Handling Non-Invertibility: Theory and Applications*

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Abstract

Existing research provides no systematic, limited information procedure for handling non-invertibility, despite the well-known inference problem it causes as well as its presence in many types of dynamic systems. Non-invertibility means that structural shocks cannot be recovered from a history of observed variables. It can arise from a form of delayed responses due to, among other things, time-to-plan, sticky information or news shocks. Structural VARs rule out non-invertibility by assumption. Inference about structural responses can, in turn, be incorrect. We develop a practical four-step procedure to partially, and sometimes fully, identify structural responses whether or not non-invertibility is present. Our method combines structural VAR restrictions, e.g. recursive identification, with ”agnostic” identification, e.g. sign restrictions and bounds on forecast error contributions. In two model-generated examples, our procedure either fully or nearly fully identifies the structural responses whereas SVARs do not. Also, we apply our procedure to real world data. We show that non-invertibility is unlikely in Fisher’s (2006) study of technology shocks in the U.S.

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"Do you mean now?" – *Baseball player and manager Yogi Berra, when asked for the time.*

1 Introduction

Suppose a police officer on foot patrol happens upon a dead man with a knife in his back. An autopsy firmly establishes that the time of death was 5:00 AM earlier that day. Detectives would like to know when he was stabbed. With no witnesses, the stabbing could have occurred at 4:59 AM with the victim dying quickly. Or, the stabbing could have occurred the previous evening with the victim dying slowly. There are other possibilities, and thus, the time of the crime is not identified.

A time series analyst often faces a similar problem. Suppose the analyst observes a series of outcomes (e.g. real GDP), each of which is indexed by a known time. Suppose the analyst does not observe the sequence of impulses (e.g. preference shocks) or their associated times. A current change in an observable might be due to immediate response to a contemporaneous impulse. Or, the current change might be a delayed response to an impulse that occurred long ago. To the analyst, this is known as the *non-invertibility identification problem*. It is distinct from the "simultaneous equation problem" that arises because of multiple simultaneous unobserved shocks[1].

The police detective and the time series analyst have different standard operating procedures for dealing with this identification problem. The police detective would look for other evidence to inform when the shock (i.e. the stabbing) occurred, such as the stiffness of the dead body. Faced with the same crime, on the other hand, the time series analyst typically would assume that the stabbing occurred at 4:59, because this is the response with the shortest delay from impulse to observable. In technical language, the analyst has dealt with the non-invertibility problem by assuming the invertible representation, i.e. the one with minimal delay, is the correct one. In non-technical terms, the analyst has done shabby police work.

In this paper, we develop a procedure for handling the identification problem with-

[1] In most problems, one must cope with both equation simultaneity and non-invertibility. Handling both is a part of our paper.

out assuming that responses to structural shocks occur with minimal delay. Rather, we follow the police detective's method. We ask whether other evidence, including the co-movement of the observable with other observables or the sign of impulse responses, are consistent or inconsistent with restrictions implied by economic theory. We wish to use as few clues given by economic theory as possible.

This paper addresses non-invertibility, also known as non-fundamentalness, in a limited information framework.[2] We treat non-invertibility in a similar manner to the one that researchers already use in SVARs to deal with the simultaneous equations identification problem. That is, compute all of the stochastic processes consistent with the data and then apply identifying restrictions from economic theory to exclude some (and potentially all but one) of these processes.

Our procedure has four steps.

Step One: *Estimate a reduced-form VARMA(1,1) on the observables.*

We begin by assuming the time series has a state-space representation. Under some general assumptions discussed in later, the observables from a state-space representation can be written as a VARMA(1,1). Many dynamic economic models is consistent with this form. To be concrete, let Y_t represent a vector of k observable, stationary variables.

Step Two: *Calculate all covariance equivalent representations.*

With k observable variables, there are at most 2^k state-space forms that have the identical covariance functions, modulus the simultaneous equations problem. One of these state-space forms will be invertible, i.e. have minimal delay. However, there is no rationale for simply choosing this one over a non-invertible representation, without further identification restrictions in hand. As such, this step records and keep tracks of each one.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

This step mimics that of the SVAR approach. A structural shock is a primative of an economic model, such as an exogenous change in technology or monetary policy. The

[2]Throughout this paper, we use the term non-invertibility rather than the equivalent *non-fundamentalness*. Using the latter can generate confusion, since economists often refer to fundamental shocks as the economically meaningful shocks, such as changes in preferences or technology. Fundamental shocks in the time series sense are not necessarily fundamental in sense of economic theory.

restriction might concern the short run, e.g. output does not respond to current monetary policy changes, or the long run, e.g. only technological change affects long-run labor productivity. This step is needed because the simultaneous equations problem exists apart from the non-invertibility issue.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out potential structural responses.*

Uhlig uses the phrase "agnostic restrictions" to describe identifying assumptions of the kind implemented for example in Faust (1998), Scholl and Uhlig (2008) and Uhlig (2005).[3] For example, a positive innovation to the structural shock might be required to: (i) have a non-negative long-run effect on a particular observable; (ii) imply a positive response to an observable at the two-year horizon; (iii) explain the variation in one variable within a certain range. In contexts outside of non-invertibility, researchers have over the past several years found agnostic restrictions very useful.[4]

After step four, the researcher is left with one or multiple structural impulse responses to the structural shock of interest. When only one response remains, the impulse response is fully identified. When multiple remain, the impulse response is partially identified. In either case, the invertible form may or may not belong to the set. If the invertible form is consistent with the restrictions from step four, then it will be a valid structural response. Importantly, our procedure does not a priori choose this response.

The problem of non-invertibility has received great attention in economics and time series analysis. In an introductory chapter of his textbook, Hamilton (1994, pg. 64) discusses the issue and presents practical reasons for preferring the invertible representation.[5] Sargent (1987) presents an early textbook discussion.[6] Fernandez-Villaverde et al (2007, FRSW hereafter) explain that non-invertibility is induced by missing variables.

Economists have pointed out that non-invertibility can arise in many environments.

[3]Other work using agnostic identification include: Cardoso-Mendonca, Medrano and Sachsida (2008), Mountford and Uhlig (2009) and Owyang (2002).

[4]Fry and Pagan (2010) contains an extensive and critical survey of one type of agnostic restriction–the sign restriction.

[5]We discuss these reasons and how our method addresses them in section two.

[6]Other textbook presentations on the invertibility of MA processes include Brockwell and Davis (2009) and Lutkepohl (2010).

Model features that can induce non-invertibility in the structural responses include: permanent income economies (Hansen and Sargent 1991, Hansen, Roberds and Sargent 1991 and FRSW 2007); learning-by-doing (Lippi and Reichlin 1993); anticipated fiscal policy shocks (Leeper, Walker and Yang 2009); anticipated technology shocks (Blanchard et. al. 2009). Alessi, Barigozzi and Capasso (2011) surveys the prevalence of non-invertibility in rational expectations models. Lippi and Reichlin (2003) discuss the possibility of misspecification due to non-invertibility in Blachard and Quah (1989). Sims (2009) is an exception to the above studies. Using data simulated from a calibrated DSGE model, he finds that non-invertibility, while present, introduces little bias in the impulse responses from a structural VAR.

Despite these extensive discussions of the problem and its practical relevance, only three categories of solutions have been offered. These are: (i) adding more observables; (ii) using full information estimation of a correctly specified DSGE model; (iii) standard SVAR estimation augmented with something akin to our Step Three. Each differs from ours in separate and important ways.

First, one could expand the observables. Most directly, researchers can try to directly observe the structural shocks. If the shock and its arrival time are known, the identification problem disappears. Case studies applied to particular changes in tax policy are well-suited for this approach. Also, Romer and Romer (2004, 2010) have used the narrative approach to create time series measures of the values of actual monetary policy shocks and actual government spending shocks. However, in most cases, shocks are not directly observed.

Even when structural shocks are not observed, adding observables potentially eliminates non-invertibility. Alessi, Barigozzi and Capasso (2010) recommend using a large number of observables and then applying structural restrictions, e.g. a Choleski decomposition, to the estimated factor-augmented VAR. Forni, Giannone, Lippi and Reichlin (2009) advocate this approach by showing that moving from a structural VAR to a factor-augmented structural VAR changes the responses of output to permanent supply shocks.[7]

Second, FRSW (2007) draws upon their discussion of the danger in using SVARs.

[7] See also Giannone and Reichlin (2006).

SVARs always choose the invertible representation of a time series. When the actual structural response is non-invertible, the SVAR leads to incorrect inference. Rather than an SVAR, they recommend correctly specifying a full dynamic, stochastic general equilibrium (DSGE) model and using a full information technique. Our limited information procedure is less likely to suffer from misspecification than using a fully specified model.

FRSW (2007) also provide a condition to use, case-by-case, to determine whether an SVAR would generate incorrect inferences. To check this condition, one uses the estimates or calibration of the DSGE model relevant for the particular time series. However, with a correctly specified DSGE model in hand, one should use all of the information in the DSGE model rather than the limited information SVAR on efficiency grounds.

In a somewhat-related way, Mertens and Ravn (2010) use DSGE models together with structural VARs in an inventive way, to address non-invertibility. They specify and calibrate a DSGE model with news shocks, and then use it to determine the placement of the non-invertibility in the system's moving-average structure, along with the magnitude of the roots associated with the non-invertibility. In their exercise, Mertens and Ravns preset the values of the roots associated with the non-invertibility.

Third, Lippi and Reichlin (1994) suggest a limited information approach. It is the closest antecedent of our work. They compute the structural impulse response using a VAR and a standard rotation restriction. The estimated structural response is by construction invertible, as discussed in FSRW. Recognizing that non-invertible solutions are also consistent with the observed data, they then do a visual inspection of roots from the estimated VAR in search of an MA structure.

Based on the inspection, they plot both non-invertible and invertible structural responses implied by their VAR. This is similar to our step three. As they explain, their method is only suitable for a two variable system. On the other hand, our procedure works for a system with more variables because we estimate the MA component directly (i.e. our step one). Also, our procedure allows us to exclude some of the potential structural responses (i.e. our step four) in a systematic manner. Moreover, their procedure can only analyze a single shock with non-invertibility, while our procedure is suitable for cases with multiple non-invertible shocks.

Our procedure has three distinct benefits not shared by the other approaches: (i) it directly estimates the model's moving average component (i.e. Step One), which is the heart of identification issue; (ii) by using the quadratic matrix equation (i.e. Step Two), it quickly and intuitively finds the entire set of covariance equivalent stochastic processes; (iii) by using agnostic restrictions (i.e. Step Four), it stays within the limited information framework of structural VARs.

First, since the entire source of non-identification is the multiplicity of moving average components of an observed covariance function, it makes sense to estimate the moving average component directly. At the same time, an autoregressive part may also be present. As such, we use a VARMA model to capture both parts. Lippi and Reichlin (1994), in contrast, estimate a VAR and then do a visual inspection for MA roots. This limits the applicability of their procedure as discussed above.

In the past, researchers have avoided estimating moving average models with good reason. There is a relatively old (circa the 1970s) concern that implementing a VARMA is so difficult as to make their use infeasible. The erstwhile approach centered on nonlinear maximization of a likelihood function over a high dimensional parameter space. While possible in theory, it can be unreliable practically.

Numerous recent advances in VARMA estimation largely ameliorate this concern. Dufour and Pelletier (2008) for example extend to the vector case the innovation-substitution method developed by Hannan and Rissanen (1982). The method involves feeding the residuals from a long-lag AR as the innovations in the estimation of an ARMA model. OTHER METHODS: Koreisha and Pukkila GLS (1990), Larimore CCA subspace (1983) and Kapetanios iterative LS (2003), Hannan and Kavalieris 3SLS (1984). We use Dufour and Pelletier's method in all of our examples. Kascha (2007) compares the above methods using a well-known macro application and shows that the innovation-substitution method dominates.

Second, we compute the entire set of structural representations using a simple formula (Potter 1966) that solves a quadratic matrix equation. We set out to develop a procedure is easy for practitioners to use. The Potter equation is easy to code and fast to run. It requires only a single matrix inversion and a single eigenvalue decomposition.

An alternative technique, Blaschke factorization, can in principle do the same job. It appears in many theoretical discussions about non-invertibility;[8] however, to our knowledge, it has never been used in applications. Perhaps this is because it is much more involved from a practical standpoint. It begins with a single eigenvalue computation that is then followed by a large number of "root flipping" steps, where each root flipping requires the calculation of the null space of a particular matrix.

Third, our paper maintains the limited information spirit of BLAH BLAH BLAH.FINISH THIS.

To set the stage, the next section contains a bivariate process where non-invertibility is present. Section 3 presents the four-step procedure along with its theoretical substructure. Section 4 applies the procedure to two sets of model-generated data and section 5 applies the procedure to a real world application. Section 6 concludes.

2 Non-invertibility in A Bivariate Example

We illustrate the nature of non-invertibility using a two variable example.[9] Suppose an economist observes y_{1t} and y_{2t}. For concreteness, call them the money growth rate and real output. Each variable has expectation zero and an own first-order autocorrelation equal to 0.01. At further lags, each has a zero autocorrelation. The two are uncorrelated with each other at every horizon. Also, suppose there are two shocks driving the system, which, for concreteness, are technology shocks and monetary policy shocks.

What VMA(1) processes are consistent with the above covariance structure? Indexing each process by j, these are

$$y_t = \Gamma_0^j \omega_t^j + \Gamma_1^j \omega_{t-1}^j$$

where Γ_0^j and Γ_1^j are square matrices of dimension two . The number of processes, or forms, modulus the simultaneous equations issue, is at most 2^k. Since $k = 2$, there are up to four forms. Figure 1 plots the impulse responses for three of these. We omit the

[8]These include Whiteman (1983), Hansen and Sargent (1991), Lippi and Reichlin (1994), Leeper, Walker and Yang (2009) and Alessi, Barigozzi and Capasso (2010).

[9]Examples using one variable are presented in Hamilton (1994) and Sargent (1987). While instructive, the scalar case cannot elucidate the important cross-covariagram implications of non-invertibility.

fourth to avoid clutter. Each row corresponds to a moving-average form and each column corresponds to a particular shock applied to a particular variable.

To deal with the simultaneity of shocks, we have imposed a short-run restriction that output does not respond contemporaneously to the monetary shock. In the figure, the period zero response of output to the monetary shock is zero in each panel of the second column of the figure. Suppose this short-run restriction holds in the underlying structural model.

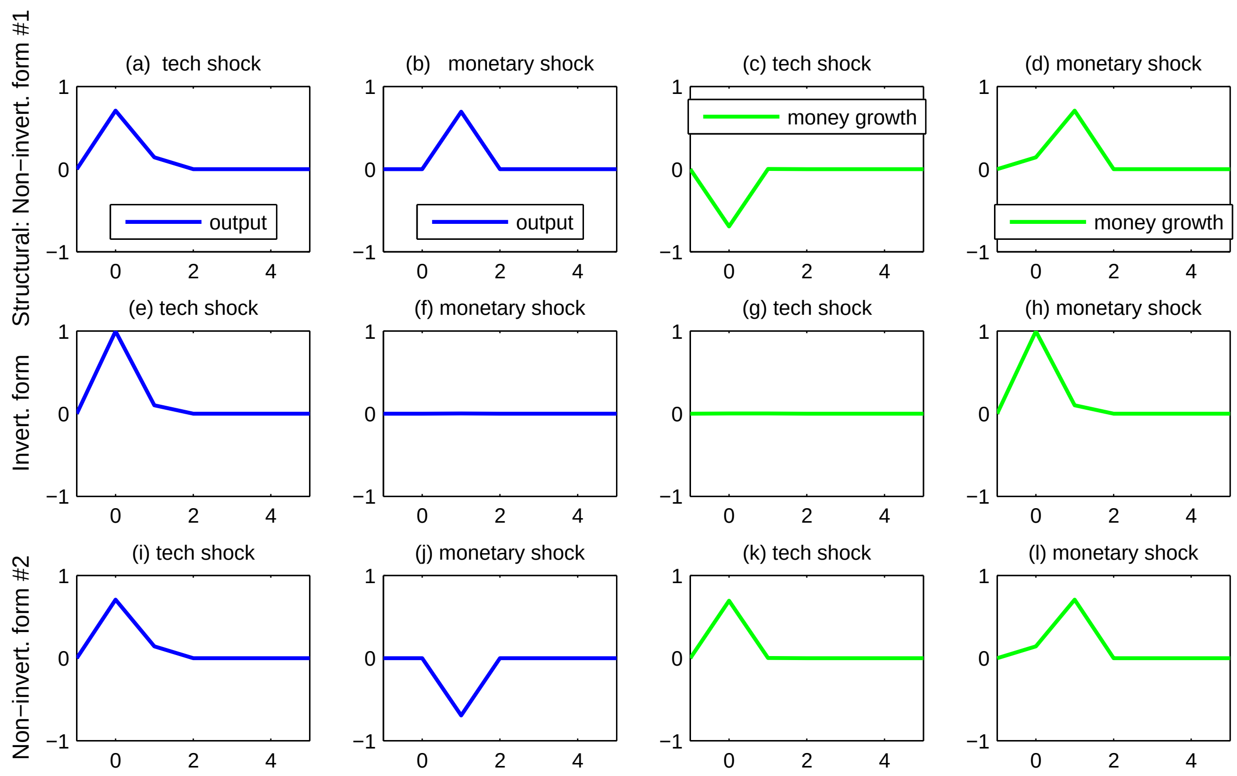
Suppose that the true structural model, or economy, that delivers the observed covariance matrices is in the first row of the figure. This economy corresponds to one of the non-invertible forms. The economy has three key features: a money growth shock is not neutral (see panel (b)), monetary policy responds counter-cyclically to technology shocks (see panel (c)), and there is a large "news component" to money growth shocks (see panel (d)). The news interpretation of panel (d) is appropriate because, although the money growth shock arrives at time zero, the most substantial increase in the money supply happens at time one. An economist that observes y_t, but does not observe either shock, may try to identify the shocks using a structural VAR, which automatically chooses the invertible form. Suppose the economist knows that the above short-run restriction is true for this economy. If the economist runs an SVAR using the restriction, she will estimate the second row of Figure 1. This is the invertible form. This economist would come away incorrectly believing that money shocks are neutral (see panel (f)) and monetary growth does not respond to technology shocks (see panel (g)).

What is going on? There is a 'covariance accounting' requirement that is satisfied for the various forms. Each form has sets of moving average coefficients that line up in a way that the corresponding second moments across forms are identical. In the next section, we provide a simple equation to construct all forms that satisfy the covariance requirement.

Armed with only the short-run restriction, the structural model is not identified. Even worse, an SVAR with only the short-run restriction will estimate the wrong model. The estimated model says money is neutral with respect to output when in reality it is not!

How can one deal with this under-identification? Our solution is to bring more *a priori* knowledge about the economic environment to the table. The goal should be to

Figure 1: Three covariance-equivalent stochastic processes



Notes: The fourth and final form, another non-invertible process, is not pictured above.

bring restrictions that are agnostic, in the sense of Uhlig (2005), as possible to reduce the set of valid forms. An alternative approach, advocated by FRSW (2007) and discussed in our introduction, is to bring a lot to the table, in the form of a fully-specified dynamic general equilibrium model. As we explained in the introduction, the dynamic general equilibrium approach goes against the spirit of the limited information technique and moreover eliminates the need for limited information anyways.

3 Theory and A Four-Step Procedure

A generic covariance-stationary stochastic process is given by:

$$\begin{aligned} s_{t+1} &= Qs_t + Ue_{t+1} \\ r_{t+1} &= Ws_t + Ze_{t+1} \end{aligned} \quad (1)$$

where e_{t+1} is k by 1 and $N(0, I)$. We refer to (Q, U, W, Z) as a *state-space form (*with associated shock process e_t) for the stochastic process $\{s_t, r_t\}$. Here, Q, U, W, Z are real-valued. Only r_t is observed by the economist. Also, we make the following assumptions.

Assumption 1: The left inverse of W, which we denote $\bar{W}$, exists.

Assumption 2: All eigenvalues of Q and $WQ\bar{W}$ are inside the unit circle.

Assumption 3: The matrix Z is invertible.

Assumption 1 requires that there are least as many observables as states. To identify the underlying system, economists need to have enough information,i.e enough observable variables. This assumption is not as restrictive as it may seem. If the economy is actually driven by a few common factors, e.g. the dynamic factors as those identified by Stock and Watson (2002) or used by Bernanke, Boivin and Eliasz (2005), most multivariate time series models have more observables than states.

Assumption 2 ensures the stationarity of observables. In our exercise, we rule out cases with non-stationary variables. However, it is straightforward to covert non-stationary variables to stationary ones by detrending them or choosing correct cointegration vectors. Our procedure then is ready to go.

Assumption 3 requires there are at least as many observables as structure shocks of concern. This assumption is for technical purposes and not restrictive, since we can add include measurement errors as structural shocks. FRSW (2007) also make this assumption.

In lieu of additional information, the time series analyst knows or can estimate the covariance generating function of the observables. Let this covariance structure be denoted $C_i = E\left(r_t r'_{t-i}\right)$ for all i.

To understand the theory that follows as we as our procedure, it is useful to compute these covariances as functions of the underlying structural form:

$$\begin{aligned}
C_0 &= WQ\bar{W}C_0\,(WQ\bar{W})' + ZZ' + WUU'W' \\
&\quad - WQ\bar{W}C_0\,(WQ\bar{W})' \\
C_1 &= WQ\bar{W}C_0 + WUZ' - WQ\bar{W}ZZ' \\
C_i &= (WQ\bar{W})^{i-1}\,C_1 \ \text{ for all } i > 1
\end{aligned}$$

In the theorem that follows, we find the number of matrix triples $\{A_j, B_j, D_j\}$ corresponding to covariance equivalent forms and also show how to conveniently compute each of them.

Moving from the structural form to an observationally equivalent one changes the amount of delay in the system, as seen in Section 2. Intuitively, this can be seen in the state space system by examining the MA representation of the original structural system. This MA representation is:

$$r_{t+1} = Ze_{t+1} + W\sum_{i=0}^{\infty} Q^i U e_{t-i}$$

Because the original and observational equivalent state-space forms differ in terms of U and Z, the corresponding impulse responses will differ in magnitude of a shock's instantaneous effect, i.e. e_{t+1}, versus its lagged effect, $e_t, e_{t-1,...}$. Moreover, as seen in the bivariate example of section 2, changing the delay in the response of one variable to a shock has implications for all of the other impulse responses because of the known covariance structure of the observables. The theorem below formalize the relation between the structural form and its covariance-equivalent cousins. Furthermore, it lays out the theoretical foundation for the practical procedure we use to tackle non-invertibilities.

Theorem 1: If r_t is a length k stochastic process with the structural state-space form (1) and assumptions 1 through 3 are satisfied, then there exists at most 2^k infinite-order covariance equivalent moving average representations for $\{r_t\}$, indexed by j, where the innovations process ε_t^j satisfies $E\left(\varepsilon_t^j \varepsilon_t^{j\prime}\right) = I_k$. Representation j is given by

$$r_{t+1} = (I - AL)^{-1} \left[D_j + B_j L\right] \varepsilon_{t+1}^j, \tag{2}$$

The coefficient matrices, A , B_j and $\tilde{C}_i,\ i = 0, 1$ are:

$$\begin{cases} A &= C_2 C_1{}^{-1} \\ \tilde{C}_1 &= C_1 - AC_0 \\ \tilde{C}_0 &= C_0 - AC_0A' - A\tilde{C}_1' - \tilde{C}_1 A' \\ B_j &= \tilde{C}_1 (D_j')^{-1} \end{cases} \tag{3}$$

where C_i is the ith order autocovariance of the observable vector. The matrix, D_j, satisfies:

(i)

$$(D_j D_j')(\tilde{C}_1')^{-1}(D_j D_j') - \tilde{C}_0 (\tilde{C}_1')^{-1}(D_j D_j') + \tilde{C}_1 = 0, \tag{4}$$

(ii) $D_j = D_j^c K_j$, where D_j^c is the lower triangular matrix generated by the Cholesky decomposition of $D_j D_j'$. The orthonormal matrix, K_j is determined by the relation between the cholesky decomposition and the identifying restriction. When we use the short-run restriction, $K_j \equiv I$. If we use the long-run restriction, K_j differs from each other.

(iii) one of the D_js is invertible and the corresponding MA form matches the Wold representation for r_t.

This theorem tells us: (i) a time series can have multiple representations; (ii) all of these forms can be backed out from a single reduced-form estimation. This multiplicity of covariance equivalent forms is one source of an identification problem with VARs.[10] Equation (4) provides a way to find all of these forms. Hence, it allows us to dramatically reduce the dimension of the identification problem.

As an aside, note that this identification difficulty is not specific to structural VARs. The difficulty can also apply if a full information method, such as maximum likelihood, is used instead. This is because the covariance equivalence of the various forms implies multiple peaks in the likelihood function.

[10] The other source is the well-known simultaneous equations problem.

From Theorem 1, we develop our four-step procedure. In sections 4 and 5, we use model-generated data and real-world data to demonstrate the procedure.

Step One: *Estimate a reduced-form VARMA(1,1) model on the observables*.

Under Assumptions 1 through 3, the structural model has a unique invertible VARMA(1,1) form. It can be consistently estimated with traditional methods.

Step Two: *Calculate all covariance equivalent representations.*

Under the same assumptions, the true model can have multiple non-invertible VARMA(1,1) forms in addition to the one invertible form. Each corresponds to a solution of a quadratic matrix equation. All can be found simultaneously using the Potter (1964) equation. This computation is simpler than the existing Blaschke method, as discussed in the introduction. Although the number of forms at this step can theoretically large,[11], this issue is mitigated in practice. As seen in the following two sections, (i) impulse response from a subset or subsets of forms is often 'clustered,' making them quantitatively indistinguishable; (ii) solutions with imaginary components are thrown out.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

When the dimension of the observable variables is k, there are at most 2^k solutions for fully specified rotation matrices. There is always at least one solution—the invertible one.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to rule out further structural representations.*

Usually there are multiple solutions after step three. More restrictions other than those on the pattern on the rotation matrix help reduce the set of covariance-equivalent forms. If only one solution remain, the structural model is fully identified, otherwise, the model is only partially identified.

4 Two Model-Based Implementations of Our Procedure

In this section, we use two model-generated examples to illustrate how our procedure identifies the structural model when the structural VAR cannot. The first example is

[11] For example, with eight observables there are potentially 256 covariance equivalent forms.

adopted from the permanent income economy in FRSW (2007). The second example is from a model of anticipated tax shocks (i.e. "news" regarding the future tax rate) in Leeper, Walker and Yang (2009).

4.1 Savings and Permanent Income in FRSW (2007)

The permanent income model is a workhorse of modern economics. FRSW (2007) show how applying structural VAR analysis to data from a permanent income model leads to an incorrect conclusion about the consumption response to an income shock. The incorrect conclusion occurs because the procedure fails to handle an inherent non-invertibility. We show how our procedure leads to the correct conclusion.

The economic model has two equations.

$$c_{t+1} = \beta c_t + \sigma_w(1 - R^{-1})w_{t+1}, \tag{5}$$

$$z_{t+1} = y_{t+1} - c_{t+1} = -c_t + \sigma_w R^{-1} w_t, \tag{6}$$

Equation (5) is the intertemporal Euler equation and equation (6) defines saving. In the model, c_t is the unobserved state, while $z_t = y_t - c_t$ is saving, the only observable in the model. This process is non-invertible, since $Q - UZ^{-1}W = \beta + R - 1 > 1$, when β is close enough to one. The ARMA(1,1) representation of the observable is given by:

$$z_{t+1} = \beta z_t + \sigma_w R^{-1} w_{t+1} - \sigma_w[1 - R^{-1} + \beta R^{-1}]w_t, \tag{7}$$

which is non-invertible. The innovation representation is:

$$\hat{c}_{t+1} = \beta \hat{c}_t + \sigma_w\left(\frac{\beta - \beta^2 + 1}{R} - \beta\right)\epsilon_{t+1} \tag{8}$$

$$z_{t+1} = -\hat{c}_t + \sigma_w\left(\frac{\beta - 1 + R}{R}\right)\epsilon_{t+1}. \tag{9}$$

Straightforwardly, the ARMA(1,1) model corresponding to the innovation representation is:

$$z_{t+1} = \beta z_t + \sigma_w\left(\frac{\beta - 1 + R}{R}\right)\epsilon_{t+1} - \frac{\sigma_w}{R}\epsilon_t. \tag{10}$$

The innovation representation is invertible, since $\hat{Q} - \hat{U}\hat{Z}^{-1}\hat{W}' = \frac{1}{R+\beta-1} \in (0,1)$. However, since the implied state variable is not the true state variable, i.e, $\hat{c}_t = E\left(c_t|z^t\right) \neq c_t$, where z^t refers to the history of the observable saving, z_t; therefore, FRSW (2007) warn that inference based on the (estimated) innovation representation is not reliable.

Suppose the economist observes a time series for savings, z_t, however, she is uninformed regarding consumption and income. She would apply our procedure as follows:

Step One: *Estimate a reduced-form ARMA(1,1) on the observable.*

Step Two: *Calculate all covariance-equivalent representations.*

With only one observable variable, there are only two covariance equivalent MA(1) representations.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

Define a positive savings shock as a disturbance that increases savings in the period of the shock. Different researchers may have different interpretations as to what exogenous factors drive savings changes, such as shocks to permanent income, transitory income or preferences. With a scalar observable and a scalar shock, there is no simultaneous equations problem. As such, an SVAR-type restriction is unnecessary.

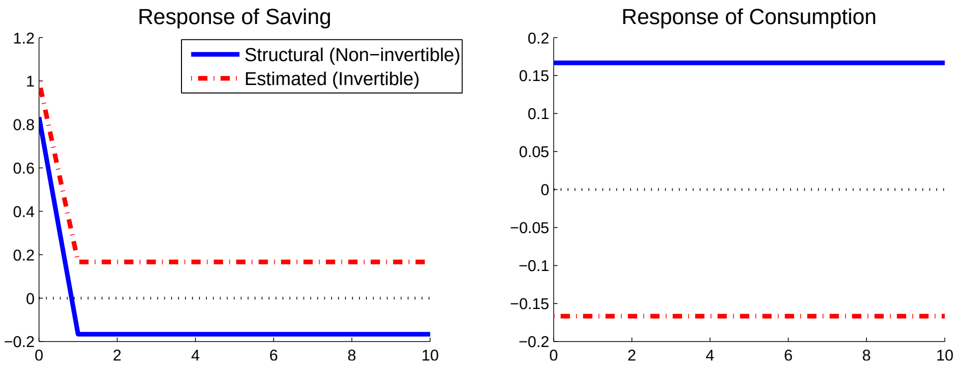
Before imposing Step Four, we plot the impulse responses that come out of Step Three. These areappear in Figure 2 both the saving level rate and the consumption. The solid and dashed lines are, respectively, the non-invertible and invertible responses. Both of these impulse response functions give the same population moments as those from (7) or (10). The non-invertible response is the true response and the invertible representation is spurious. As FRSW (2007) explain, a structural VAR always selects the invertible representation; therefore, it would lead to the incorrect impulse responses.

Step Four: *Impose an agnostic restriction on each representation, delivered from Step Three, to rule out further potential structural responses..*

Rather than a priori select the invertible form, we impose an agnostic restriction based on economic theory. We will impose the standard idea that people save now in order to consume more later. Formally, we require that: *if savings is non-zero in at least one period,* *then it must switch signs at least once.*

Examining Figure 2, only the invertible response satisfies the agnostic restriction. After Step Four, we have a single structural impulse response, which is the true response from the economic model. It is exactly the structural model's impulse response.

Figure 2: Covariance-equivalent impulse responses to a positive savings shock



Notes: From the permanent income model with $r = 0.2$. Impulse responses to a one unit shock from step three and before application of step four.

In a wide class of models, an individual increases current savings in order to finance greater future consumption. The use of agnostic restrictions is, in our view, very powerful exactly because it implies transparency regarding the source of identification.

4.2 An Anticipated Fiscal Shock in Leeper, Walker and Yang (2009)

The second model-generated example is based on Leeper, Walker and Yang (2009, LWY hereafter). This example has an anticipated fiscal shock: changes in the tax rate are announced two quarters before their implementation.

Consider a neoclassical model with fixed labor supply and full capital depreciation. The capital stock k_t is the single endogenous state variable. In equilibrium, it satisfies

$$(1 - \alpha L)(1 - \theta L^{-1})k_t = -\frac{\tau}{1-\tau} E_t\left(\tau_{t+1}\right) + a_t - \theta E_t\left(a_{t+1}\right)$$

where every variable is the log deviation from its steady-state value. The variables τ_t and a_t are the tax rate and technology level.

LWY assume there is a random component to the tax rate, which is announced two periods before the tax implementation. This news is denoted by $\epsilon_{\tau,t}$. The equilibrium law of motion for capital, consumption c_t and output y_t are:

$$
\begin{aligned}
k_{t+1} &= \alpha k_t + a_{t+1} - \frac{\tau}{1-\tau}(1-\theta)[\theta\epsilon_{\tau,t+1} + \epsilon_{\tau,t}], && (11)\\
c_{t+1} &= \alpha k_t + a_{t+1} + \frac{\tau}{1-\tau}\theta[\theta\epsilon_{\tau,t+1} + \epsilon_{\tau,t}], && (12)\\
y_{t+1} &= \alpha k_t + a_{t+1}. && (13)
\end{aligned}
$$

LWY show that non-invertibility affects not only the identification of fiscal shocks, but also the identification of the technology shock. They assume that the tax rate has both the above anticipated random component as well as a contemporaneous response to technology. The tax rate is $\tau_t = \psi a_t + \epsilon_{\tau,t-2}$.

LWY demonstrate the non-invertibility problem using a structural VAR where τ_t and k_t observed. In this case, the shocks identified by the structural VAR are not the true shocks, but rather combinations of the technology and tax/news shocks.

Our four-step procedure can identify, at least partially, the structural shocks in the model. It is applied step-by-step below. We require having enough observable variables, hence, we augment the observable space with consumption, c_t and the shocks with u_t, a measurement error on consumption. The addition of consumption does not remove the non-invertibility.

The state-space representation is:

$$
\begin{aligned}
\overbrace{\begin{bmatrix} k_{t+1} \\ \epsilon_{\tau,t+1} \\ \epsilon_{\tau,t} \end{bmatrix}}^{s_{t+1}} &= \overbrace{\begin{bmatrix} \alpha & -\frac{\tau(1-\theta)}{1-\tau} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} k_t \\ \epsilon_{\tau,t} \\ \epsilon_{\tau,t-1} \end{bmatrix}}^{s_t} + \overbrace{\begin{bmatrix} 1 & -\frac{\tau\theta(1-\theta)}{1-\tau} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} a_{t+1} \\ \epsilon_{\tau,t+1} \\ u_{t+1} \end{bmatrix}}^{e_{t+1}} \qquad (14)\\
\overbrace{\begin{bmatrix} \tau_{t+1} \\ k_{t+1} \\ c_{t+1} \end{bmatrix}}^{r_{t+1}} &= \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ \alpha & -\frac{\tau(1-\theta)}{1-\tau} & 0 \\ \alpha & \frac{\tau\theta}{1-\tau} & 0 \end{bmatrix}}^{W} \overbrace{\begin{bmatrix} k_t \\ \epsilon_{\tau,t} \\ \epsilon_{\tau,t-1} \end{bmatrix}}^{s_t} + \overbrace{\begin{bmatrix} \psi & 0 & 0 \\ 1 & -\frac{\tau\theta(1-\theta)}{1-\tau} & 0 \\ 1 & \frac{\tau\theta^2}{1-\tau} & 1 \end{bmatrix}}^{Z} \overbrace{\begin{bmatrix} a_{t+1} \\ \epsilon_{\tau,t+1} \\ u_{t+1} \end{bmatrix}}^{e_{t+1}}
\end{aligned}
$$

Our analysis requires setting values for the parameters. We follow LWY for most parameters.[12] In addition, the fiscal shock has unit standard deviation and $\sigma_a = 0.1$, The standard deviation of the measurement error is 0.05.[13]

By checking the "poor man's invertibility condition" from FRSW (2007), we see that the system is non-invertible. This is because the matrix $Q - UZ^{-1}W$ has eigenvalues outside the unit circle for our parameterization. The three eigenvalues of $Q - UZ^{-1}W$ are .33, -8.98 and -0.45; therefore, there is one dimension of non-invertibility.

The structural VAR approach ignores the embedded non-invertibility. On the other hand, our procedure takes all possible non-invertibilities into consideration.

Step One: *Estimate a reduced-form VARMA(1,1) on the observables.*

Denote the VARMA(1,1) representation of the structural model as $r_{t+1} = \overbrace{WQ\bar{W}}^{A} r_t + \overbrace{Z}^{D} e_{t+1} + \overbrace{(WU - WQ\bar{W}Z)}^{B} e_t$ with the following matrices:

$$A = \begin{bmatrix} 0 & \frac{(\tau-1)}{\tau} & \frac{(1-\tau)}{\tau} \\ 0 & \alpha & 0 \\ 0 & \alpha & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \psi\sigma_a & 0 & 0 \\ \sigma_a & \frac{\tau\theta(\theta-1)}{1-\tau} & 0 \\ \sigma_a & \frac{\tau\theta^2}{1-\theta} & \sigma_u \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \theta & \frac{(1-\tau)}{\tau}\sigma_u \\ 0 & \frac{\tau(1-\theta)}{\tau-1} & 0 \\ 0 & \frac{\tau\theta}{1-\theta} & 0 \end{bmatrix}$$

The traditional structural VAR approach can only give the innovation representation, $r_{t+1} = Ar_t + \hat{D}\hat{e}_{t+1} + \hat{B}\hat{e}_t$, of the true model. The AR coefficient matrix, A is consistently identified, but $\hat{D}$ and $\hat{B}$ are biased. In our numerical example, the true VARMA(1,1) representation is:

$$A = \begin{bmatrix} 0 & -3 & 3 \\ 0 & .36 & 0 \\ 0 & .36 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} .12 & 0 & 0 \\ .12 & .065 & 0 \\ .12 & -.024 & .05 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -.27 & -.15 \\ 0 & .24 & 0 \\ 0 & .89 & 0 \end{bmatrix}.$$

[12] We choose $\alpha = .36$, $\beta = .99$, $\tau = .25$.

[13] The size of technology shock is set up to allow the contribution of technology shocks and tax shocks on the variance of consumption is equalized in the long run. This parameterization is purely for analytical simplicity, and it does not affect the result qualitatively

The estimated innovation representation, on the other hand, is:[14]

$$A = \begin{bmatrix} 0 & -3 & 3 \\ 0 & .36 & 0 \\ 0 & .36 & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} .29 & 0 & 0 \\ .21 & .14 & 0 \\ -.01 & -.01 & .15 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & -.12 & -.08 \\ 0 & .13 & .03 \\ 0 & -.04 & 0.01 \end{bmatrix}.$$

In the true VARMA(1,1) representation, there are eigenvalues of BD^{-1} outside the unit circle, while every eigenvalue of $\hat{B}\hat{D}^{-1}$ in the innovation representation is inside the unit circle.[15]

Step Two: *Calculate all covariance equivalent representations*

This step finds all the representations with the same autocovariance structure. Each covariance equivalent form has an associated triple $\{A_j, D_j, B_j\}$. It is easy to verify that $A_j = A$ and every pair of $\{D_j, B_j\}$ satisfies the following equations:

$$\begin{aligned} D_jD_j' + B_jB_j' &= \qquad (15) \\ & \begin{bmatrix} \psi^2\sigma_a^2 + \theta^2 + (\frac{\sigma_u}{\kappa})^2 & \psi\sigma_a^2 + \kappa\theta(1-\theta) & \psi\sigma_a^2 - \kappa\theta^2 \\ \psi\sigma_a^2 + \kappa\theta(1-\theta) & \sigma_a^2 + \kappa^2(1+\theta^2)(1-\theta)^2 & \sigma_a^2 - \kappa^2\theta(1-\theta)(1+\theta^2) \\ \psi{\sigma_a}^2 - \kappa\theta^2 & \sigma_a^2 - \kappa^2\theta(1-\theta)(1+\theta^2) & \sigma_a^2 + \kappa^2\theta^2(1+\theta^2) + \sigma_u^2 \end{bmatrix} \\ B_jD_j' &= \begin{bmatrix} 0 & \kappa\theta^2(1-\theta) & -\kappa\theta^3 - \frac{\sigma_u^2}{\kappa} \\ 0 & \kappa^2\theta(1-\theta)^2 & -\kappa^2\theta^2(1-\theta) \\ 0 & -\kappa^2\theta^2(1-\theta) & \kappa^2\theta^3 \end{bmatrix}, \end{aligned}$$

where $\kappa = \tau/(1-\tau)$. The equation system (16) can be equivalently converted into a quadratic matrix equation in D_jD_j'. The solution of this quadratic matrix equation is given in Potter (1964). Since D_jD_j' is a 3×3 matrix for each j, there are at most $2^3 = 8$ different lower triangular matrices solving the quadratic matrix equation. Under this current parameterization, D_jD_j' has one pair of complex eigenvalues. As such, there are only four sets of real-valued structural responses.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each*

[14] Here we only show the result after imposing a short run restriction.

[15] The true model has two eigenvalues outside the unit circle, which are complex conjugates of each other.

representation.

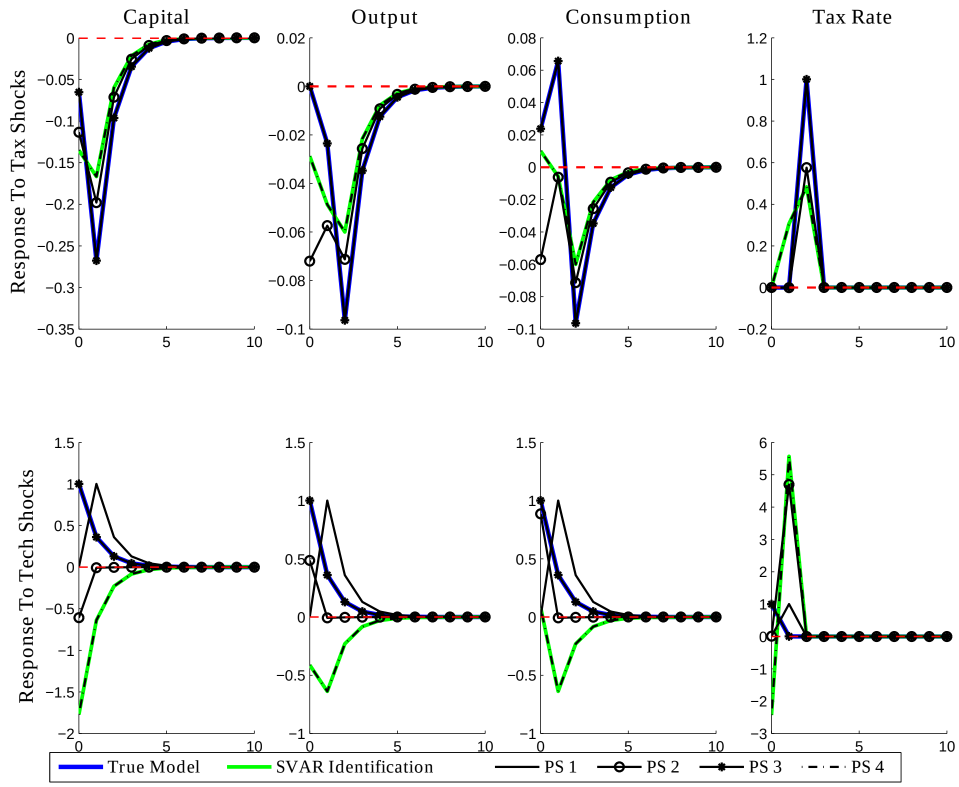
A positive technology shock is defined as a shock which increases consumption and does not reduce the tax rate. Consumption increases because of the positive effect of technology shocks on production capacity. Obviously, a positive tax shock increases the tax rate as well; however, the way it affects capital and consumption is not clear. One possible way to separate the positive tax shock from the positive technology shock is by assuming that an anticipated tax rate change cannot changes the current tax rate. Since we know that measurement error only affects the measurement of consumption, it should not affect the tax rate or capital on impact. Based on the definitions, we can impose the following short-run restriction: a valid D matrix should be lower triangular.

Figure 3 plots the impulse responses to a positive tax shock (upper panel) and those to a positive technology shock (lower panel) in all the four possible cases after imposing the short-run restriction. One of them overlaps with the VAR-based inference, which is the (invertible) innovation representation of the model. In response to a positive tax shocks, capital and output falls in all four possible cases and tax rate increases in all of them. The only difference is the magnitude of responses. When studying the responses to a positive technology shock, capital falls in two cases but rises in other two. Output falls in the innovation representation but rises in all the other three cases. The fall in output seems to contradict traditional wisdom, however, there are evidences in existing research to show technology shocks are contractionary. At this stage, we cannot rule out any of the four cases without further justification.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out structural responses.*

Two agnostic restrictions are imposed. Both are based on the short-term forecast error variance decomposition. In order to identify the true impulse responses, we employ multiple criteria based on reasonable economic intuition. First, measurement errors should not be important factors to explain volatilities in any of the variables, especially in the longer term. Therefore, we setup a quantitative threshold of 30% for the average contribution of measurement errors on all observable variables (*criterion one*). Second, technol-

Figure 3: Response To Tax and Technology Shocks (after step three)



Notes: upper panel responses to a positive tax shock; lower panels responses to a positive technology shock. *PS i*: the *i*th solution based on the Potter equation.

Table 1: Identification Based on Short-Term Variance Decomposition

	Model One	Model Two	Model Three	Model Four
The average contributions on different horizons of identified measurement errors on variables				
tax rate	0	34.82	0	14.78
capital	0	39.32	0	0.51
consumption	7.84	39.45	7.84	70.51
The average contributions of technology on tax rate at different horizons				
	1.42	35.05	1.42	53.24
The contribution of technology shocks on capital and consumption when $h = 1$				
capital	0	37.55	79.11	71.01
consumption	0	48.01	83.23	0.09

ogy shocks should not be the dominant factor to explain the volatilities in the tax rate, especially in longer time horizons. Quantitatively, we set up the threshold value to be 50% when the the time horizon is longer than two quarters (*criterion two*). The result of this variance decomposition exercise is shown in Table 1.

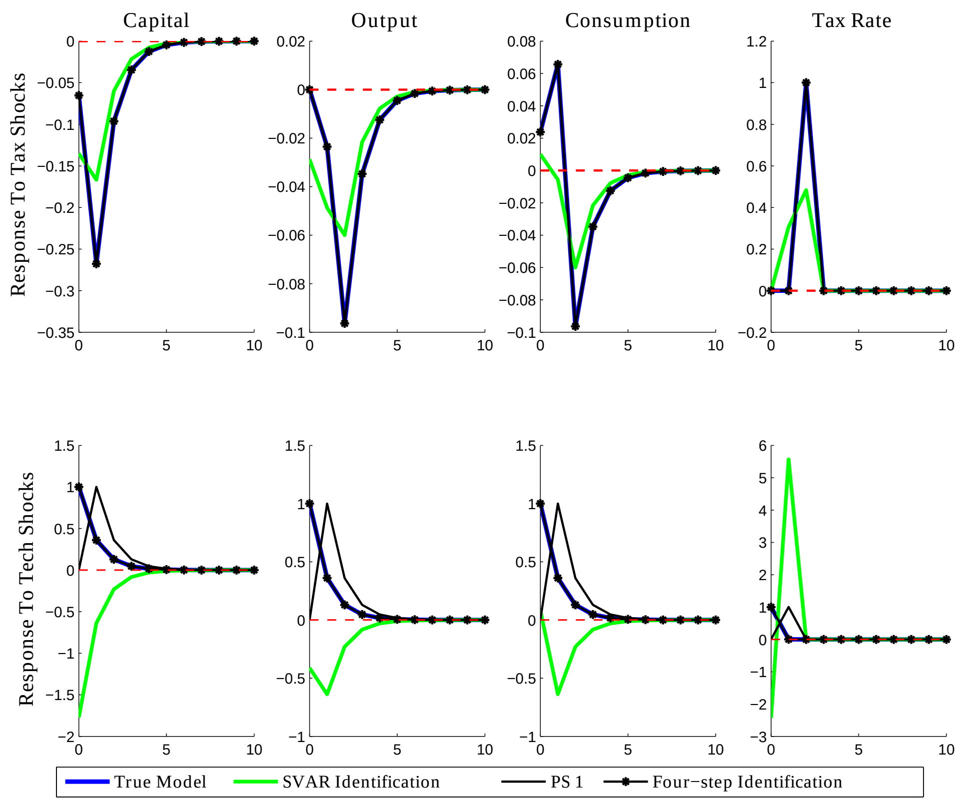
Based on criterion one, case 2 and case 4 are ruled out, since these two cases attribute too much variation to measurement errors. In this model, case 4 corresponds to the innovation representation, in other words, the model identified with traditional SVAR methods. This specification can be ruled out based on our second criterion as well, since technology shocks should not be the main driving force for tax rates. The reason why we can use variance decompositions to identify the correct model is that covariance-equivalent representations other than true models are likely to mix different shock together. Therefore, the variance decomposition is distorted in those representations. Leeper et al (2009) makes a similar point from a different perspective. They view this as a failure of identification with traditional SVAR methods. Our procedure goes one step further: some mis-identification will give wildly implausible variance decomposition. Therefore, we can rule out such mis-identified models. Such identification scheme share the same spirit as the identification methods proposed by Faust (1997) and Uhlig (2005). As long as economic theory gives us enough restrictions on the model, e.g, the variance decomposition, the sign of impulse responses or the sign of magnitude of a particular coefficient, we can always apply them to rule out mis-identified models.

However, we still cannot achieve full identification here. As shown in Table 1, we cannot choose between case one and case three based on the first two criteria we proposed. Until this step, we achieve partial identification. Figure 4 compares the impulse responses implied by the remaining solutions to those of the true model and by the innovation representation. Both solutions recover the true responses to a positive tax shock in the structural model. One of them (the *"identified model"*) recovers the true responses to technology shocks as well.

In this example, we cannot uniquely pin down the true model. The reason is that the first solution based on our procedure only mis-specifies the timing or invertibility of the technology shock, but it does disentangle tax shocks and technology shocks effectively. To further refine the result, we require more restrictions. For instance, if we have a strong belief that the transmission of technology shocks is fast enough, then the technology shock should explain the bulk of changes in capital and consumption in the short term. Hence, we might add a third agnostic restriction: the contribution of technology shocks to the one step forecast error variances in consumption and capital should be higher than 30%. With this extra restriction, we uniquely pin down the model as shown in Table 1. In the true model, capital and output fall in response to an anticipated tax shock. Consumption rises on impact but falls in following period. The initial rise is due to the substitution effect induced by higher tax rate in the future while the following decrease is because of the drop in production capacity.

When the model is identified correctly, capital, output and consumption all rise in response to a positive technology shock, while the innovation representation shows capital and output falls in response to it. Adding this third criterion, the true model is uniquely identified. From our perspective, criteria three is too strong to be used. Here, our procedure is not a slam dunk.

Figure 4: Response To Tax and Technology Shocks (after step four)



Notes: upper panel responses to a positive tax shock; lower panels responses to a positive technology shock

5 Application with Real Data: Technology Shocks in the U.S.

Fisher (2006) uses a three-variable model to study the effect of technology shocks on the U.S. economy in the second half of the twentieth century. In his exercise, the investment-specific shock, which is captured by surprise changes in the relative price of investment, is important to explain the variation in output and working hours in U.S.

Recently, studies on the effect of ”news shocks”, which is the anticipated component in technology shocks, have drawn more and more attentions of economists, since the seminal work by Beaudry and Portier (2006). They show that technology shocks identified by traditional long run restrictions can be well replicated by another shock originated in the stock index but are orthogonal to contemporaneous technology changes. They argue that this piece of evidence shows technology shocks are anticipated (”news shocks”) and they further show this news shock is important to explain business fluctuations. Jaimovich and Rebelo (2009) show that certain real frictions, including habit persistence in consumption, investment adjustment costs and costly capacity utilization, are important to the propagation of news shocks in a real business cycle model. Christiano et al (2009) estimate a dynamic general equilibrium model featuring nominal and real frictions for the U.S. economy and show that news shocks are important sources of business fluctuations. However, Sims (2009) uses traditional SVAR methods to identify news shocks in a large scale VAR model and finds that news shocks fail to generate co-movement in macro variables, so news shocks cannot be a valid candidate for the main driving force of business cycles.

To shed light on the effect of anticipated technology shocks or news shocks on the economy, we estimate a small scale VARMA model similar to Fisher (2006). There are three variables in the model: the growth rate of real equipment price, the growth rate of labor productivity and the log index of average working hours. The rationale behind this exercise is as follows: if there is a significant anticipated component in either the investment specific technology shock or the neutral technology shock, the implied time series becomes non-invertible. With our four-step procedure, we should be able to identify the true model with enough reasonable restrictions, no matter it is non-invertible or not. The application of the four-step procedure is given as follows:

Step one: *Estimate a reduced-form VARMA(1,1) on the observables*

First, we estimate a VARMA(1,1) model on the data. In practice, there are at least two advantages of this VARMA(1,1) setup over the traditional long VAR models: (i) the model requires less parameters, which relieves the concern on too many estimated parameters to some extent; (ii) the VARMA(1,1) setting is more consistent with the DSGE models studied in macroeconomics.[16] The VARMA model is estimated in a two-step manner. The first step is estimating a long VAR model to obtain a residual series. In the second step, we estimate a VARMA(1,1) model by adding the residual series from the first step as a regressor and check for convergence.[17] After obtaining the estimated VARMA(1,1) model, we get variance matrix of error terms, $\hat{\Omega}$, which is the estimate of $D_j D_j'$, and the MA coefficient matrix, N, which is the estimate of $B_j D_j^{-1}$. These moment estimates are used in the second step.

Step two: *Calculate all covariance equivalent representations*

Second, we compute all covariance equivalent representations. As we show in section three, all the covariance equivalent representations are solutions of the Potter equation defined by the moments of observable variables, and the true model should be one of them. In the current application, the Potter equation is given by:

$$\begin{aligned} D_j D_j' + B_j B_j' &= \hat{\Omega} + N\hat{\Omega}N' \\ B_j D_j' &= N\hat{\Omega}. \end{aligned} \tag{16}$$

Step three: *Define the structural shocks of interest and impose an SVAR-type restrictions on each representation.*

Following Fisher (2006) and Altig et al (2009), a positive investment specific shock is defined as the only shock which lowers the real equipment price in the long run, while a

[16] See for example Kehoe (2007).

[17] The efficiency of estimation could be improved by employing a 3SLS procedure or iterated 2SLS procedure. Kascha (2007) gives a good survey on estimation methods of the VARMA models.

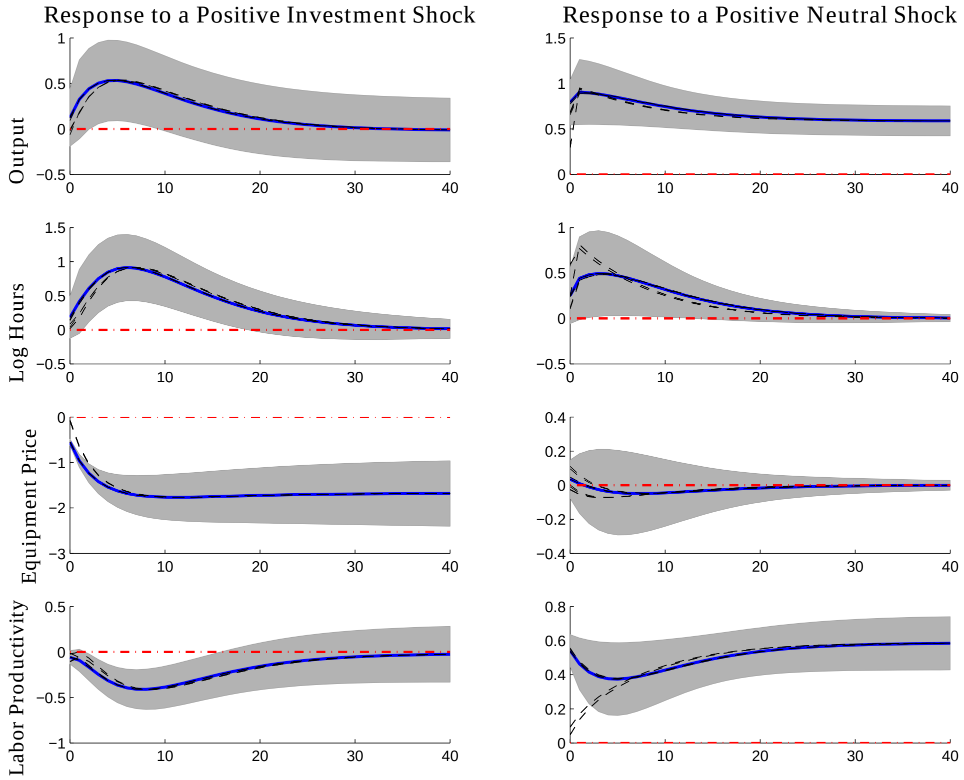
positive neutral technology shock is define as the other shock which increases labor productivity in the long run apart from the positive investment specific shock. Based on the definitions, two long run restrictions are imposed on the estimated model to identify the two technology shocks. There are eight structural representations satisfying the Potter equation as well as the two long run restrictions.

Figure 5 shows the impulse responses of all eight cases along with the point estimate and the confidence interval based on the innovation representation. The latter is the counterpart of the traditional VAR identification in our VARMA(1,1) setup. In the invertible case, the estimated effect of identified shocks are in line with existing research: in response to a positive investment shock, hours and output increase prominently, however, labor productivity falls for a long period after the shock. Output and labor hours increase less significantly in the case with a positive neutral technology shock. In non-invertible cases, the responses to the investment shocks are similar to those in the invertible case. In response to the neutral technology shock, hours rise faster and stronger in some non-invertible cases, but the response of output on impact becomes weaker. In those cases, labor productivity increases gradually, instead of jumping up as shown in the invertible case. If technology is only disseminated slowly in the economy, we should observe the slow buildup of labor productivity in response to technology shocks as shown here. The strong response of hours in can be readily explained by strong intertemporal substitution effect as in Jaimovich and Rebelo (2009). Up to this step, economic theory cannot distinguish between the invertible and the invertible models. Therefore, we need additional selection criteria to pin down the true model, which is the purpose of the fourth step in our procedure.

Step four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out structural responses.*

In this step, we impose agnostic restrictions on variance decompositions: (i) the investment shock should explain the long run variance in the growth of real equipment price at least 10%; (ii) the neutral technology shock contributes the long run variance on the growth of labor productivities at least 10%; (iii) the third shock, with is a combination of

Figure 5: Response To Technology Shocks (All Cases)



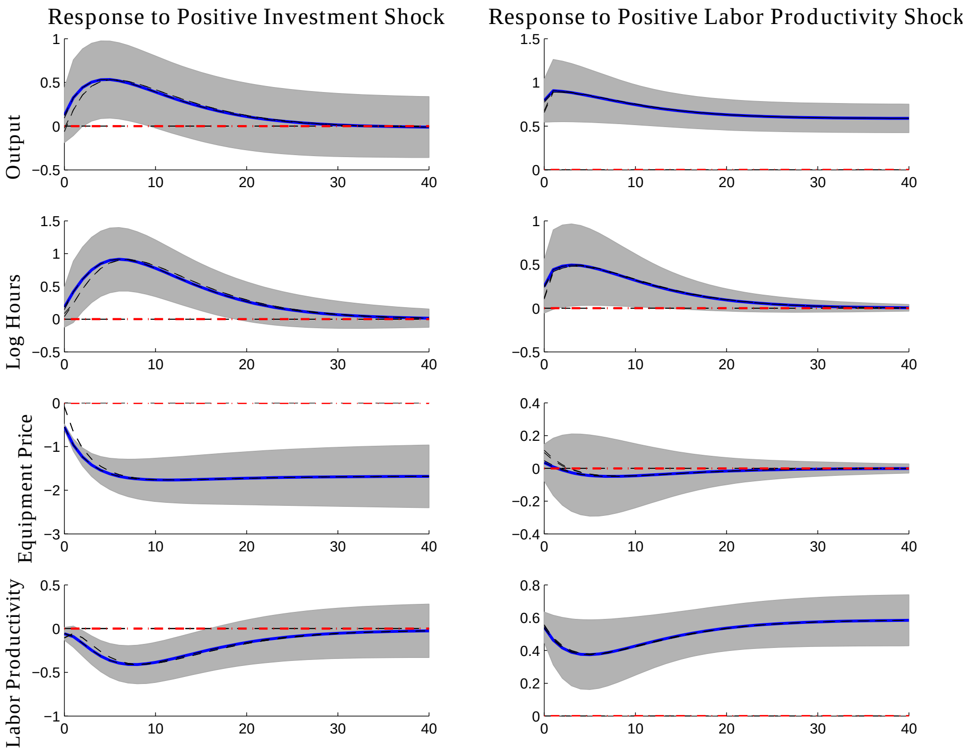
Notes: solid blue line: the point estimate of impulse responses in the innovation representation; gray area: 90% confidence interval in the innovation representation; dashed black lines: impulse responses from the solutions of the Potter equation

other non-technology shocks and measurement errors, should not contribute more then 30% to the long run volatility in either the real equipment price or the labor productivity. The result of the variance decomposition is summarized in table 2.

As shown in the table, we successfully rule out some cases. Based on the third criterion, we can rule out case models 1, 3, 5 and 7. In all the four cases, the contribution of other non-technology shocks on the growth of technology in the long run are unreasonably large. However, we cannot refine the outcome further, in other words, we only achieve a partial identification in this example.

Figure 6 plot the responses of models satisfying the agnostic restrictions based on variance decompositions along with the invertible case. In all the four valid cases, impulse responses are very similar to each other. Furthermore, the invertible case is among the four cases we keep. The variance decomposition analysis also show similar result in all the four cases. Therefore, we can reach the conclusion that the inference based on analysis on an invertible VAR model is valid and reliable. In other words, news or anticipated components in technology shocks does not play important roles when studying the effect of these two types of technology shocks. Between the two technology shocks, the investment specific shock is more important to explain the dynamics in labor hours. In additional, we notice that the remaining cases actually ”cluster” based on our identification. It might indicate all the identification on technology shocks are correct, while the identification of the third shock might differ. If our interest is only on technology shocks, we probably can keep all of them.

Figure 6: Response To Technology Shocks (Identified)



Notes: solid blue line: the point estimate of impulse responses in the innovation representation; gray area: 90% confidence interval in the innovation representation; dashed black lines: impulse responses from the solutions of the Potter equation

Table 2: Identification Based on Short-Term Variance Decomposition

	Model One	**Model Two**	**Model Three**	**Model Four**	**Model Five**	**Model Six**	**Model Seven**	**Model Eight**
contribution of investment shocks in the long run								
variable 1	97.32	96.46	97.32	96.07	98.24	98.88	98.31	99.22
variable 2	8.87	9.93	8.86	10.10	8.06	7.55	8.08	7.51
variable 3	51.66	51.35	51.66	51.51	51.68	51.69	51.62	51.50
contribution of neutral shocks in the long run								
variable 1	.35	2.84	0.32	2.23	0.40	0.69	0.31	0.46
variable 2	5.69	74.50	6.14	70.08	5.74	76.77	6.14	72.24
variable 3	17.29	13.07	17.23	12.96	17.19	13.31	17.22	13.21
contribution of other shocks in the long run								
variable 1	2.33	0.70	2.35	1.69	1.36	0.43	1.38	0.32
variable 2	85.44	15.57	85.00	19.82	86.21	16.68	85.77	20.24
variable 3	31.04	35.58	31.11	35.99	31.11	35.00	31.18	35.28

variable 1: the growth rate of real equipment price; variable 2: the growth rate of labor productivity; variable 3: labor hours

6 Conclusion

Traditional limited information econometric methods, including the widely applied structural VAR approach, cannot handle non-invertiblility embedded in many business cycle models. However, researchers need not abandon the limited information approach, which is the power and soul of the structural VAR. We show that non-invertible time series can be recovered with its invertible counterpart. That is, there is always an invertible innovation representation corresponding to a non-invertible model. The invertible innovation representation shares the same population moment with the structural model. Therefore, we can recover all the valid models through those consistently estimated moments, regardless of invertibility.

Based on the theory developed in this paper, we propose a four step procedure to handle non-invertibility in practice. This four steps are: (i) estimate a reduced form VARMA(1,1); (ii) compute all VARMA(1,1) models with the same autocovariance structure using Potter's (1964) algorithm; (iii) use the outcomes from step two and an SVAR-type restriction to find a finite number of valid structural impulse responses; (iv) use agnostic restriction implied by economic theory to identify, at least partially, the true model.

We then apply this procedure to two model-generated examples. In both the permanent income model of FRSW (20007) and the anticipated fiscal shock model in LWY, our procedure recovers the true model. We further apply our method to cases with real data. We find that result in Fisher (2006)'s study on technology shocks holds even when we consider possible non-invertibilities in the model. It indicates that anticipated component technology shocks or "news shocks" do not spoil the inference of the transmission mechanism of technology shocks.

Appendix

A Proofs

Proof of Theorem 1: First, we prove equations (2) -(4) are necessary for any MA representation to be covariance equivalent to the structural form. That is, every MA representation of the structural form satisfies these conditions. The structural form has an MA representation in the same format as (2).

Let $\bar{W}$ be the left inverse of W, which exists by Assumption 1. The MA representation of s_{t+1} is:

$$s_{t+1} = (I - QL)^{-1} U e_{t+1} = \sum_{i=0}^{\infty} Q^i U e_{t+1-i}. \tag{A. 1}$$

Substituting (A. 1) in the observer equation from the state-space form is:

$$r_{t+1} = W \sum_{i=0}^{\infty} Q^i U e_{t-i} + Z e_{t+1}, \tag{A. 2}$$

Premultiplying both side by $\bar{W}L$ and rearranging,

$$\sum_{i=0}^{\infty} Q^i U e_{t-1-i} = \bar{W}(r_t - Z e_t).$$

Hence, (A. 2) can be rewritten as:

$$\begin{aligned} r_{t+1} &= W[U e_t + Q\bar{W}(r_t - Z e_t)] + Z e_{t+1} \qquad \text{(A. 3)} \\ &= WQ\bar{W} r_t + Z e_{t+1} + (WU - WQ\bar{W}Z) e_t, \end{aligned}$$

The MA representation of (A. 3) is given by:

$$r_{t+1} = [I - WQ\bar{W}L]^{-1} [Z + W(U - Q\bar{W}Z)L] e_t, \tag{A. 4}$$

In the next step, we prove that $WQ\bar{W} = A$ and $W(U - Q\bar{W}Z) = \tilde{C}_1 (Z')^{-1}$.

Next, we show that the MA representation (A. 4) satisfies (3) and (4). Define C_i to be the ith order autocovariance matrix of r_t.

Suppose y_t is a general VARMA(p,q), $y_t = M(L)y_t + N(L)w$ with $w_t \sim \mathcal{N}(0, I)$. The autocovariance-generating function of y_t is

$$G_y(z) = [I - M(z)]^{-1} N(z)N(z^{-1})' \left[I - M(z^{-1})'\right]^{-1}$$

Therefore, we have:

$$\begin{aligned}
C_0 &= E\left(r_t r_t'\right) \\
&= WQ\bar{W}C_0(WQ\bar{W})' + ZZ' + WUU'W' \\
&\quad -WQ\bar{W}ZZ'(WQ\bar{W})' && \text{(A. 5)} \\
C_1 &= E\left(r_t r_{t-1}\right) \\
&= WQ\bar{W}C_0 + WUZ' - WQ\bar{W}ZZ' && \text{(A. 6)} \\
C_i &= E\left(r_t r_{t-i}{}'\right) \\
&= (WQ\bar{W})^{i-1}C_1 \text{ for } i \geq 2 && \text{(A. 7)}
\end{aligned}$$

Simplifying notation, let $A = WQ\bar{W}$, $B = WU - AZ$ and $D = Z$. Then,

$$A = WQ\bar{W} = C_2 C_1{}^{-1}$$

Based on the definitions, $\tilde{C}_0$ and $\tilde{C}_1$ satisfy:

$$\begin{aligned}
\tilde{C}_1 &= C_1 - AC_0 = BD' \\
\tilde{C}_0 &= C_0 - AC_0A' - A\tilde{C}_1' - \tilde{C}_1 A' = DD' + BB'
\end{aligned}$$

Therefore, we have:

$$B = W(U - Q\bar{W}Z) = \tilde{C}_1 Z'^{-1}$$

We further substitute B in the equation for $\tilde{C}_0$,

$$\tilde{C}_0 = ZZ' + \tilde{C}_1 (ZZ')^{-1} \tilde{C}_1'$$

Premultiplying both sides with $\tilde{C}_1^{-1} ZZ'$,

$$(ZZ')(\tilde{C}_1')^{-1}(ZZ') - \tilde{C}_0(\tilde{C}_1')^{-1}(ZZ') + \tilde{C}_1 = 0, \tag{A. 8}$$

Thus, ZZ' satisfies (4). Also, since ZZ' is a symmetric positive semi-definite matrix, its Cholesky decomposition generates a lower triangular matrix Z^c such that $Z^c Z^{c\prime} = ZZ'$. Based on Uhlig (2005), there always exists an orthonormal matrix K such that $K = (Z^c)^{-1} Z$.

In our final step, we show that (2)-(4) are also sufficient for a valid covariance equivalent representation: every process satisfying (2)-(4) is covariance equivalent the structural form.

It is obvious that the proposed representations have the same first moments as the structural form. Hence, if the second moments of the proposed processes are also the same as those implied by the structural form, then the proposed forms are covariance equivalent..

Based on the construction, the general form of each candidate is:

$$\hat{r}_{t+1} = A\hat{r}_t + Z_j \varepsilon_{t+1}^j + \tilde{C}_1 (Z_j')^{-1} \varepsilon_t^j \tag{A. 9}$$

where $\varepsilon_t^j \sim$ is $\mathcal{N}(0, I)$ and A, Z_j and $\tilde{C}_1$ are determined by (3) and (4). The autocovariance of $\hat{r}_t$ is:

$$\begin{aligned}
\hat{C}_0 &= E\left(\hat{r}_{t+1}\hat{r}_{t+1}'\right) \\
&= A\hat{C}_0 A' + AZ_j(Z_j)^{-1}\tilde{C}_1 + \left(AZ_j(Z_j')^{-1}\tilde{C}_1'\right)' + Z_j Z_j' + \tilde{C}_1 (Z_j Z_j)^{-1} \tilde{C}_1' \\
\hat{C}_1 &= E\left(\hat{r}_t \hat{r}_{t-1}'\right) = A\hat{C}_0 + \tilde{C}_1 (Z_j')^{-1} Z_j \\
\hat{C}_i &= E\left(\hat{r}_t \hat{r}_{t-i}'\right) = (A)^{i-1} \hat{C}_1 \text{ for } i \geq 2
\end{aligned}$$

Since $Z_j Z_j'$ is a solution to (A. 8),

$$\tilde{C}_0 = (Z_j Z_j') + \tilde{C}_1 (Z_j Z_j')^{-1} \tilde{C}_1' \tag{A. 10}$$

Therefore, the equation for $\tilde{C}_0$ becomes:

$$\hat{C}_0 = A\hat{C}_0 A' + A\tilde{C}_1 + \tilde{C}_1 A' + \tilde{C}_0 \tag{A. 11}$$

Hence, the solution of $\hat{C}_0$ is given by

$$vec(\hat{C}_0) = [I - (A \otimes A)]^{-1} vec(A\tilde{C}_1 + \tilde{C}_1 A' + \tilde{C}_0) \tag{A. 12}$$

where $vec(\bullet)$ is the vectorization operation turning an m by n matrix into an mn by 1 vector. Based on the definition of $\tilde{C}_0$ and $\tilde{C}_1$,

$$vec(C_0) = [I - (A \otimes A)]^{-1} vec(A\tilde{C}_1 + \tilde{C}_1 A' + \tilde{C}_0) \tag{A. 13}$$

Therefore,

$$\hat{C}_0 = C_0. \tag{A. 14}$$

Given the equivalence between C_0 and $\hat{C}_0$, it is easy to see that

$$\hat{C}_1 = A\hat{C}_0 + \tilde{C}_1 = AC_0 + \tilde{C}_1 = C_1 \tag{A. 15}$$

and

$$\hat{C}_i = A^{i-1}\hat{C}_1 = A^{i-1} C_1 = C_i, \ \forall i \geq 2. \tag{A. 16}$$

Hence, we if a representation satisfies (2)-(4), it is covariance equivalent to the structural form.

As for the number of valid Z_js, there are $\begin{pmatrix} 2k \\ k \end{pmatrix}$ solutions to equation (c). The form of $Z_j Z_j'$ requires it to be symmetric and positive definite; thus, the valid solution is less than $\begin{pmatrix} 2k \\ k \end{pmatrix}$. With an alternative approach, we can show there are a total of 2^k valid representations. Furthermore, we show that among all the valid covariance-equivalent representations, there is one presentation which is invertible. The detail of this alternative approach is included in appendix B. **Q.E.D**

B The equivalence between Blaschke Matrices and the Potter Equation

Lippi and Reichlin (1994) show every noninvertible stationary VARMA(p,q) model has one invertible representation by multiplying an appropriate Blaschke matrix. A Blaschke matrices, $B(z)$, is a special matrix satisfying the following property:

$$B(z)B(z^{-1})' = I. \tag{B. 17}$$

As we know, every orthonormal matrix is a Blaschke matrix. In the remaining part of this section, we show how to use Blaschke matrices to get an invertible representation and how this alternative procedure is related to the proposed procedure in the main text.

Lemma *Every covariance-equivalent form can be achieved by multiplying an appropriate Blaschke matrix on the original model*

Proof:

$$\begin{aligned} r_{t+1} &= WI - QL^{-1}Ue_t + Ze_{t+1} \\ &= W\sum_{i=0}^{\infty} Q^i U e_{t-i} \\ &= WQ\bar{W}(r_t - Ze_t) + WUe_t + Ze_{t+1} \\ &= WQ\bar{W}y_t + Ze_{t+1} + (WU - WQ\bar{W}Z)e_t. \end{aligned} \tag{B. 18}$$

For simplicity in notations, define $M = WQ\bar{W}$, $N_0 = Z$ and $N_1 = WU - WQ\bar{W}Z$. Therefore, we have the autocovariance generating function of r_t is given by:

$$G_r(z) = ([I - Mz])^{-1}(N_0 + N_1 z)(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \tag{B. 19}$$

Equation () is a VARMA(1,1) representation of the structural model, which might be invertible or non-invertible. Next, we show that there is an alternative VARMA(1,1) representation of the same model, and furthermore, this representation is invertible. To this end, we construct a square matrix A(L) of dimension m. This matrix depends on the matrix lag polynomial $N(L) = N_0 + N_1 L$. More specifically, let $\{\lambda_i\}_{i=1}^{m}$ be the eigenvalues of $N(L)$. Define a matrix $R(\lambda_i, z)$ as follows:

$$R(\lambda_i, z) \;=\; \begin{cases} \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \frac{1-\bar{\lambda}_i z}{1-\lambda_i z} & 0 \\ 0 & 0 & I_{m-i} \end{pmatrix}, & |\lambda_i| > 1 \\ I_m, & \text{otherwise} \end{cases} \tag{B. 20}$$

The matrix $R(\lambda_i, z)$ is known as a Blaschke matrix. It satisfies the property $R(\lambda_i, z)R'(\bar{\lambda}_i, z^{-1}) = I$. Now, we defines another matrix K_i. This matrix is an orthonormal matrix, whose ith column is the normalized solution of $N(\lambda_i)x = 0$.

Firstly, we can construct another lag polynomial $N^i(L) = N_0^i + N_1^i L = (N_0 + N_1 L)K_i R(\lambda_i, L)$. By right multiplying $N(L)$ with K_i, one can move all the entries containing the factor $1 - \lambda_i L$ on the ith column. By further right multiplying $R(\lambda_i, L)$, one replaces $1 - \lambda_i L$ with $\lambda_i - L$ but leave other elements untouched, in other words, "flips" a particular eigenvalue of the lag polynomial. At the same time, we even have:

$$\begin{aligned} G_r^i(z) &= ([I - Mz])^{-1}(N_0^i + N_1^i z)(N_0^i + N_1^i z^{-1})'[I - M'^{-1}]^{-1} \\ &= ([I - Mz])^{-1}(N_0 + N_1 z)K_i R_i(\lambda_i, L)R'(\bar{\lambda}_i, L^{-1})K_i'(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \\ &= ([I - Mz])^{-1}(N_0 + N_1 z)(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \\ &= G_r(z) \end{aligned} \tag{B. 21}$$

Therefore, we construct another VARMA(1,1) representation of the structural model:

$$r_{t+1} = Mr_t + N_0^i e_{t+1}^i + N_1^i e_t^i. \tag{B. 22}$$

Compared to the model in equation (B), model (B. 22) has the same variance-covariance structure and the same likelihood. Based on construction, we know that the eigenvalues of the covariance-equivalent forms are either the eigenvalues of the structural form or the reciprocal of them. Therefore, if there are eigenvalues outside the unit circle (non-invertible), there has to be a covariance-equivalent form "flipping" all the explosive eigenvalues while keeping the stable eigenvalues untouched.

Q.E.D

Lemma *The method with Blaschke matrices gives the same result as the procedure based on the Potter equation*

Proof: The proof applies to a general $VARMA(p,q) model$, $M(L)x_t = N(L)w_t$, where $M(L)$ is stable. *(i) Any solution implied by Blaschke matrices is a solution implied by the Potter equation.* This is obvious. Based on construction, a representation generated by using Blaschke matrices have the same covariance structure as the structural form. Hence, it is satisfies conditions (2) to (4)

Any solution satisfying conditions (2) to (4) is a solution by using Blaschke matrices This is based on Theorem 2 in Lippi and Reichlin (1994). Assume the invertible $VARMA(p,q)$ model is given by $M(L)x_t = N(L)u_t$. an arbitrary solution from the potter equation is given by $M(L)x_t = \tilde{N}(L)w_t$. Based on definition, $x_t = M(L)^{-1}\tilde{N}(L)w_t$ is a MA representation of the original VARMA model. Therefore, we have to have $M(L)^{-1}\tilde{N}(L) = M(L)^{-1}N(L)B(L)$, where $B(L)$ is a Blaschke matrix. Thus, $\tilde{N}(L) = N(L)B(L)$.

Q.E.D

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