Sequential Innovation, R&D Intensity, and Patent Policy*

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Abstract

This paper studies optimal patent policy in a dynamic setting with sequential innovation. Firms innovate by undertaking "research" activities to generate new ideas and by undertaking "developement" activities to commercialize these ideas. To provide firms with incentive to invest in R&D, the social planner grants a monopoly right (patent) to an innovator to exclude its opponent from producing the goods based on its innovation for a certain period of time. If the idea arrival process is exogenous, the optimal policy is stationary and a new idea is patentable if its quality is better than a constant cutoff. If idea arrival is endogenously determined by firms' "research" investment, the optimal policy is in general nonstationary and characterized by two cutoffs, and the policy switches from one cutoff to the other cutoff only once.

1 Introduction

The cumulative nature lies in the heart of innovation activities. Most new technologies are results of a sequence of incremental improvements. Two well-documented examples from British industrial revolution are the blast furnace in the iron industry (Allen, 1983) and the Cornish pumping engine in the mining industry (Nuvolari, 2004). More recent examples include the invention of laser and spreadsheet (Scotchmer, 2004). The laser was inspired by its predecessor – the maser, and Microsoft Excel spreadsheet built on its predecessor Lotus 1-2-3, while Lotus itself was an improvement upon the first electronic spreadsheet – VisiCalc.

*Very preliminary. Comments welcome.

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The cumulative nature of innovation gains growing importance in the past century, especially for modern industries such as semiconductor and biotechnology.

Patent system plays an important role in providing firms with incentives to innovate. By granting an innovator the monopoly rights to produce and market the patented product for a certain period of time, a patent allows the innovator to recover and profit from its investment in innovation. In industries where innovation is cumulative or sequential, a social planner who is restricted to use patents to reward innovation faces two important trade-offs. First, the total discounted horizon of monopoly rights as innovation reward is finite, so a stronger patent today constrains the social planner to reward future innovators tomorrow. Second, in order to fulfill the promise of a stronger patent protection today, the social planner has to increase the breath and scope of the patent, denying many more ideas the opportunity to file a patent in the future. This may discourage future innovators from investing in research to generate new ideas.

We introduce a model of sequential innovation where firms compete in "research" activities to generate ideas and in "development" activities to commercialize these ideas. The terms "research" and "development" are often used together or interchangeably in the literature and practice when referring to innovation activities. This paper will treat research and development separately. In particular, we use research to refer activities undertaken primarily to acquire new knowledge, while we use development to refer to activities that use existing knowledge gained from research to produce new products or install new processes, or to improve existing products or processes. We study these two important trade-offs faced by the social planner and characterize the optimal patent policy.

We model the line of current and future products as a quality ladder. Firms compete in innovation to improve the quality of the product so that they can climb high on the quality ladder. The size of the improvement depends on the idea quality and subsequent development effort. New ideas arrive sequentially according to a Poisson process and the arrival rate depends on the firm's investment on research.

When a firm gets an idea, it decides how much to invest in order to develop the idea into a viable product. Better ideas are associated with lower development costs. For simplicity, we assume that the quality of idea is publicly observable.[1] However, we assume that its subsequent development investment are private information of the firm. We also assume that the patents held by the incumbent expire immediately whenever the entrant successfully files a new patent. The goal of the paper is to investigate the implication of the nature of cumulative innovation on the optimal patent policy.

To provide firms with incentives to innovate, the authority grants a monopoly right

[1] Under a standard sorting condition, one can derive a mechanism to induce truth-telling when the idea of quality is not observable. See, for example, Hopenhayn, Llobet and Mitchell (2006).

(patent) to an innovator to exclude its opponent from producing the goods based on its innovation for a certain period of time. Since innovation is cumulative, future innovations will necessarily build on the current technology. This leads to two important trade-offs. First, the total discounted duration of monopoly rights available to the authority is finite, so a longer duration of market power assigned to the current innovator limits the size the possible reward to future innovators. Therefore, the social planner must trade-off rewarding current innovators versus rewarding future innovators. Second, a stronger patent must be associated with a sequence of higher cutoffs, which means that only very good ideas are patentable. Therefore, the entrant will have less incentive to invest in research, leading to a lower arrival rate of new ideas.

Our main result is a characterization of the optimal patent policy in our setting. We show that if the idea arrival process is exogenous, then the optimal patent policy is stationary. Under the optimal policy, future ideas are patentable as long as their quality exceeds a constant cutoff which is a function of the strength of the current patent protection. If the arrival rate of ideas is endogenously determined by firms' investment in research activities, then the optimal policy is in general nonstationary. This nonstationary optimal policy consists of two cutoffs that determine the set of patentable ideas and the optimal policy switches from one cutoff to the other cutoff only once. For this general case, we characterize these two cutoffs, the switching time, and the welfare implications.

This paper is closely related to the patent design literature that is pioneered by Nordhaus (1969), who considers how to set the length of the patent to provide social optimal incentives for innovation. Built on Nordhaus (1969), Gilbert and Shapiro (1990) and Klemperer (1990) add the scope (or breadth) of the patent to the designer's toolkit and investigate the trade-off between the length and breadth of a patent in providing rewards to innovators. In particular, Gilbert and Shapiro shows that the optimal length of patent is infinite under fairly general conditions in a static environment. In our dynamic setting, the optimal patent length is also infinite, but it is contingent on possible arrival of future innovations.

Cornelli and Schankerman (1999) and Scotchmer (1999) introduce mechanism design approach to the literature, and show that when the patent granting authority does not have information about the value of an innovation, a patent may be superior than the alternative incentive tools, because it can tie the value the innovation to the length and scope of patent protection. Cornelli and Schankerman (1999) show that it is optimal for the government to offer firms a menu of patent lives and associated lump-sum patent fees. Hopenhayn and Mitchell (2001) showed that, under certain sorting condition, such a scheme may be dominated by a menu that trades off the breadth and length of a patent and with zero patent fees.

The nature of cumulative innovation gains a lot of attention and related patent research grows quickly. Built on the idea of Green and Scotchmer (1995), O'Donoghue, Scotchmer and Thisse (1998) study the trade-off between the patent length and the patent breadth in the sequential innovation case with licensing. They find that a broad patent with short patent length improves the diffusion of new products, but a narrow patent with long patent life can lower R&D cost. Bessen and Maskin (2006) develop a model of cumulative and complementary innovation in a differentiated product market and show that the patent protection may discourage innovation and reduce social welfare.

Lastly, our work is also related to recent analysis of state-contingent patent policies by Acemuglu and Akcigit (2008) and Hopenhayn and Mitchell (2010). In a general equilibrium framework with cumulative innovation, Acemuglu and Akcigit (2008) find that welfare maximizing intellectual property rights is state-dependent. Hopenhayn and Mitchell (2010) characterizes the optimal patent policy in a setting where two long lived firms compete in innovation. Both firms can innovate repeatedly, but they assume that the idea arrival process is exogenous and the ideas are homogenous in quality. Both papers find that the optimal patent policy may favor the technology leader rather than the follower.

2 The Model

Consider a sequential innovation environment where new technologies must build on the old ones. Firms compete in innovation to improve the quality of the current product in the market. Each innovation starts with an "idea". Ideas are heterogenous in quality. The idea quality z is distributed according to a cumulative distribution $\Phi(z)$ with support $[0, \infty)$ and a density $\phi > 0$. New ideas arrive according to a Poisson process, and the arrival rate is increasing in the amount of investment in "research" activities. The *flow* cost of generating new ideas is denoted by an increasing function $K(\lambda)$, where λ is the arrival rate of new ideas.

Once a firm has an idea, it has to decide whether and how much to invest in "development" activities that transform the idea into a new product. The *lump-sum* cost of development is $c(\triangle, z)$, which is decreasing in z and increasing in the size of quality improvement $\triangle$ over the current technology. The idea quality z is publicly observable but the investment size $\triangle$ is private information of the innovator. An idea is either innovated immediately or lost, i.e., banking ideas is not possible.[2] We exclude the possibility of licensing, so a firm with a new idea will undertake "development" activities only if the new idea is patentable. We assume that $c(\triangle, z)$ is increasing and convex in $\triangle$, decreasing in z, and its cross derivative is negative (i.e., $\frac{\partial^2 c(\triangle,z)}{\partial \triangle \partial z} < 0$).

Firms engage in price competition in the product market. A product can be freely

[2] See Erkal and Scotchmer (2009) for an analysis allowing innovators to bank ideas for future use.

imitated unless patented, so a patent holder can charge price $p = \triangle$ on patented product until its patent expires. We assume that there is only one consumer who will buy the product at the market price. By assuming away the static distortion due to monopoly, we can focus on the dynamic trade-off purely due to innovation race. Firms discount future with discount rate r.

For exposition purpose, we assume for now that, at each point of time, one incumbent, one active challenger, and many potential entrants compete in innovation. The incumbent is the current patent holder, technology leader and producer who charges price $p = \triangle$ for its patented product. The challenger actively invests in "research" to generate new ideas to improve upon the current patented technology. If the challenger successfully files a patent based on a new idea, the current patent expires immediately. We define an innovation as the process of successfully obtaining a patentable idea and transforming it into a new commercial product. That is, an innovation consists of activities of both research and development. We assume that each firm can innovate only once, so once a patent expires the current incumbent (and patent holder) exits the market. The challenger makes lump-sum investment in developing its new patent into a new product, and becomes the new incumbent. In addition, one of the potential entrants enters the market and becomes the new challenger.

We are interested in the optimal patent policy when the planner is restricted to reward innovators by granting the exclusive right to produce and sell the patented products. Since the discounted duration of monopoly rights is scarce, the social planner has to trade off between rewarding current innovation and rewarding future innovations, and trade off between rewarding "development" and rewarding "research".

Let t denote the calender time since last innovation and is reset to zero when a new patent is filed. Immediately after a new patent z is filed, a patent policy must specify the set of patentable ideas for every future period. We assume that the set of patentable ideas takes the form of cutoffs: in period t only ideas with quality above cutoff $\widehat{z}_t$ are patentable.[3] Note that a patent holder values a patent because it gives the exclusive right to produce the patented product. Therefore, a patent holder does not directly concern the sequence of cutoffs $\{\widehat{z}_s\}_{s=0}^{\infty}$ per se but rather the expected discounted duration of this exclusive rights implied by these cutoffs. This duration, denoted by $d(z)$, summarizes the strength of the initial protection assigned to patent z implied by the cutoffs $\{\widehat{z}_s\}_{s=0}^{\infty}$. In general, the strength of patent protection may vary over time. If we let D_t denote the remaining protection for patent z conditional on that no new patent is filed before time t, then the remaining sequence of cutoffs $\{\widehat{z}_s\}_{s=t}^{\infty}$ must be consistent with the value of D_t.

Moreover, the current patent holder will be indifferent between two different remaining

[3] Under our model setting, we show it is optimal to use cutoffs to screen ideas. A proof is given in the appendix.

sequences of cutoffs as long as they deliver the same value of D_t. Therefore, for the resource constraint faced by the social planner, the remaining patent protection D_t can be used as a state variable at time t as the remaining patent protection she has promised to the current patent holder.

The sequence of cutoffs $\{\widehat{z}_s\}_{s=t}^{\infty}$ affects D_t in two ways. First, for a given arrival rate of new ideas, a lower cutoff $\widehat{z}_t$ directly increases the likelihood of an idea arriving at time t to be patentable. Since the filing of a new patent leads to the expiration of the current patent, a lower $\widehat{z}_t$ implies a lower expected discounted duration D_t. Second, a lower cutoff $\widehat{z}_t$ also gives the challenger a stronger incentive to invest in research activities to increase the arrival rate of new ideas. A higher arrival rate of new ideas speeds up the arrival of a new patent and thus the expiration of the current patent.

We focus on the class of patent policies that have the following features. First, the current patent expires immediately whenever a new patent is filed, so there is always only one active patent. Second, the reward function $d(z)$ is stationary: it does not depend on t. Therefore, a patent policy consists of a reward function $d(\cdot)$ and, for each patentable idea z, a sequence of cutoffs $\{\widehat{z}_s\}_{s=0}^{\infty}$ consistent with $d(z)$. Given a patent policy $d(\cdot)$ and $\{\widehat{z}_s\}_{s=0}^{\infty}$, immediately after the patent is approved ($t = 0$), the current patent holder (the incumbent) chooses the size of quality improvement $\triangle$ once-and-for-all. The challenger firm, on the other hand, chooses the flow investment $K(\lambda_t)$ in each period t to generate new ideas with arrival rate λ_t. If the challenger gets an idea z' with $z \geq \widehat{z}_t$ at time t, the challenger files a new patent, and the old patent expires immediately. As a result, the old incumbent is out, and the challenger becomes the new incumbent and decides how much to invest in development additivities to transform the new patent into a commercial product. A potential entrant enters the market and the process repeats.

The initial promised duration D_0 at $t = 0$ and the remaining promised duration D_t at time t are endogenously determined by the future arrival rates and the sequence of cutoffs. We derive a formula to link durations to the future arrival rates and the remaining sequence of cutoffs.

Lemma 1 $D_0 = \int_0^{\infty} e^{-\int_0^s (\lambda_\tau (1-\Phi(\widehat{z}_\tau))+r) d\tau} ds$, *and* $D_t = \int_t^{\infty} e^{-\int_t^s (\lambda_\tau (1-\Phi(\widehat{z}_\tau))+r) d\tau} ds$.

Proof. The remaining duration D_t can be written recursively as

$$D_t = dt + \lambda_t (1 - \Phi(\widehat{z}_t))\, dt \cdot 0 + e^{-rdt} [1 - \lambda_t (1 - \Phi(\widehat{z}_t))\, dt]\, D_{t+dt}.$$

Subtracting both sides by $e^{-rdt} D_t$, dividing both sides by dt, and let $dt \to 0$, we have

$$\dot{D}_t - (\lambda_t (1 - \Phi(\widehat{z}_t)) + r)\, D_t = -1.$$

We can use the integrating factor $e^{-\int_0^t (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau}$ to find the solution to the above ordinary differential equation:

$$e^{-\int_0^t (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau} D_t - D_0 = -\int_0^t e^{-\int_0^s (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau} ds.$$

Note that $D_\infty < 1/r$, so we have

$$e^{-\int_0^t (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau} D_t = e^{-rt} \cdot e^{-\int_0^t \lambda_\tau(1-\Phi(\widehat{z}_\tau))d\tau} D_t \to 0,$$

as $t \to \infty$. Therefore,

$$D_0 = \int_0^\infty e^{-\int_0^t (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau} dt \quad \text{and} \quad D_t = \int_t^\infty e^{-\int_t^s (\lambda_\tau(1-\Phi(\widehat{z}_\tau))+r)d\tau} ds.$$

■

Note that given patent policy $\{\widehat{z}_s\}_{s=0}^\infty$ and its implied patent protection $d(z)$, the incumbent chooses the size of increment $\triangle$ to maximize its profit. That is

$$\pi(d(z), z) \equiv \max_\triangle [d(z)\triangle - c(\triangle, z)].$$

On the other hand, the challenger chooses λ_t at time t to maximize

$$\lambda_t \int_{\widehat{z}_t}^\infty \pi(d(\xi), \xi)\, \phi(\xi)\, d\xi - K(\lambda_t),$$

where $\pi(d(z), z)$ is the expected profit (excluding cost of research investment) defined above. Therefore, we can write the optimal choice of arrival rate λ_t as a function of the cutoff $\widehat{z}_t$: $\lambda(\widehat{z}_t)$.

For an "development" of size $\triangle$ based on idea z at time t, the social value (excluding cost of research investment) is given by

$$R(d(z), z) = \frac{1}{r}\triangle(d(z), z) - c(\triangle, z).$$

By comparing the social value and the private value of "development", one can see that each "development" benefits the society with discounted duration $1/r$, but benefits the individual innovator only in the period $d(z)$ that the product is protected. Since the discounted duration $d(z)$ cannot be higher than $1/r$, the private incentive for "development" is always lower than the social incentive for "development". As a result, the social planner should set the patent policy to encourage as much "development" as possible.

Although the social gain from an innovation of fixed size $\triangle$ is higher than the individual gain, it is not necessarily true that the individual incentives to invest in "research" activities are always higher than the social incentives. This is because there is a negative externality of "research" investment. To see this, consider an increase in investment K. Since a higher K implies a higher λ and shortens the expected waiting time to obtain the next new idea, the planner must raise the cutoffs in order to keep the promise $d(z)$. Therefore, some ideas that were previously patentable are now lost. This negative externality may give excessive incentives for firms to invest in "research".

Suppose at time $t = 0$, a new idea z_0 is filed for patent and is rewarded with a protection D_0 which is a number. The social planner needs to specify the reward function $d(\cdot)$ for future patentable ideas and a sequence of cutoffs that is consistent with the promised protection D_0 to the current patent. We can write the social planner's problem as

$$\max_{d(\cdot),\{\widehat{z}_t\}_{t=0}^{\infty}} \int_0^\infty \left\{ \lambda(\widehat{z}_t) \int_{\widehat{z}_t}^\infty \left[R(d(\xi),\xi) + V(d(\xi)) \right] \phi(\xi)\, d\xi - K(\lambda(\widehat{z}_t)) \right\} e^{-\int_0^t [\lambda(\widehat{z}_\tau)(1-\Phi(\widehat{z}_\tau))+r]d\tau} dt$$

subject to

$$\int_0^\infty e^{-\int_0^s (\lambda(\widehat{z}_\tau)(1-\Phi(\widehat{z}_\tau))+r)d\tau} ds \geq D_0.$$

To understand the objective function, note that, under policy $\{\widehat{z}_s\}_{s=0}^{\infty}$, the arrival process of patentable ideas is a non-homogeneous Poisson process with arrival rate at time t being $\lambda(\widehat{z}_t)(1-\Phi(\widehat{z}_t))$. Therefore, we use the distribution of waiting time for the next *patentable* idea to form expectation, because only when a new patentable idea arrives, we will have static social value from new innovation and the corresponding continuation value.

3 The Optimal Patent Policy

In this section, we will characterize the optimal sequence of cutoffs to support a given reward function D_0. For a given protection D_0 assigned to the current patent, the planner's dynamic optimization problem is to choose the sequence of cutoffs $\{\widehat{z}_t\}_{t=0}^{\infty}$ to maximize the expected total social welfare:

$$\max_{\{\widehat{z}_t\}_{t=0}^{\infty}} \int_0^\infty \left\{ \lambda(\widehat{z}_t) \int_{\widehat{z}_t}^\infty \left[R(d(\xi),\xi) + V(d(\xi)) \right] \phi(\xi)\, d\xi - K(\lambda(\widehat{z}_t)) \right\} e^{-\int_0^t [\lambda(\widehat{z}_\tau)(1-\Phi(\widehat{z}_\tau))+r]d\tau} dt$$

subject to the constraint that these cutoffs are consistent with protection D_0

$$\int_0^\infty e^{-\int_0^s [\lambda(\widehat{z}_\tau)(1-\Phi(\widehat{z}_\tau))+r]d\tau} ds \geq D_0.$$

It is useful to introduce the arrival rate of patentable ideas $u(\widehat{z}_t)$ as

$$u(\widehat{z}_t) \equiv \lambda(\widehat{z}_t)\left[1 - \Phi(\widehat{z}_t)\right].$$

Since both $\lambda(\widehat{z}_t)$ and $[1 - \Phi(\widehat{z}_t)]$ are decreasing functions of $\widehat{z}_t$, there is one-to-one mapping between $u(\widehat{z}_t)$ and $\widehat{z}_t$. Now we want to reformulate the planner's problem as choosing $u(\widehat{z}_t)$

instead of $\widehat{z}_t$. To simplify notations, we use u_t to denote $u(\widehat{z}_t)$. Define a function $Q(u_t)$ as

$$Q(u_t) \equiv \lambda(\widehat{z}_t(u_t)) \int_{\widehat{z}_t(u_t)}^{\infty} [R(d(\xi), \xi) + V(d(\xi))] \phi(\xi) d\xi - K(\lambda(\widehat{z}_t(u_t))).$$

Then we can rewrite the social planner's optimization problem as

$$\max_{\{u_t\}} \int_0^{\infty} Q(u_t) e^{-\int_0^t (u_\tau + r) d\tau} dt \text{ subject to } \int_0^{\infty} e^{-\int_0^s (u_\tau + r) d\tau} ds \geq D_0.$$

Let's define u_0 and u_* as follows:

$$\begin{aligned} u_0 &= \frac{1}{D_0} - r \text{ so that } D_0 = \frac{1}{u_0 + r} \\ u_* &\in \arg\max_{u>0} \frac{Q(u)}{u+r} \end{aligned}$$

It is clear that u_0 is uniquely defined for each D_0. If $Q(u)$ is strictly concave, u_* is unique. Otherwise, there may exist several u-values that maximizes $\frac{Q(u)}{u+r}$, we define u_* as the smallest of these u-values. That is, u_* is the smallest arrival rate of patentable ideas that maximizes $\frac{Q(u)}{u+r}$.

Theorem 1 (Binding Constraint) *(1) If $u_* \leq u_0$, the optimal patent policy is stationary with constant u_* and the promise-keeping constraint is redundant. (2) If $u_* > u_0$, then the promise-keeping constraint must be binding in optimum.*

Proof. (1) Note that our objective function is

$$\begin{aligned} & \int_0^{\infty} Q(u(t)) e^{-\int_0^t [u(\tau)+r] d\tau} dt \\ = & \int_0^{\infty} \frac{Q(u(t))}{u(t)+r} [u(t)+r] e^{-\int_0^t [u(\tau)+r] d\tau} dt \\ \leq & \left\{ \max_u \frac{Q(u)}{u+r} \right\} \int_0^{\infty} [u(t)+r] e^{-\int_0^t [u(\tau)+r] d\tau} dt \\ = & \max_u \frac{Q(u)}{u+r}. \end{aligned}$$

The last equality follows from the fact that

$$\int_0^{\infty} [u(t)+r] e^{-\int_0^t [u(\tau)+r] d\tau} dt = 1.$$

If $u_* \leq u_0$, then the upperbound $\max_u \frac{Q(u)}{u+r}$ is achieved by setting $u^*(t) = u_*$ and the constraint is not binding because

$$\int_0^{\infty} e^{-\int_0^t [u_* + r] d\tau} dt = \frac{1}{u_* + r} \geq \frac{1}{u_0 + r} = D_0.$$

(2) (Sketch and incomplete) Now assume $u_* > u_0$. Suppose, by contradiction, that under the optimal policy $u^*(t)$ the constraint is not binding. Then the optimal $u^*(t)$ cannot be equal to u_* for all t, otherwise, the promise-keeping constraint would have violated. Moreover, in order to meet the constraint, we must have $u^*(t) < u_*$ for some interval with positive measure. Consider an interval $[t, t+dt]$ where $u^*(t) < u_*$, and replace it by u_*. Then by the defintion of u_*, such a deviation will improve the social value. Because the promise-keeping constraint is not binding, such a deviation is feasible. ■

Therefore, from now on, we only need to consider the case where the resource constraint is binding, that is, $u_* > u_0$.

3.1 Optimal Policy with Concave Q Function

We start with the simple case when function $Q(u_t)$ is concave in u_t. We prove that the optimal policy is stationary with constant cutoffs over time.

We first reformulate our optimal control problem. Let's define the state variable $x(t)$ as

$$x(t) = e^{-\int_0^t [u(\tau)+r]d\tau}$$

with boundary condition as $x_0 = 1$ and $x_\infty = 0$. Then we have

$$\dot{x}(t) = -[u(t) + r]\, x(t).$$

Let's introduce another state variable $y(t)$ which is defined as

$$y(t) = \int_0^t x(s)\, ds.$$

Then the promise keeping constraint $\int_0^\infty x(t)\, dt = D_0$ can be replaced by

$$\dot{y}(t) = x(t) \text{ with boundary condition } y_0 = 0 \text{ and } y_\infty = D_0.$$

Therefore, using $u(t)$ as the control variable and $(x(t), y(t))$ as the state variables, we rewrite the control problem as

$$\begin{aligned}
\text{Program C} \quad &: \quad \max_u \int_0^\infty Q(u(t))\, x(t)\, dt \\
\text{subject to} \quad &: \quad \dot{x}(t) = -[u(t) + r]\, x(t) \\
&: \quad \dot{y}(t) = x(t) \\
&: \quad x_0 = 1 \text{ and } x_\infty = 0 \\
&: \quad y_0 = 0 \text{ and } y_\infty = D_0.
\end{aligned}$$

From now on, we will refer to this concave optimal control problem as Program C.

Theorem 2 (Concave Q) *The optimal solution to Program C is* $u^*(t) = u_0$ *for all* t.

Proof. The Hamiltonian is defined as

$$H(x(t), y(t), u(t), p_1(t), p_2(t), t) = Q(u(t))x(t) - p_1(t)[u(t) + r]x(t) + p_2(t)x(t).$$

Let $(x^*(t), y^*(t))$ denote the optimal path corresponding to control u_0. We first find a pair of continuous and piecewise continuously differentiable function $p_1(t)$ and $p_2(t)$ such that

$$\begin{aligned}
\dot{p}_1(t) &= -\frac{\partial H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t)}{\partial x} = -Q(u_0) + p_1(t)(u_0 + r) - p_2(t) \quad (1) \\
\dot{p}_2(t) &= -\frac{\partial H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t)}{\partial y} = 0 \quad (2)
\end{aligned}$$

and

$$H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) \geq H(x^*(t), y^*(t), u, p_1(t), p_2(t), t) \quad \text{for all } u \in U. \quad (3)$$

The last condition implies that

$$\frac{\partial H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t)}{\partial u} = [Q'(u_0) - p_1(t)]x^*(t) = 0 \quad (4)$$

Therefore, $p_1(t) = Q'(u_0)$, which implies that $\dot{p}_1(t) = 0$. As a result, from (1) and (2), we have

$$\begin{aligned}
p_1(t) &= Q'(u_0), \\
p_2(t) &= Q'(u_0)(u_0 + r) - Q(u_0).
\end{aligned}$$

In order to show that our candidate policy u_0 indeed maximizes the objective function, it is sufficient to show that, for all $u(t) \in U$,

$$\triangle \equiv \int_0^\infty Q(u_0)x^*(t)\,dt - \int_0^\infty Q(u(t))x(t)\,dt \geq 0.$$

By definition of the Hamiltonian, we have

$$\begin{aligned}
Q(u(t))x(t) &= H(x(t), y(t), u(t), p_1(t), p_2(t), t) + p_1(t)[u(t) + r]x(t) - p_2(t)x(t) \\
&= H(x(t), y(t), u(t), p_1(t), p_2(t), t) - p_1(t)\dot{x}(t) - p_2(t)\dot{y}(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\triangle &= \int_0^\infty \{H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) - H(x(t), y(t), u(t), p_1(t), p_2(t), t)\}\,dt \\
&\quad + \int_0^\infty p_1(t)[\dot{x}(t) - \dot{x}^*(t)]\,dt + \int_0^\infty p_2(t)[\dot{y}(t) - \dot{y}^*(t)]\,dt \\
&= \int_0^\infty \{H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) - H(x(t), y(t), u(t), p_1(t), p_2(t), t)\}\,dt \\
&\quad + Q'(u^*)\int_0^\infty [\dot{x}(t) - \dot{x}^*(t)]\,dt + [Q'(u^*)(u^* + r) - Q(u^*)]\int_0^\infty [\dot{y}(t) - \dot{y}^*(t)]\,dt \\
&= \int_0^\infty \{H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) - H(x(t), y(t), u(t), p_1(t), p_2(t), t)\}\,dt
\end{aligned}$$

The last equality uses the boundary conditions that $x(0) = x^*(0)$, $x(\infty) = x^*(\infty)$, $y(0) = y^*(0)$, and $y(\infty) = y^*(\infty)$.

Consider the integrand of the last expression:

$$\begin{aligned}
& H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) - H(x(t), y(t), u(t), p_1(t), p_2(t), t) \\
= \;& Q(u_0) x^*(t) - p_1(t)(u_0 + r) x^*(t) + p_2(t) x^*(t) \\
& - [Q(u(t)) x(t) - p_1(t)[u(t) + r] x(t) + p_2(t) x(t)] \\
= \;& \{Q(u_0) - Q'(u_0)(u_0 + r) + Q'(u_0)(u_0 + r) - Q(u_0)\} x^*(t) \\
& - \{Q(u(t)) - Q'(u_0)[u(t) + r] + Q'(u_0)(u_0 + r) - Q(u_0)\} x(t) \\
= \;& - \{Q(u(t)) - Q'(u_0)[u(t) - u_0] - Q(u_0)\} x(t)
\end{aligned}$$

Since $Q(u)$ is weakly concave in u, for any $u(t) \neq u_0$, we have

$$Q(u(t)) \leq Q(u_0) + Q'(u_0)[u(t) - u_0].$$

Together with the fact that $x(t) > 0$ for all t, we have

$$H(x^*(t), y^*(t), u_0, p_1(t), p_2(t), t) - H(x(t), y(t), u(t), p_1(t), p_2(t), t) \geq 0.$$

Therefore, $\triangle \geq 0$. ■

3.2 Optimal Policy with Exogenous Idea Arrival Process

Suppose the arrival rate λ of new ideas is exogenous and is independent of investment K. Therefore, firms will set $K = 0$. The social planner's optimization problem becomes

$$\max_{\{u_t\}} \int_0^\infty Q_E(u_t) e^{-\int_0^t (u_\tau + r) d\tau} dt \;\; \text{subject to} \;\; \int_0^\infty e^{-\int_0^s (u_\tau + r) d\tau} ds = D_0,$$

where

$$Q_E(u_t) = \lambda \int_{\widehat{z}_t(u_t)}^\infty [R(d(\xi), \xi) + V(d(\xi))] \phi(\xi) d\xi$$

Theorem 3 (Exogenous Arrival) *Suppose the arrival process of new ideas is exogenous with arrival rate λ. Let z_0 be the unique solution to the equation $u_0 = \lambda[1 - \Phi(z_0)]$, Then the optimal policy is a stationary policy with $\widehat{z}_t = z_0$ for all t.*

Proof. By Theorem 2, it is sufficient to show that $Q^E(u_t)$ is concave in u_t. If we write $\eta(u) \equiv \widehat{z}_t(u)$, then by definition, $u = \lambda[1 - \Phi(\eta(u))]$, which implies $1 = -\lambda\phi(\eta(u))\eta'(u)$.

Therefore,

$$\begin{aligned} Q_E'(u) &= -\lambda \left[R\left(d(\eta\left(u\right)),\eta\left(u\right)\right)+V\left(d\left(\eta\left(u\right)\right)\right)\right]\phi\left(\eta\left(u\right)\right)\eta'\left(u\right) \\ &= R\left(d(\eta\left(u\right)),\eta\left(u\right)\right)+V\left(d\left(\eta\left(u\right)\right)\right) \\ Q_E''(u) &= -\frac{d\left[R\left(d(\eta\left(u\right)),\eta\left(u\right)\right)+V\left(d\left(\eta\left(u\right)\right)\right)\right]}{d\eta\left(u\right)}\frac{1}{\lambda\phi\left(\eta\left(u\right)\right)} \\ &= -\frac{d\left[R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right)\right]}{d\widehat{z}_t}\frac{1}{\lambda\phi\left(\widehat{z}_t\right)} \end{aligned}$$

This implies that Q_E is concave if $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right)$ is increasing in $\widehat{z}_t$.

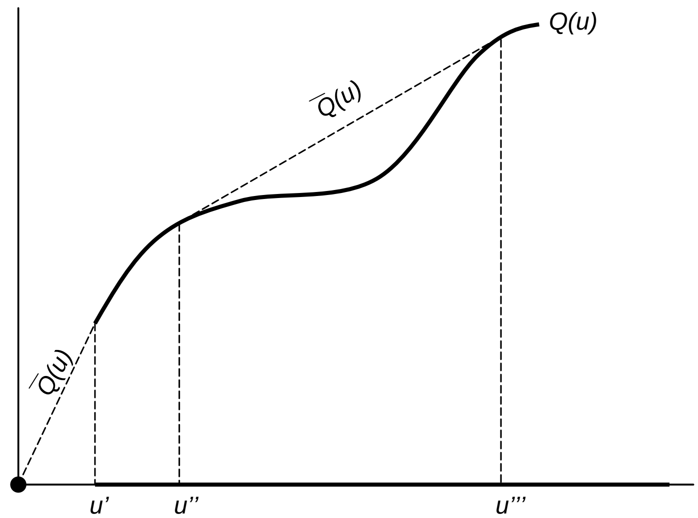
To show that $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right)$ is increasing in $\widehat{z}_t$, suppose the opposite the true. That is, $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right)$ is strictly decreasing around some $\widehat{z}_t$. Therefore, we have a $\widehat{z}_t$ and a $\widehat{z}_t' > \widehat{z}_t$, such that $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right) > R\left(d(\widehat{z}_t'),\widehat{z}_t'\right)+V\left(d\left(\widehat{z}_t'\right)\right)$. Consider the following deviation: If idea quality $\widehat{z}'$ arrives at time t, provide it with protection $d\left(\widehat{z}_t\right)$ instead of $d\left(\widehat{z}_t'\right)$. Due to the lower innovation cost associated with higher idea quality, it follows that $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right) < R\left(d(\widehat{z}_t),\widehat{z}_t'\right)+V\left(d\left(\widehat{z}_t\right)\right)$. Thus $d\left(\widehat{z}_t'\right)$ cannot be optimal, which means that $R\left(d(\widehat{z}_t),\widehat{z}_t\right)+V\left(d\left(\widehat{z}_t\right)\right)$ is monotonically increasing. Thus $Q_E(u)$ is concave in a model with exogenous λ. ■

3.3 Optimal Policy with General Q Function

Concavity of the $Q\left(u\right)$ function is a restrictive assumption as it uncompatible with typical caracteristics of research cost structures. There are often returns to scale in research activities, which gives rise to discontinuities in the investment functions $\lambda\left(\widehat{z}_t\right)$. This implies that as $\widehat{z}_t$ decreases, eventually the return of research become so small that reasearch is unprofitable at any scale, in which case λ jumps to zero. Since $u\left(\widehat{z}_t\right) \equiv \lambda\left(\widehat{z}_t\right)\left[1-\Phi\left(\widehat{z}_t\right)\right]$, this imples that there are value of u that are not feasible, there are empty segments in u. More generally, the cost function of reasearch activities can be complex, and give rise to convex segments in the $Q(u)$ function. In both cases a stationary policy may not be optimal. However, as shown below, the optimal policy can always be described as a two-point policy. As part of the proof, we use the *convex envelope* of the $Q(u)$ function.

Suppose we form a concave function $\overline{Q}\left(u\right)$ from the original $Q\left(u\right)$ by replacing all its non-concave segments by corresponding linear segments as illustrated in the graph below. For illustrative purposes, the figure includes empty u-segments, $[0,u']$ and non-concave $Q(u)$

segments $[u'', u''']$.



Let $\overline{Q}(u)$ denote the obtained concave function from the original function $Q(u)$.

Let $V(D_0)$ denote the optimal value obtained from the following maximization problem

$$V(D_0) \equiv \max_{\{u_t\}} \int_0^\infty Q(u_t)\, e^{-\int_0^t (u_\tau + r) d\tau} dt \text{ subject to } \int_0^\infty e^{-\int_0^s (u_\tau + r) d\tau} ds = D_0.$$

Let $\overline{V}(D_0)$ denote the optimal value obtained from the following maximization problem

$$\overline{V}(D_0) \equiv \max_{\{u_t\}} \int_0^\infty \overline{Q}(u_t)\, e^{-\int_0^t (u_\tau + r) d\tau} dt \text{ subject to } \int_0^\infty e^{-\int_0^s (u_\tau + r) d\tau} ds = D_0.$$

By Theorem 2, the optimal solution to the second maximization problem is $u_t = u_0$ for all t.

It is clear that the optimal value $\overline{V}(D_0)$ achieved with the convex envelope $\overline{Q}(u)$ is weakly higher than the optimal value $V(D_0)$ achieved with original function $Q(u)$. The next theorem shows show that, in fact, $\overline{V}(D_0) = V(D_0)$.

Theorem 4 (Optimality) $\overline{V}(D_0) = V(D_0)$.

Proof. First note that $\overline{V}(D_0) = V(D_0)$ if $Q(u_0) = \overline{Q}(u_0)$, because $\overline{V}(D_0)$ is attained under $Q(u)$ by simply setting $u^* = u_0$. Now suppose $Q(u_0) \neq \overline{Q}(u_0)$. Then $\overline{Q}(u_0)$ must lie on a linear segment of $\overline{Q}(u)$. Let $\overline{Q}(u_1)$ and $\overline{Q}(u_2)$ denote the two end points of this linear segment, with $u_1 < u_0 < u_2$. Then $\overline{Q}(u_1) = Q(u_1)$ and $\overline{Q}(u_2) = Q(u_2)$. Geometrically, we must have

$$\overline{Q}(u_0) = Q(u_1) + \frac{Q(u_2) - Q(u_1)}{u_2 - u_1}(u_0 - u_1).$$

Therefore,

$$\overline{V}(D_0) = D_0 \left[Q(u_1) + \frac{Q(u_2) - Q(u_1)}{u_2 - u_1}(u_0 - u_1) \right].$$

Now consider a (u_1, u_2) policy under $Q(u)$. The transition time t is such that the promise-keeping constraint is satisfied:

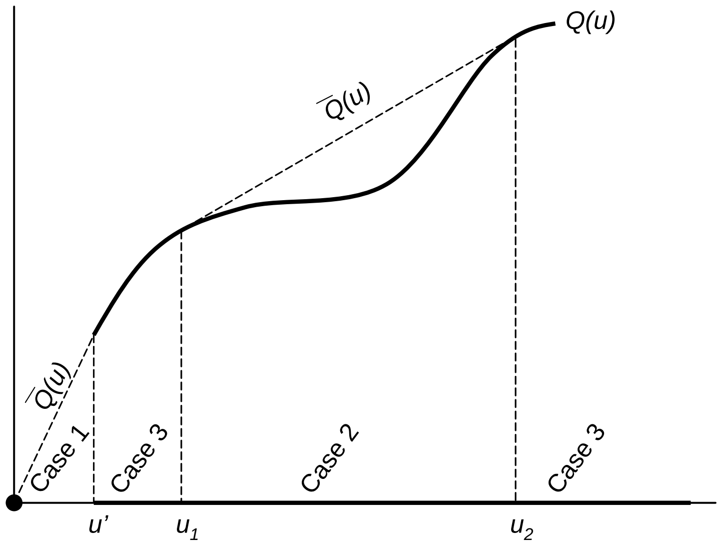
$$\begin{aligned} D_0 &= \left(1 - e^{-(u_1+r)t}\right) \frac{1}{u_1 + r} + e^{-(u_1+r)t} \frac{1}{u_2 + r} \\ \Longrightarrow \quad e^{-(u_1+r)t} &= \frac{\frac{1}{u_1+r} - D_0}{\frac{1}{u_1+r} - \frac{1}{u_2+r}}. \end{aligned}$$

Then

$$\begin{aligned} V(D_0) &= \int_0^t Q(u_1) e^{-[u_1+r]s} ds + e^{-[u_1+r]t} \int_t^\infty Q(u_2) e^{-(u_2+r)(s-t)} ds \\ &= Q(u_1)\left(1 - e^{-(u_1+r)t}\right) \frac{1}{u_1+r} + Q(u_2) e^{-(u_1+r)t} \frac{1}{u_2+r} \\ &= Q(u_1)\left[D_0 - e^{-(u_1+r)t} \frac{1}{u_2+r}\right] + Q(u_2) e^{-(u_1+r)t} \frac{1}{u_2+r} \\ &= Q(u_1) D_0 - \left[Q(u_1) - Q(u_2)\right] \frac{\frac{1}{u_1+r} - D_0}{\frac{1}{u_1+r} - \frac{1}{u_2+r}} \frac{1}{u_2+r} \\ &= D_0 \left[Q(u_1) + \frac{Q(u_2) - Q(u_1)}{u_2 - u_1} (u_0 - u_1)\right] \end{aligned}$$

Therefore, $V(D_0) = \overline{V}(D_0)$. ■

As a result, the optimal patent policy under original $Q(u)$ can be stated as follows. (1) If $u_0 \geq u_*$, then the optimal policy is stationary with arrival rate u_*, and the promise-keeping constraint is redundant. (2) If $u_0 < u_*$ and $\overline{Q}(u_0) = Q(u_0)$, then the optimal policy is again stationary, but the arrival rate is u_0 and the promise-keeping constraint is binding. (3) If $u_0 < u_*$ and $\overline{Q}(u_0) > Q(u_0)$, then the optimal policy is a two point policy (u_1, u_2, τ) where $\overline{Q}(u_1)$ and $\overline{Q}(u_2)$ are the end points of the linear segment that $\overline{Q}(u_0)$ lies on, with $u_1 < u_0 < u_2$. The transition time τ is chosen such that the promise keeping constraint is satisfied. The following graph summarizes the above three possible cases.



Remark 1 (Neutrality Principle) *When u_0 lies in the linear segment of the convex envelope, the optimal policy is a two-point patent policy. For example in case 2 of the above figure, the policy will use u_1, u_2 and a switching time t to replicate the value of $\overline{Q}(u_0)$. There are two ways to implement these two-point policy. The planner could implement u_1 first and then switch to u_2 at some time t_1. Alternatively, the planner could implement u_2 first and then switch to u_1 at some time t_2. It turns out that these two ways of implementation are equivalent in terms of social welfare and the expected players' payoffs. We call this fact as Neutrality Principle.*

Remark 2 (Incentives for Research and Development) *The incentive for the incumbent to invest in development activities is always socially insufficient, because the social gain from quality improvement is always higher than the individual gain from quality improvement. In contrast, the incentive for the challenger firm to invest in research activities to generate ideas is not necessarily socially insufficient due to the negative externality of research activities imposes on the promised patent protection.*

4 An Illustrating Example

This section solves a simple specification of our model to illustrate the optimal two-point policy. First, we assume that ideas are homogeneous in quality, although we will use the variable z to index ideas. Therefore, the cost of development $c(\triangle)$ will be the function of quality improvement $\triangle$ only, which is assumed to be strictly increasing and convex. Second, we assume that the challenger firm's research decision is binary: either invest or not invest.

If the firm invests, new ideas arrive with a given arrival rate $\overline{\lambda}$; if the firm does not invest, no ideas arrive. The cost of the investment in research is K.

Since ideas are homogenous, all ideas will be rewarded with the same duration D_0. The static social value of an innovation is independent of z:

$$R\left(D_0\right) = \frac{1}{r} \triangle\left(D_0\right) - c\left(\triangle\left(D_0\right)\right),$$

where

$$\triangle\left(D_0\right) = \arg\max_{\triangle}\left[\triangle \cdot D_0 - c\left(\triangle\right)\right].$$

The first-order condition for $\triangle\left(D_0\right)$ is given by $D_0 - c'\left(\triangle\left(D_0\right)\right) = 0$. Let use $\pi\left(D_0\right)$ denote a firm's expected profit given protection D_0:

$$\pi\left(D_0\right) = \triangle\left(D_0\right) \cdot D_0 - c\left(\triangle\left(D_0\right)\right).$$

Recall that the variable z is used as an index for new ideas, so we can still use cutoffs $\{\widehat{z}_t\}$ to select patentable ideas in all future period t. The term $(1 - \Phi(\widehat{z}_t))$ will be the probability that a new idea is patentable at time t. The arrival rate of patentable ideas u_t will be defined as before:

$$u_t = \lambda_t\left(1 - \Phi(\widehat{z}_t)\right) = \begin{cases} \overline{\lambda}\left(1 - \Phi(\widehat{z}_t)\right) & \text{if} \quad \text{firm invests } K \text{ at time } t \\ 0 & \text{if} \quad \text{firm does not invest } K \text{ at time } t \end{cases}.$$

The firm invests in research at time t if $u_t \geq K/\pi\left(D_0\right)$. Note that $u_t \in (0, K/\pi)$ is not feasible, because if $u_t \in (0, K/\pi)$ the firm will not invest and thus $u_t = 0$.

Note that the $Q\left(u\right)$ function in this setting reduces to

$$\begin{aligned} Q\left(u\right) &= \lambda\left(u\right) \int_{\widehat{z}(u)}^{\infty} \left[R\left(D_0\right) + V\left(D_0\right)\right] \phi\left(\xi\right) d\xi - K\left(\lambda\left(u\right)\right) \\ &= u\left[R\left(D_0\right) + V\left(D_0\right)\right] - K\left(\lambda\left(u\right)\right) \\ &= \begin{cases} \overline{\lambda}\left(1 - \Phi(\widehat{z})\right)\left[R\left(D_0\right) + V\left(D_0\right)\right] - K & \text{if} \quad u \geq K/\pi \\ 0 & \text{if} \quad u = 0 \end{cases} \end{aligned}$$

Therefore,

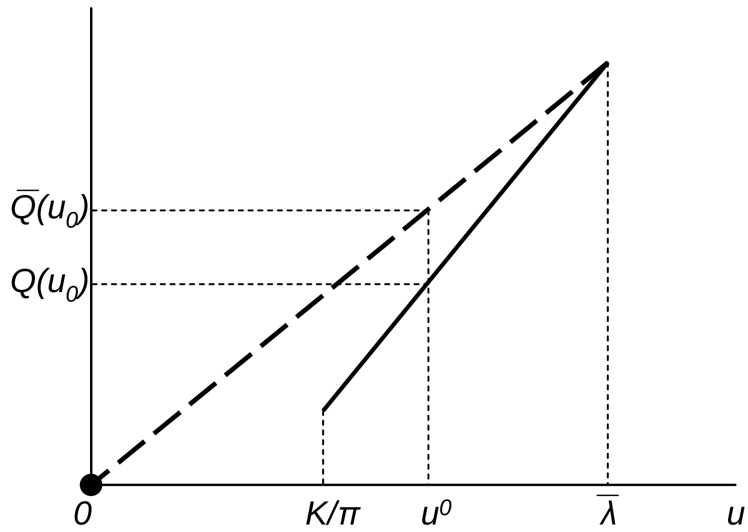
$$\frac{Q(u)}{u + r} = \begin{cases} \frac{u\left[R(D_0) + V(D_0)\right] - K}{u + r} & \text{if} \quad u \geq K/\pi \\ 0 & \text{if} \quad u = 0 \end{cases}$$

Note that when $u \geq K/\pi$, $Q(u)/\left(u + r\right)$ is strictly increasing in u everywhere. Therefore, by definition $u_* = \arg\max Q(u)/\left(u + r\right)$, we have $u_* = \overline{\lambda}$.

It is clear that $Q\left(u\right)$ is not concave in u, but we can easily construct its convex envelope as

$$\overline{Q}(u) = u\left[R\left(D_0\right) + V\left(D_0\right) - \frac{K}{\overline{\lambda}}\right].$$

The $Q(u)$ function and the convex envelope $\overline{Q}(u)$ are illustrated in the following figure.



The $Q(u)$ function consists of the origin and the solid line starts from $u = K/\pi$. The upper dash line represents the convex envelope $\overline{Q}(u)$.

The optimal policy with respect to the convex envelope $\overline{Q}(u)$ is stationary with cutoff u_0. However, since $\overline{Q}(u_0) > Q(u_0)$, the optimal policy with respect to the function $Q(u)$ is a two-point policy: two cutoffs $(\widehat{z}_1 = \infty, \widehat{z}_2 = 0)$ and a switching time $\overline{t}$. This implies that $u_1 = 0$ and $u_2 = \overline{\lambda}$, and the switching time $\overline{t}$ is implicitly determined by the reward constraint:

$$D_0 = \frac{1}{u_0 + r} = \left(1 - e^{-r\overline{t}}\right)\frac{1}{r} + e^{-r\overline{t}}\frac{1}{\overline{\lambda} + r}.$$

Remark 3 *Alternatively, the optimal two-point policy could set* $(\widehat{z}_1 = 0, \widehat{z}_2 = \infty)$ *and a switching time* $\overline{t}'$*, where* $\overline{t}'$ *is chosen to maintain the reward constraint. According to neutrality principle, these two policies are welfare equivalent.*

However, if the firm, in addition to the flow cost K*, has to pay a fixed start-up cost* K_0 *at the point in time it starts to invest in research. Let* V_{01} *denote the value under policy* $(\infty, 0, \overline{t})$ *and let* V_{10} *denote the value under policy* $\left(0, \infty, \overline{t}'\right)$*. Then we have*

$$\begin{aligned} V_{01} &= [R + V_{01} - K]\frac{u_0}{u_0 + r} - e^{-r\overline{t}}K_0 \\ &= [R + V_{01} - K]\frac{u_0}{u_0 + r} - \frac{u_0}{u_0 + r}(1 + r)K_0 \end{aligned}$$

That is

$$V_{01} = \frac{u_0\left[R - K - (1 + r)K_0\right]}{r}$$

Similarly, under policy $\left(0, \infty, \overline{t}'\right)$

$$V_{10} = [R + V_{10} - K]\frac{u_0}{u_0 + r} - K_0$$

That is

$$V_{10}=\frac{u_0\left[R-K\right]-(1+r)K_0}{r}$$

Therefore, the policy $\left(\infty,0,\overline{t}\right)$ *dominates policy* $\left(0,\infty,\overline{t}'\right)$ *because*

$$V_{01}-V_{10}=\left(1-u_0\right)\frac{(1+r)K_0}{r}>0.$$

According to Theorem 4, the optimal two-point policy achieves $\overline{V}\left(D_0\right)$. Therefore,

$$\begin{aligned} V(D_0) &= \overline{V}(D_0)=\frac{\overline{Q}\left(u_0\right)}{u_0+r} \\ &= \frac{u_0}{u_0+r}\left[R(D_0)+V(D_0)-\frac{K}{\overline{\lambda}}\right] \\ &= \left(1-rD_0\right)\left[R(D_0)+V(D_0)-\frac{K}{\overline{\lambda}}\right] \end{aligned}$$

We can solve $V\left(D_0\right)$ as

$$V(D_0)=\left(\frac{1}{rD_0}-1\right)\left[R(D_0)-\frac{K}{\overline{\lambda}}\right].$$

Next we can evaluate the welfare loss when the social planner is restricted to choose stationary policies. The optimal stationary policy is to implement $u_0=\left(1/D_0-r\right)$. To make the comparison interesting, we assume that the challenger firm will invest K under cutoffs $\widehat{z}_t=\widehat{z}$ such that $u_0=\overline{\lambda}\left[1-\Phi\left(\widehat{z}\right)\right]$. The social welfare under the stationary policy u_0 is

$$\begin{aligned} V^S(D_0) &= \int_0^{\infty}\left\{u_0\left[R\left(D_0\right)+V^S\left(D_0\right)\right]-K\right\}e^{-(u_0+r)s}ds \\ &= \frac{u_0\left[R\left(D_0\right)+V^S\left(D_0\right)\right]-K}{u_0+r} \\ &= \left(1-rD_0\right)\left[R\left(D_0\right)+V^S\left(D_0\right)\right]-D_0K \end{aligned}$$

Therefore, we can solve

$$V^S(D_0)=\left(\frac{1}{rD_0}-1\right)R\left(D_0\right)-\frac{K}{r}$$

Therefore, the welfare loss is given by

$$V(D_0)-V^S(D_0)=\frac{\left(\overline{\lambda}+r\right)D_0-1}{\overline{\lambda}rD_0}K.$$

Note that if $D_0\leq 1/\left(\overline{\lambda}+r\right)$, then the reward constraint is not binding, so the optimal patent policy is stationary. Therefore, it is no loss for the social planner to restrict to the stationary policy. If the reward constraint is binding, $D_0 > 1/\left(\overline{\lambda}+r\right)$, then there is always a loss by restricting to the stationary policy. Moreover, the size of the loss is larger when K is larger or the reward constraint is tighter in the sense that $D_0 - 1/\left(\overline{\lambda}+r\right)$ is larger.

So far we assume that D_0 is exogenously fixed. We now consider the optimal D_0^* which is defined as

$$D_0^* \in \arg\max_{D_0} \left\{ V(D_0) = V^S(D_0) + \frac{\left(\overline{\lambda}+r\right) D_0 - 1}{\overline{\lambda} r D_0} K \right\}.$$

If the planner is restricted to adopt stationary policy, the optimal D_0^S will solve the following maximization problem:

$$D_0^S \in \arg\max_{D_0} V^S(D_0).$$

From the theorem of maximum, it is clear that the solutions to both maximization problem exist. Note that the extra term in $V(D_0)$ is increasing in D_0, which implies that the maximum of $V(D_0)$ lies to the right of the maximum of $V^S(D_0)$. That is, $D_0^* > D_0^S$.

The intuition is the following. Consider the impact on social welfare of a marginal increase in duration D_0. For both stationary policy and two-point optimal policy, a higher D_0 increases a firm's incentive to invest in development, but it also leads to a longer expected waiting time for the next innovation. However its impact on *average research cost per unit of discounted time* differs between the two policy regimes. Under the optimal policy, increasing D_0 means that the period of no research is extended in time. Under the stationary policy, the research investment flow is the same as before, the higher reward is realized by an increase in the cutoff. Hence, for every unit increase in duration, relative to the stationary policy, the optimal policy saves the flow of research cost K for the entire extended period.

5 Extensions and Future Research

We can incorporate entry into this model. We can also embed this model to a general equilibrium model of industry dynamics. To be added.

6 Conclusions

To be added

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7 Appendix

7.1 Proof of the Optimality of the Cutoff Policy (sketch)

Consider the following deviation policy. Immidatly after the present patent is provided, the planner deviates by introducing a cut off for the next idea. This is done as follows: By construction the probability distribution of promises are equal under the orginal policy and the deviation policy. Since the idea distribution under the deviation policy first order dominates the deviation policy under the original policy, the deviation policy can be constructed such that any d value is mapped with a weakly better idea under the deviation policy than under the original policy.

The cut of set such that $u_t^0 = u_t^1$ at each point in time, where 0 refers to the original policy, and 1 to the deviation policy.

Since the idea distribution is replicated it is convenient to integrate over d instead of z. First assume that d is monotone. Let $F(d)$ be the probability distribution, and let $p(z)$ be the probability that an idea of quality z is patentable. Since the deviation policy maps a weakly better idea to each d, it follows that $z_1(d) \geq z_0(d)$.

The firm maximizes

$$\lambda \int_0^\infty \Pi\left(d, z(d)\right) dF(d) - K(\lambda)$$

with respect to λ. Since $z_1 d) \geq z_0(d)$ we have that $\Pi\left(d, z_1(d)\right) \geq \Pi\left(d, z_0(d)\right)$, thus $\lambda_1 \geq \lambda_0$.

Then we have

$$\begin{aligned}
& \int_0^\infty \left\{ \lambda_0 \int_0^\infty \left[R\left(d(\xi),\xi\right) + V\left(d\left(\xi\right)\right)\right] p(\xi)\phi\left(\xi\right) d\xi - K\left(\lambda_0\right) \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt \\
= & \int_0^\infty \left\{ \lambda_0 \int_0^\infty \left[R\left(d, z_0(d)\right) + V\left(d\right)\right] dF(d) - K\left(\lambda_0\right) \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt \\
= & \int_0^\infty \left\{ \begin{array}{c} \lambda_0 \int_{\widehat{z}_t}^\infty \left[R\left(d, z_0(d)\right) + V\left(d\right) - \Pi\left(d, z_0(d)\right)\right] dF(d) \\ +\lambda_0 \int_0^\infty \Pi\left(d, z_0(d)\right) dF(d) - K\left(\lambda_0\right) \end{array} \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt \\
= & \int_0^\infty \left\{ \begin{array}{c} \lambda_0 \int_0^\infty \left[\left(\frac{1}{r} - d\right) \Delta(d, z_0(d)) - C(\Delta(d, z_0(d)), z_0(d)) + V\left(d\right)\right] dF(d) \\ +\lambda_0 \int_0^\infty \Pi\left(d, z_0(d)\right) dF(d) - K(\lambda_0) \end{array} \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt \\
\leq & \int_0^\infty \left\{ \begin{array}{c} \lambda_1 \int_0^\infty \left[\left(\frac{1}{r} - d\right) \Delta(d, z_1(d)) - C(\Delta(d, z_1(d)), z_1(d)) + V\left(d\right)\right] dF(d) \\ +\lambda_0 \int_0^\infty \Pi\left(d, z_1(d)\right) dF(d) - K(\lambda_0) \end{array} \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt \\
\leq & \int_0^\infty \left\{ \begin{array}{c} \lambda_1 \int_0^\infty \left[\left(\frac{1}{r} - d\right) \Delta(d, z_1(d)) - C(\Delta(d, z_1(d)), z_1(d)) + V\left(d\right)\right] dF(d) \\ +\lambda_1 \int_0^\infty \Pi\left(d, z_1(d)\right) dF(d) - K(\lambda_1) \end{array} \right\} e^{-\int_0^t [u_\tau + r] d\tau} dt
\end{aligned}$$

The first inequality follows from the first order dominance $z_1(d) \geq z_0(d)$, which also explains $\lambda_1 \geq \lambda_0$. The second inequality follows from the definition of λ_1 as argmax. If the initial policy is not a cut off policy, we have $z_1(d) > z_0(d)$ for some d, thus the weak inequalities can be replaced by strict.

If $d(z)$ is not monotone, the same argument applies, but $F(d)$ must be interpreted differently as follows: d must be ordered not according to value, but according to the z variable - it can be done correcly, but very messy. If we can prove that $d(z)$ is monotone, we avoid this.

7.2 Proof of Monotonicity of $d(z)$ (sketch)

(1) The proof consists of two steps. We first show that $d\left(z\right)$ is weakly increasing in z , and then show that the monotonicity is strict.

Step One. Assume on the contrary that $d\left(z\right)$ is strictly decreasing in z over some interval. Consider two ideas in this interval with $z_0 \leq z_L < z_H$. Then $d\left(z_L\right) \geq d\left(z_H\right)$. Define

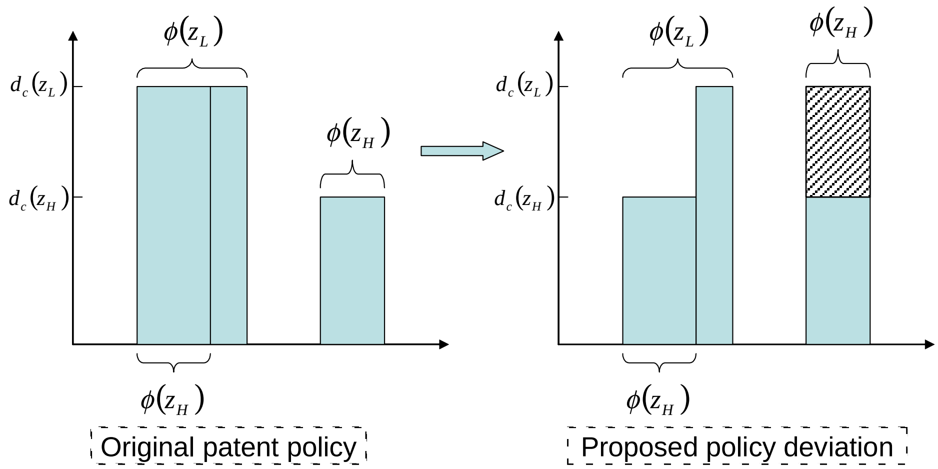
$$p_H = \phi(z_L)/\max[\phi(z_H), \phi(z_L)],$$

and

$$p_L = \phi(z_H)/\max[\phi(z_H), \phi(z_L)].$$

Consider the following policy deviation: if z_H arrives, with probability p_H provide it with protection $d\left(z_L\right)$ and with probability $(1-p_H)$ provide it with protection $d\left(z_H\right)$; if z_L arrives, with probability p_L provide it with protection $d\left(z_H\right)$ and with probability $(1-p_L)$

provide it with protection $d(z_L)$. Since the deviation occurs only if z_L or z_H arrives we can ignore all other z-values. The follow graph illustrates the effect of the policy deviation on d_c when $\phi(z_L) > \phi(z_H)$ so that $p_H = 1$ and $p_L < 1$:



Observe that the above policy deviation simply switches probabilistically between the original property rights assignment with idea z_H and the assignment with idea z_L, in a way such that the distribution of the promised expected durations $d(z)$ is unchanged. Therefore, the deviation has no effect on continuation values.

Consider the effect on firm's reseach λ. The firm's first order condtion is

$$\int_{\widehat{z}_t}^{\infty} \pi\left(d(\xi), \xi\right) \phi\left(\xi\right) d\xi = K'(\lambda_t) \tag{5}$$

Now consider the net expected effect on the left hand side of (5):

$$\begin{aligned} & p_H \phi(z_H) \left[\pi(d(z_L), z_H) - \pi(d(z_H), z_H)\right] \\ & - p_L \phi(z_L) \left[\pi(d(z_L), z_L) - \pi(d(z_H), z_L)\right] \\ = \; & \frac{\phi(z_H)\phi(z_L)}{\max[\phi(z_H), \phi(z_L)]} \left\{ \begin{array}{c} \left[\pi(d(z_L), z_H) - \pi(d(z_H), z_H)\right] \\ - \left[\pi(d(z_L), z_L) - \pi(d(z_H), z_L)\right] \end{array} \right\} \\ = \; & -\frac{\phi(z_H)\phi(z_L)}{\max[\phi(z_H), \phi(z_L)]} \int_{z_L}^{z_H} \int_{z_L}^{z_H} \pi_{12}(d(s), \zeta) \frac{\partial d(s)}{\partial s} ds d\zeta. \end{aligned} \tag{6}$$

Note that $\frac{\partial d(s)}{\partial s}$ is non-positive by assumption, thus the above expression is positive if $\pi_{12} > 0$. But

$$\pi_{12}(d, z) = d\triangle_{12} - \triangle_1 c_{12}.$$

The first term is positive by the sorting assumption. The second term is positive since $c_{12} < 0$ and $\triangle_1 > 0$. Thus the deviation (cet.par) induces the firm to increase λ. To comply with initial promises, the cut off must increase. Consider the following construction: for any given increase in the cut off, the $d(z)$ function is adjusted such that the probability distribution for d is replicated. Since the cut off is increased, this implies that each given d is attached to a weakly better idea.

Finally, consider the impact on social surplus. From the previous proof we know that the increase in the cut off is welfare improving. Thus it remains to show that the intitial switch between z_L and z_H is beneficial. Replacing π with R in (6), the same argument applies (oberve by construction, the deviation has no impact on continuation values).

Therefore, under Assumption 1 and 2, the suggested policy deviation increases social welfare: a contradiction to the optimality of $d(z)$. Therefore, $d(z)$ is weakly increasing in z if $d(z) > 0$.

Step Two. We only need to show that $d(z)$ can not be constant over some interval if $d(z) > 0$. Assume on the contrary that $d(z)$ is constant over some interval. Consider two ideas in this interval with $z_0 \leq z_L < z_H$ with $d(z_L) = d(z_H) = \overline{d}$. Then

$$\overline{d} \in \arg\max_{d(z)} \{R(z, d(z)) - V(d(z))\}$$

where

$$R(z, d(z)) = \frac{1}{r} \triangle (d(z), z) - c(\triangle, z)$$

is the static welfare which depends on duration $d(z)$ and idea quality z, and $V(d(z))$ is the continuation value which depends on duration $d(z)$ only. Notice that a slight increase in d increases static social welfare $R(z, d)$ but decreases the continuation value $V(d)$. The necessary first-order conditions for the optimality of $\overline{d}$ imply that

$$0 = R_2\left(z_L, \overline{d}\right) - V'\left(\overline{d}\right) = R_2\left(z_H, \overline{d}\right) - V'\left(\overline{d}\right).$$

Therefore, $\triangle_1\left(\overline{d}, z_L\right) = \triangle_1\left(\overline{d}, z_H\right)$. But this contradicts our sorting condition – Assumption 2. Therefore, $d(z)$ is strictly increasing in z if $d(z) > 0$.