitive equilibrium with positive entry satisfies the first-order condition and is therefore the optimum. Denote the competitive equilibrium $\{\mathbf{V},\mathbf{U},\theta^*(x,y,s)\}$. Guess the following Lagrange multipliers:

$$\mu_t(y^t,s^t)=\mathbf{U}(y_t,s_t)$$
$$\eta_t(y^t,s^t,\theta)=x \text{ s.t } x=\theta^{*-1}(\theta,y_t,s_t).$$

In particular, notice that the Lagrange multipliers only depend on the current aggregate state of the economy (y_t,s_t) and not on its entire history anymore. One may worry here about the feasibility of inverting the equilibrium function θ^*, but we know thanks to lemma 1 that there always exists a corresponding promised utility x for all values of θ in $[0,\infty)$[15]. Given this guess, we can now recognize that the planner's objective is to sum the joint-surplus $\mathbf{V}$ of incumbent and entering firms and the utility of unemployed workers $\mathbf{U}$. Each of these problems can be solved independently and

[15]The bounds $[\underline{x},\overline{x}]$ are chosen so that the optimal x lies in the interior, so that we are not constraining the equilibrium.

we know that the policies obtained in the competitive equilibrium maximize each of them. To see this, let us have a look at the part corresponding to an existing firm given our choice of Lagrange multipliers:

$$\max_{\tau_t,\theta_{xt},d_t,v_t,\theta_{Xt}} \mathrm{E}_{y,s,z}\sum_t \beta^t\Big[(1-d_t)\big(F(n_t)-k_f-cv_t-q(\theta_{Xt})v_tX_t(\theta_{Xt})$$
$$+n_{t-1}(1-\tau_t)\lambda p(\theta_{xt})x_t(\theta_{xt})\big)+n_{t-1}(d_t+(1-d_t)\tau_t)\mathbf{U}(y_t,s_t)\Big]$$

which corresponds exactly to the surplus maximization problem in the competitive equilibrium. Turning to firms entering at date t:

$$\max_{\tau_{t'},\theta_{xt'},d_{t'},v_{t'},\theta_{Xt'},t'\geq t} h_t\Big\{\mathrm{E}_{g_z}\mathrm{E}_{y,s,z}\sum_{t'\geq t}\beta^{t'-t}\Big[(1-d_{t'})\big(F(n_{t'})-k_f-cv_{t'}-q(\theta_{Xt'})v_{t'}X_{t'}(\theta_{Xt'})$$
$$+n_{t'-1}(1-\tau_{t'})\lambda p(\theta_{xt'})x_{t'}(\theta_{xt'})\big)+n_{t'-1}(d_{t'}+(1-d_{t'})\tau_{t'})\mathbf{U}(y_{t'},s_{t'})\Big]-k_e\Big\}.$$

This is exactly the free-entry problem solved in the competitive equilibrium. The planner will increase the number of entrants h_t as long as the expected surplus from entering is equal to the entry cost k_e. Now, comparing the part related to unemployed workers:

$$\max_{\theta_{ut},u_t}\sum_t\beta^t\Big[u_tb-\mathbf{U}(y_t)\big(u_t-u_{t-1}(1-p(\theta_{ut}))\big)+u_{t-1}p(\theta_{ut})x_t(\theta_{ut})\Big].$$

The first-order condition with respect to u_{t+1} and θ_{ut} are equal to

$$(u_t)\quad b-\mathbf{U}(y_t,s_t)+\beta\mathrm{E}_{y,s}[p(\theta_{u,t+1})\mathbf{U}(y_{t+1},s_{t+1})+p(\theta_{u,t+1})x_{t+1}(\theta_{u,t+1})]=0$$
$$(\theta_{ut})\quad -\mathbf{U}(y_t,s_t)p'(\theta_{ut})+p'(\theta_{ut})x_t(\theta_{ut})+p(\theta_{ut})x'(\theta_{ut})=0.$$

We recognize in the first equation the Bellman equation faced by unemployed workers, and in the second equation the first-order of the maximization of $p(\theta_{ut})(x_t(\theta_{ut})-\mathbf{U}(y_t,s_t))$ which is exactly the problem they face. Therefore, the policies obtained from the competitive equilibrium maximize the planner's problem given our choice of Lagrange multipliers. The first-order conditions are thus satisfied. We can now conclude that if a block-recursive equilibrium exists with positive entry, it is also the unique efficient equilibrium. □

E.3 Part 3.3

Proof of proposition 4. (i) Let us write the utility of the worker under the new contract $\{w+\Delta,\hat{\tau},\hat{x},\hat{d},\hat{W}-a\Delta\}$:

$$\mathbf{W}(y,s,z;\{w+\Delta,\hat{\tau},\hat{x},\hat{d},\hat{W}-a\Delta\})=w+\Delta+\beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{\hat{d}\mathbf{U}(\hat{y},\hat{s})+(1-\hat{d})\Big(\hat{\tau}\mathbf{U}(\hat{y},\hat{s})$$
$$+(1-\hat{\tau})\lambda p(\hat{x})\hat{x}+(1-\hat{\tau})(1-\lambda p(\hat{x}))(\hat{W}-a\Delta)\Big)\Big\}$$
$$=\mathbf{W}(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\})+\underbrace{\Delta-\beta(1-\lambda p(\hat{x}))(1-\hat{\tau})(1-\hat{d})a\Delta}_{=0}$$

The worker's utility is unchanged. His promise-keeping constraint is thus still satisfied. Now, write the firm's expected utility:

$$\mathbf{J}(y,s,z,n,\varphi)=\mathrm{e}^{y+z}F(n)-k_f-\int w\mathrm{d}\varphi+\beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{(1-\hat{d})\Big(-c\hat{v}+\mathbf{J}(\hat{y},\hat{s},\hat{z},\hat{n},\hat{\varphi})\Big)\Big\}$$
$$=\mathrm{e}^{y+z}F(n)-k_f-\int w\mathrm{d}\varphi+\beta\mathrm{E}_{\hat{y},\hat{s},\hat{z}}\Big\{(1-\hat{d})\Big(-c\hat{v}+\mathbf{V}(\hat{y},\hat{s},\hat{z},\hat{n})-\int\hat{W}\mathrm{d}\hat{\varphi}\Big)\Big\}$$

Under the new contract, since only the contract for workers receiving utily W, the firm's next distribution $\hat{\varphi}_\Delta$ is:

$$\forall W', \hat{\varphi}_\Delta(W') = \hat{\varphi}(W') + (1-\lambda p(\hat{x}(W))(1-\hat{\tau}(W))(-\mathbb{1}(W' \geq \hat{W}) + \mathbb{1}(W' \geq \hat{W} - a\Delta))$$

Therefore, the firm's utility changes by

$$[-\Delta - \beta(1-\lambda p(\hat{x}))(1-\hat{\tau})(1-\hat{d})(-a\Delta)]\mathrm{d}\varphi(W) = 0.$$

The new contract leaves the firm and workers indifferent.
(*ii*) First, recall that:

$$p(\hat{x}) = p \circ q^{-1}\left(\frac{c}{\kappa - \hat{x}}\right)$$

It is easy to show that under assumption 2, $p(\hat{x})$ is a strictly decreasing, strictly concave function on $[\underline{x}, \kappa(y) - c]$.
Let us first show that for each worker, there is a unique $\hat{x} \in [\underline{x}, \kappa(\hat{y}) - c]$ that maximizes the surplus. The terms that depend on $\hat{x}$ in the joint surplus are the following:

$$\int (1-\hat{\tau})\lambda p(\hat{x})\hat{x}\mathrm{d}\varphi + \mathbf{V}\left(\hat{y}, \hat{s}, \hat{z}, \int (1-\hat{\tau})(1-\lambda p(\hat{x}))\mathrm{d}\varphi + \hat{n_e}\right)$$

Proceed to the change of variable $\hat{\xi} = p(\hat{x})$, which is well defined since p is continuous strictly increasing. The term to maximize as a function of $\hat{\xi}$ is:

$$\int (1-\hat{\tau})\lambda\hat{\xi}p^{-1}(\hat{\xi})\mathrm{d}\varphi + \mathbf{V}\left(\hat{y}, \hat{s}, \hat{z}, \int (1-\hat{\tau})(1-\lambda\hat{\xi})\mathrm{d}\varphi + \hat{n_e}\right) \quad (19)$$

p being decreasing, strictly concave, so p^{-1} is also decreasing, strictly concave. Since $\mathbf{V} \in \mathcal{V}$, i.e concave in n, the whole expression is strictly concave in ξ.
Let us show now that all workers have the same $\hat{\xi}$. Assume by contradiction that two workers have different $\hat{x_1} \neq \hat{x_2}$, i.e $\hat{\xi_1} \neq \hat{\xi_2}$. Then, since p^{-1} is strictly concave, the corresponding first term in (19) is:

$$(1-\hat{\tau}_1)p^{-1}(\hat{\xi_1})\hat{\xi_1} + (1-\hat{\tau}_2)p^{-1}(\hat{\xi_2})\hat{\xi_2} < [(1-\hat{\tau}_1) + (1-\hat{\tau}_2)]p^{-1}\left(\hat{\xi}'\right)\hat{\xi}'$$

where $\hat{\xi}' = \frac{(1-\hat{\tau}_1)\hat{\xi}_1 + (1-\hat{\tau}_2)\hat{\xi}_2}{2-\hat{\tau}_1-\hat{\tau}_2}$ The surplus can be strictly higher if we offer a contract $(\hat{x}', \hat{\tau}')$ to both workers with the same $\hat{x}' = p^{-1}(\hat{\xi}')$, and a firing probability $\hat{\tau}'$ such that:

$$(1-\hat{\tau_1})(1-\lambda\hat{\xi_1}) + (1-\hat{\tau_2})(1-\lambda\hat{\xi_2}) = 2\left(1-\lambda\hat{\xi}'\right)(1-\hat{\tau}').$$

Therefore, the rest of the expression in (19) is left equal to the case with the initial contract. The new contract $(\hat{x}', \hat{\tau}')$ that sets the two workers identical strictly increases the total surplus. $\hat{x}$ must therefore be the same across workers.
(*iii*) $\hat{x}$ being the same for all workers, we can rewrite the only terms in the surplus that depend on $\hat{\tau}$ as:

$$\mathbf{U}(\hat{y}, \hat{s}) \int \hat{\tau}\mathrm{d}\varphi + \lambda p(\hat{x})\hat{x} \int (1-\hat{\tau})\mathrm{d}\varphi + \mathbf{V}\left(\hat{y}, \hat{s}, \hat{z}, (1-\lambda p(\hat{x})) \int (1-\hat{\tau})\mathrm{d}\varphi + \hat{n_e}\right)$$

This clearly shows that only the total number of layoffs $\int \hat{\tau}\mathrm{d}\varphi$ is determined by the maximization. Any combination of $\hat{\tau}$ that keeps this summation constant yields the same joint surplus. □

E.4 Part 3.4

Proof of proposition 5. I will prove the result in two steps. I will first show that if the firm can choose any continuing utility $\hat{W} \in \mathbb{R}$, then there exists a unique $\hat{W}(\hat{x}^*)$ that makes the worker choose $\hat{x}^*$ exactly. We will then show that this continuing utility must satisfy the participation constraint, i.e $\lambda p(\hat{x}^*)\hat{x}^* + (1-\lambda p(\hat{x}^*))\hat{W}(\hat{x}^*) \geq \mathbf{U}(\hat{y},\hat{s})$.

Step 1. Recall that workers solve the problem

$$\hat{x}^* = \underset{\hat{x}\in[\underline{x},\kappa(\hat{y},\hat{s})-c]}{\text{argmax}} \; p(\hat{x})(\hat{x}-\hat{W}).$$

Define

$$\tilde{D}(\hat{x},\hat{W}) = p(\hat{x})(\hat{x}-\hat{W}) \text{ and } \begin{cases} D(\hat{W}) = \underset{\hat{x}\in[\underline{x},\kappa(\hat{y},\hat{s})-c]}{\max} \tilde{D}(\hat{x},\hat{W}) \\ C(\hat{W}) = \underset{\hat{x}\in[\underline{x},\kappa(\hat{y},\hat{s})-c]}{\text{argmax}} \tilde{D}(\hat{x},\hat{W}) \end{cases}$$

$\tilde{D}$ is a continuous function of $\hat{x}$ and $\hat{W}$. It reaches its maximum in $\hat{x}$ on $[\hat{W},\kappa(\hat{y},\hat{s})-c]$. Assumption 2 guarantees that $\tilde{D}$ is strictly concave in $\hat{x}$ on $[\hat{W},\kappa(\hat{y},\hat{s})-c]$. The Theorem of the Maximum tells us therefore that $D(\hat{W})$ and $C(\hat{W})$ are continuous functions of $\hat{W}$. p being strictly positive over $[\underline{x},\kappa(\hat{y},\hat{s})-c)$, D is strictly decreasing on $[-\infty,\kappa(\hat{y},\hat{s})-c]$. Therefore, C is strictly increasing on $[-\infty,\kappa(\hat{y},\hat{s})-c]$, as can be seen from the following: take $\hat{W}_1 < \hat{W}_2 \leq \kappa(\hat{y},\hat{s})-c$. Denote $\hat{x}_i = C(\hat{W}_i), i = 1,2$. Then the following is true:

$$p(\hat{x}_2)(\hat{W}_2-\hat{W}_1) < p(\hat{x}_1)(\hat{x}_1-\hat{W}_1) - p(\hat{x}_2)(\hat{x}_2-\hat{W}_2) < p(\hat{x}_1)(\hat{W}_2-\hat{W}_1)$$

Therefore, $\hat{x}_2 > \hat{x}_1$ and C is strictly increasing.
Now, let us show that C reaches $\underline{x}$ and $\kappa(\hat{y},\hat{s})-c$. For $\hat{W} = \kappa(\hat{y},\hat{s})-c$, function $\tilde{D}$ trivially reaches its maximum at $\hat{x} = \hat{W} = \kappa(\hat{y},\hat{s})-c$. The derivative of $\tilde{D}$ with respect to $\hat{x}$ is

$$p'(\hat{x})(\hat{x}-\hat{W}) + p(\hat{x}).$$

Due to concavity, this is a decreasing function on $\hat{x} \in [\hat{W},\kappa(\hat{y},\hat{s})-c]$. For a low value of $\hat{W}$ (ex. $\hat{W} \leq \underline{x} + p(\underline{x})/p'(\underline{x})$), this function can be made negative everywhere. In that case, the maximum of $\tilde{D}$ is reached at $\hat{x} = \underline{x}$. Therefore, $C(\hat{W})$ is a continuous strictly increasing function that reaches $\underline{x}$ and $\kappa(\hat{y},\hat{s})-c$. By the Intermediate Value Theorem, for any $\hat{x}^* \in [\underline{x},\kappa(\hat{y},\hat{s})-c]$, there exists a unique $\hat{W}^{IC}(\hat{x}^*)$ such that $\max_{\hat{x}} \tilde{D}(\hat{x},\hat{W}^{IC}(\hat{x}^*))$ is reached at $\hat{x}^*$ exactly. In other words, there exists a unique continuation utility $\hat{W}^{IC} \in [-\infty,\kappa(\hat{y},\hat{s})-c]$ that makes the worker choose exactly $\hat{x}^*$. To finish this first-step, we must choose the rest of the contract. Set $\hat{\tau}^{IC} = \hat{\tau}^*$ and $\hat{d}^{IC} = \hat{d}^*$. Now, in an optimal allocation, w^{IC} must be chosen so that the promise-keeping constraint is binding. The worker's expected utility is:

$$\mathbf{W}(y,s,z;\{w,\hat{\tau},\hat{x},\hat{d},\hat{W}\}) = w + \beta \mathbf{E}_{\hat{y},\hat{s},\hat{z}}\Big\{\hat{d}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{d})\Big(\hat{\tau}\mathbf{U}(\hat{y},\hat{s}) + (1-\hat{\tau})\lambda p(\hat{x})\hat{x} + \\ + (1-\hat{\tau})(1-\lambda p(\hat{x}))\hat{W}\Big)\Big\}$$

Given $\{\hat{\tau}^{IC},\hat{d}^{IC},\hat{W}^{IC}\}$, there exists a unique wage w^{IC} that matches exactly the promised utility. This does not affect the joint surplus, which is maximized by assumption. From proposition 1, the firm's profit is maximized when the level of promised utility is exactly achieved. We have thus found a contract that implements the optimal allocation.

Step 2. I will now proceed to the second step of the proof and show that the participation constraint is satisfied by $\hat{W}(\hat{x}^*)$. Let us first have a look at the problem faced by the worker choosing whether or not to leave the firm at the time of separation. The participation constraint is satisfied if

$$\max_{\hat{x}} \lambda p(\hat{x})\hat{x} + (1-\lambda p(\hat{x}))\hat{W} \geq \hat{U}(\hat{y},\hat{s}).$$

We can derive the first order condition for the worker:

$$\lambda p'(\hat{x})(\hat{x}-\hat{W}) + \lambda p(\hat{x}) = 0.$$

Turning back to the joint surplus maximization, the terms related to $\hat{x}$ and $\hat{\tau}$ are:

$$n\hat{U}(\hat{y},\hat{s})\int \hat{\tau}\mathrm{d}j + \lambda p(\hat{x})\hat{x}\int(1-\hat{\tau})\mathrm{d}j + \mathbf{V}(\hat{y},\hat{s},\hat{z},(1-\lambda p(\hat{x}))\int(1-\hat{\tau})\mathrm{d}j + \hat{n}_i).$$

To simplify the notation, write $n\hat{T} = \int \hat{\tau} dj$, $\hat{T}$ being the total fraction of layoffs. We can rewrite the problem as:

$$n\hat{T}\hat{U}(\hat{y},\hat{s}) + n(1-\hat{T})\lambda p(\hat{x})\hat{x} + \mathbf{V}\Big(\hat{y},\hat{s},\hat{z},n(1-\hat{T})(1-\lambda p(\hat{x})) + \hat{n}_i\Big)$$

The first-order condition with respect to $\hat{x}$ is

$$n(1-\hat{T})\lambda p'(\hat{x})\Big(\hat{x} - \mathbf{V}_n\big(\hat{y},\hat{s},\hat{z},n(1-\hat{T})(1-\lambda p(\hat{x})) + \hat{n}_i\big)\Big) + n(1-\hat{T})\lambda p(\hat{x}) = 0$$

Notice that it is possible to identify $\hat{W}(\hat{x}^*)$ from the two first-order conditions. The incentive compatible contract must be such that:

$$\hat{W}(\hat{x}^*) = \mathbf{V}_n\Big(\hat{y},\hat{s},\hat{z},n(1-\hat{T}^*)(1-\lambda p(\hat{x}^*)) + \hat{n}_i\Big)$$

To verify whether the participation constraint is satisfied, it is informative to look at the first-order condition with respect to $\hat{T}$ (ignoring the irrelevant case where $\hat{T}=1$):

$$n\hat{U}(\hat{y},\hat{s}) - n\Big(\lambda p(\hat{x}^*)\hat{x}^* + (1-\lambda p(\hat{x}^*))\mathbf{V}_n\Big) \leq 0$$

which is exactly equivalent to the participation constraint

$$\lambda p(\hat{x}^*)\hat{x}^* + (1-\lambda p(\hat{x}^*))\hat{W}(\hat{x}^*) \geq \hat{U}(\hat{y},\hat{s}).$$

The incentive compatible contract therefore satisfies the participation constraint. □