Aggregate Production Functions with Micro Foundations

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This paper presents a geometric derivation of an aggregate production function from simple Edgeworth exchange and production box diagrams. The production box is shown for two firms, each using labor and capital to produce one of the two goods produced in the economy. Mapping the Pareto optimal combinations of labor and capital into the output space yields a production possibilities curve and a continuum of Edgeworth exchange boxes (for a two-consumer economy). Solving the exchange optimization problem for given output prices yields a solution on the grand utility frontier. With appropriate choice of the numeraire good, the intercept of the iso-profit line tangent to the production possibilities frontier is GDP. As labor expands with capital held constant, combinations of labor and GDP trace out the short run aggregate production. This construct is useful in that it provides microeconomic and welfare foundations for the aggregate production function and aggregate labor demand curves often used in teaching macroeconomics. Animated graphics show how expanding Edgeworth production boxes and production possibilities frontiers yield increases in value added. The animations can be used to see subtle differences between changes in capital stock and changes in technology. The effects of changes in consumer preferences and output prices are also simulated. The model can also be used to explore the welfare effects of various macroeconomic disturbances. *JEL* classification: A10, A22, E23.

1. Introduction

Aggregate production functions have many uses in macroeconomics, including growth models, neoclassical aggregate supply curves and aggregate labor market models. Models employing aggregate production functions are popular in spite of the stringent aggregation conditions that must be satisfied to guarantee their existence (e.g., see Houthakker[9], Fisher[5, 6, 7, 8], Johansen[10],Levhari[12], Sato[18]). Jesus Felipe and Franklin Fisher[2] provide an excellent survey of the uses and misuses of aggregate production functions and conclude that none of the reasons for using aggregate production functions in applied work can be readily justified. A heated discussion of this "capital controversy" once ensued in the literature–see Ferguson[3, 4], Robinson[13, 14, 15, 16], Solow[20], Fisher[7, 8]– with Joan Robertson ultimately accusing C. E. Ferguson sophistry, believing in the impossible and setting up smoke screens; Robert Solow of believing in parables; and Franklin Fisher of sawing off the bough he was sitting on, but expecting to remain in the air all the same.

This paper diagrammatically develops a simple short run aggregate supply curve from the basic constructs of modern welfare economics. This is done without entering into the above-outlined controversy in the literature on aggregate production functions; once the construct is understood it can be used (or not used) for good or ill. This work is presented in the spirit of the Walrasian vision of microeconomics exemplified in the work of Francis Bator[1], Paul Samuelson[17] and Donald Katzner[11]. The ideas are clarified with animated graphics generated with *Mathematica*. A web page accompanies this paper at the url

`http://personal.stthomas.edu/csmarcott` .

The page will provide updated versions of the paper and graphics. The *Mathematica* code to produce the images is included and animated gifs that will run in any browser are available to those who are allergic to *Mathematica*.

2. A Simple Model of Short Run Aggregate Production with Attention to Welfare Analytics

Consider an economy with only two consumers and two goods produced with labor and capital in separate industries. The prices of the goods, p_1, p_2, are taken as given. The production of the goods is determined by production functions $f^1(\cdot)$ and $f^2(\cdot)$. If firm one uses labor and capital of (L_1, K_1) and firm two uses the input vector (L_2, K_2), aggregate value added or GDP is

$$GDP \equiv p_1 f^1(L_1, K_1) + p_2 f^2(L_2, K_2).$$

If the firms each maximize profits the value added of the economy will maximized subject to the total capital and labor constraints (see Silberberg and Suen[19], chapter 16). For the economy as a whole the constrained optimization problem is given by the following Lagrangian:

$$\mathcal{L}(L_1, L_2, K_1, K_2, \lambda_L, \lambda_K) = p_1 f^1(L_1, K_1) + p_2 f^2(L_2, K_2) + \lambda_L(L - L_1 - L_2) + \lambda_K(K - K_1 - K_2);$$

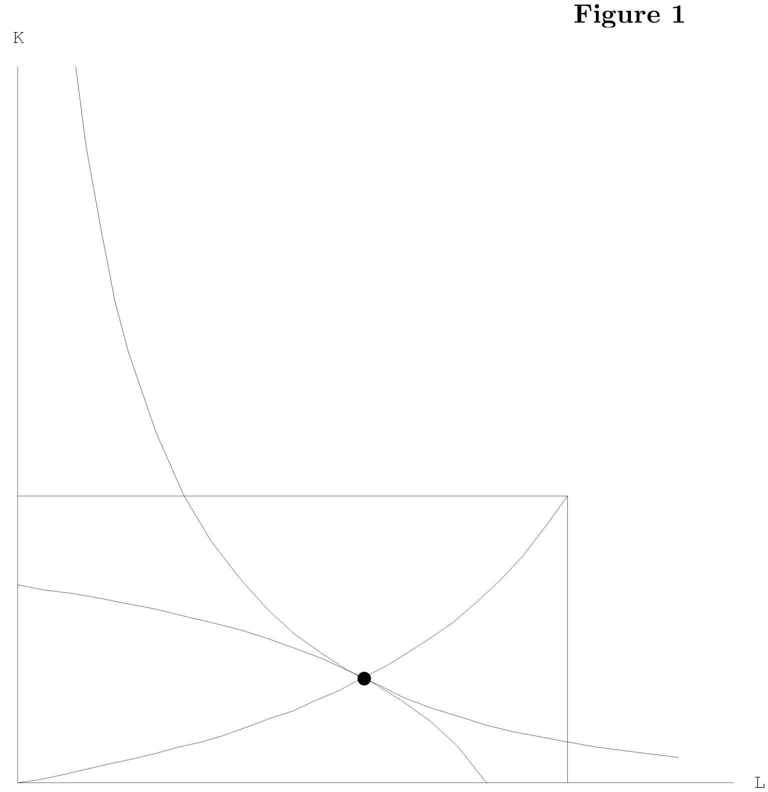
where L and K are the initial endowments of labor and capital available to the two industries.

After eliminating the Lagrange multipliers and output prices, the necessary conditions for this maximization problem can be expressed as

$$\frac{f_L^1}{f_K^1} = \frac{f_L^2}{f_K^2}$$
$$L = L_1 + L_2$$
$$K = K_1 + K_2.$$

The first of these is the familiar condition that the isoquants associated with the two industry production functions must be tangent. The last two conditions state that all available labor and output must be employed. Graphically the production equilibrium can be represented in the usual Edgeworth production box diagram shown in figure 1. The set of all Pareto optimal allocations of labor and capital is also shown.

Figure 1



In this setting there are no input prices, but the Lagrange multipliers are interpreted (in the usual way) as the shadow prices of the constraints. Under the necessary and sufficient conditions the implicit function theorem can be invoked to obtain continuous and differentiable (i.e., in some neighborhood around the solution to the first-order conditions) "choice functions": $L_1^*(p_1, p_2, L, K)$, $K_1^*(p_1, p_2, L, K)$, $L_2^*(p_1, p_2, L, K)$, $K_2^*(p_1, p_2, L, K)$, $\lambda_L^*(p_1, p_2, L, K)$ and $\lambda_K^*(p_1, p_2, L, K)$. When these choice functions are substituted into the first-order (i.e., necessary) conditions, the implicit function theorem insures that these conditions will hold as identities in some neighborhood around the solution to the first-order conditions. Substituting the four choice functions for labor and capital back into the objective function gives the indirect objective function:

$$\begin{aligned}\text{GDP}^*(p_1, p_2, L, K) \equiv\, & p_1 f^1(L_1^*(p_1, p_2, L, K), K_1^*(p_1, p_2, L, K)) \\ & + p_2 f^2(L_2^*(p_1, p_2, L, K), K_2^*(p_1, p_2, L, K)).\end{aligned}$$

By the envelope theorem

$$\frac{\partial \text{GDP}^*}{\partial L} = \mathcal{L}_L = \lambda_L^*(p_1, p_2, L, K)$$

and

$$\frac{\partial \text{GDP}^*}{\partial K} = \mathcal{L}_K = \lambda_K^*(p_1, p_2, L, K).$$

Thus, $\lambda_L^*(\cdot)$ is the additional GDP associated with a small increase in the labor endowment and $\lambda_K^*(\cdot)$ is the additional GDP associated with a small increase in the capital endowment.

The second-order (i.e., necessary and sufficient) conditions are that the bordered Hessian matrix

$$H = \begin{pmatrix} p_1 f_{LL}^1 & p_1 f_{LK}^1 & 0 & 0 & -1 & 0 \\ p_1 f_{KL}^1 & p_1 f_{KK}^1 & 0 & 0 & 0 & -1 \\ 0 & 0 & p_2 f_{LL}^2 & p_2 f_{LK}^2 & -1 & 0 \\ 0 & 0 & p_2 f_{KL}^2 & p_2 f_{KK}^2 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

is negative definite subject to constraint. The determinant test for this is that the naturally-ordered border-preserving principal minors of order three or greater alternate sign, beginning with negative; that is, $H_{44} < 0$ and $|H| > 0$. Since

$$H_{44} = p_2 f_{LL}^2 + p_1 f_{LL}^1$$

this condition will hold whenever both production functions exhibit diminishing marginal products of labor. Similarly, it must be the case that $p_1 f_{KK}^2 + p_2 f_{KK}^1 < 0$, which will hold for the case of diminishing marginal products of capital.

The second and last of these border-preserving principal minors is the determinant of the bordered Hessian. This determinant is evaluated as

$$|H| = p_1^2 \left(f_{LL}^1 f_{KK}^1 - (f_{LK}^1)^2 \right) + p_2^2 \left(f_{LL}^2 f_{KK}^2 - (f_{LK}^2)^2 \right) + p_1 p_2 \left(f_{LL}^1 f_{KK}^2 - 2 f_{LK}^1 f_{LK}^2 \right)$$

If the production functions are concave (i.e., $f_{LL}^i < 0$ and $f_{LL}^i f_{KK}^i - (f_{LK}^i)^2 > 0$, $i = 1, 2$) the first two terms of this determinant will be positive. This, however is not enough to insure that $|H| > 0$. If the mixed partial derivatives of the production functions are of opposite signs and of sufficient magnitude $|H|$ could be negative, even with concave production functions in both industries. This is the case of a convex production possibilities, and is sometimes missed by even the most punctilious authors.

In order to produce specific diagrams and animations the production functions are assumed to be CES:

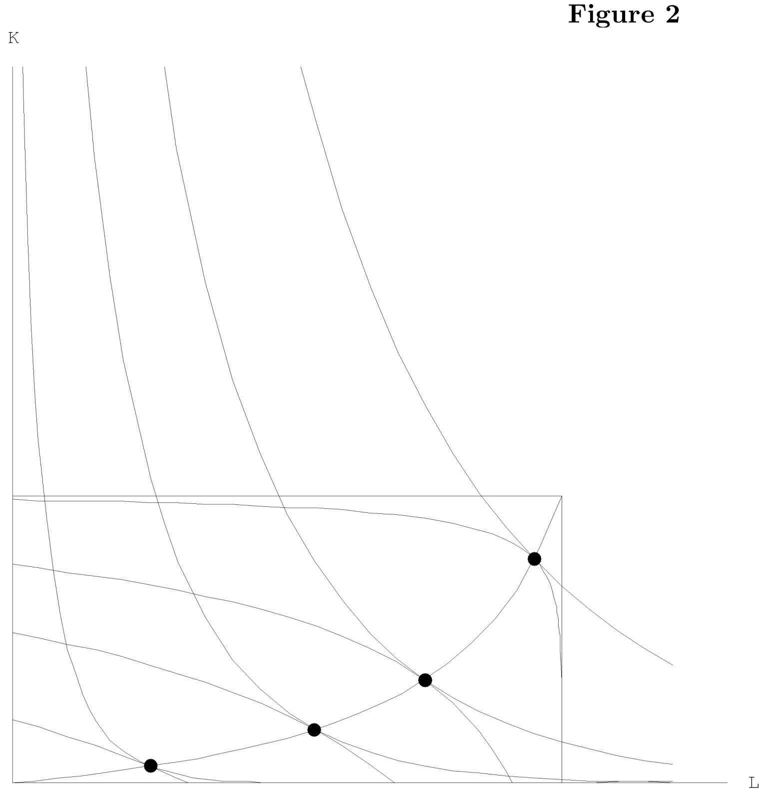
$$f^1(L_1, K_1) = (\alpha_1 L_1^\rho + (1 - \alpha_1) K_1^\rho)^{1/\rho}$$

and

$$f^2(L_2, K_2) = (\alpha_2 L_2^\rho + (1 - \alpha_2) K_2^\rho)^{1/\rho};$$

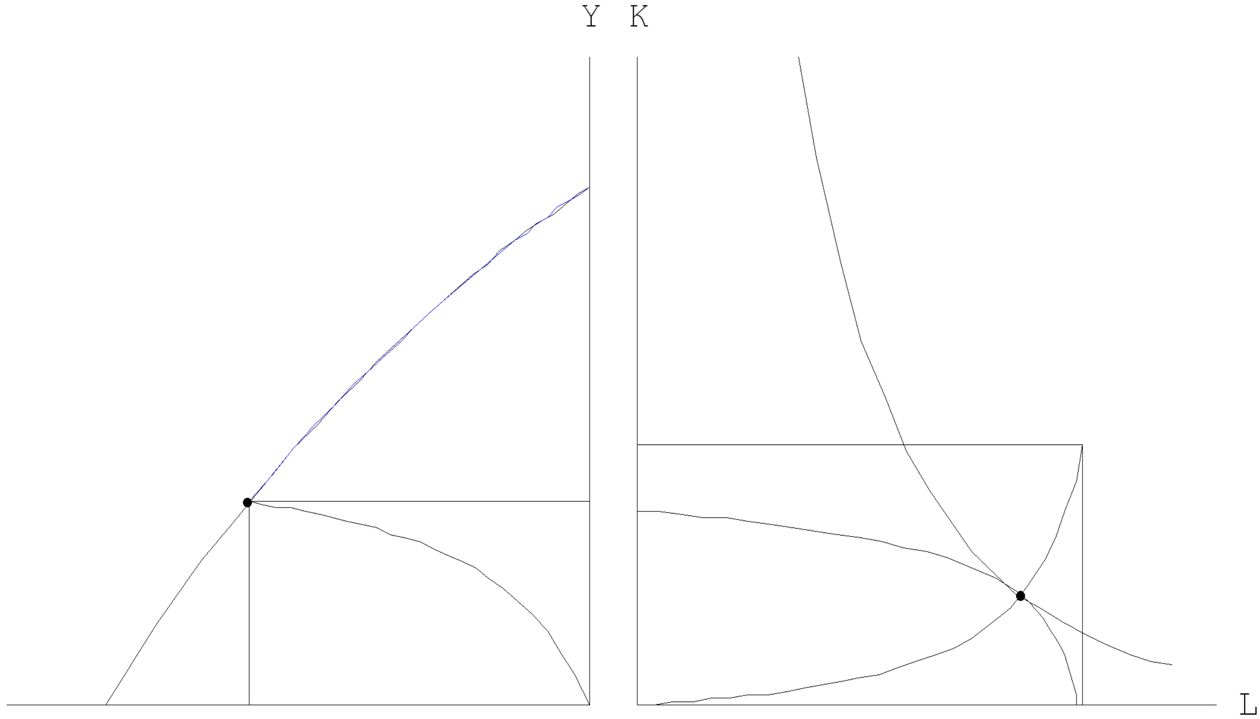
where $0 < \alpha_1 < 1$, $0 < \alpha_1 < 1$, and $-1 < \rho < \infty$. Letting ρ differ for the two firms requires the use of numerical techniques and creates other problems, so it is assumed that the two production functions share a common value of ρ. This assumption gives a closed-form solution for the set of Pareto optimal combinations of labor and capital. It should be noted that the CES function does not necessarily satisfy the second-order conditions (i.e., $|H| = 0$). For given α_1 and α_1, a common value of ρ the Pareto optimal allocations of ρ map into a convex production possibilities frontier for some range of ρ. The graphics for this paper have avoided such values of ρ. Figure 2 shows four different points on the set of Pareto optimal allocations. There is also an animated version of this graph available at the web page for this paper.

Figure 2



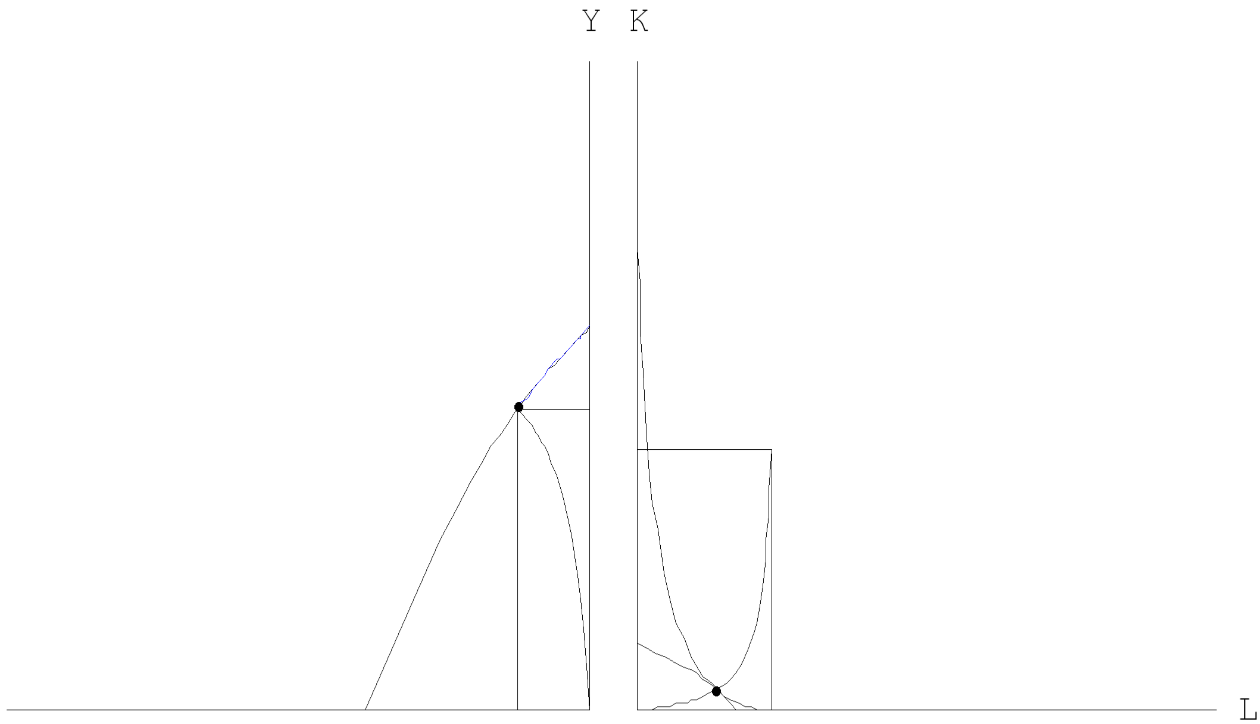
If the set of efficient production allocations of labor and capital are mapped back into the production functions, the production possibilities frontier is obtained. Figure three shows this for a single production optimum point. An Edgeworth exchange box is shown for this point on the production possibilities curve. The set of Pareto optimal exchange combinations is shown (without the distraction of the indifference curves at this point). Here CES utility functions for the two consumers (or types of consumers) are used. Just as in the production problem the two utility functions share a common value of the parameter ρ. This, again, allows for a closed-form characterization of the set of exchange optima. The reader is encouraged to view the animated version of figure 3 (either the animated gif version or the *Mathematica* animation itself). Each production optimum is associated with a different point on the production possibilities curve and a different exchange box in the left quadrant.

Figure 3



Now suppose the amount of the labor endowment increases. Assume, for now, that the additional labor is divided between the industries at a constant ratio equal to the initial division of labor. The Edgeworth exchange box begins to expand and the set of Pareto optimal input allocations sags. When the points on the new contract curves are mapped into the output space the production possibilities frontier expands. This also affects the exchange box and the set of Pareto optimal combinations of the two goods. The animated graphic, figure 4, shows all of the action. The static version of figure 4 shows the first and last frames of the animation.

Figure 4



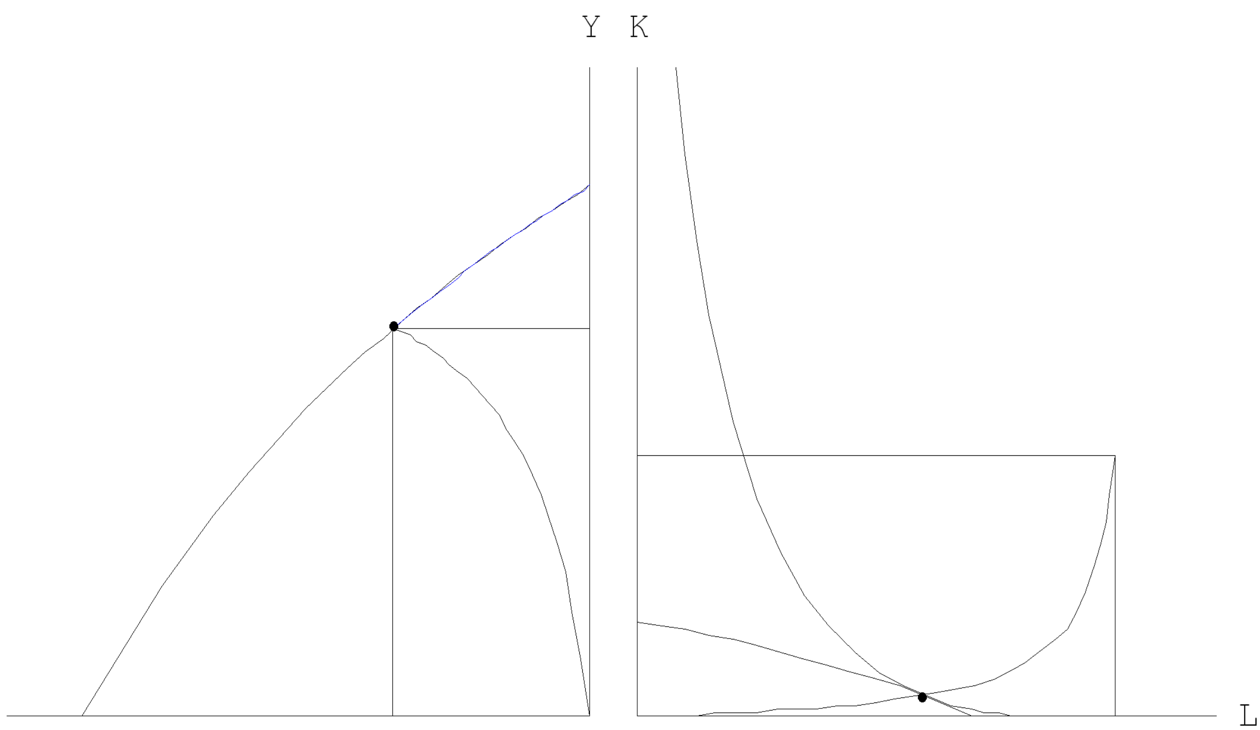


Figure 5

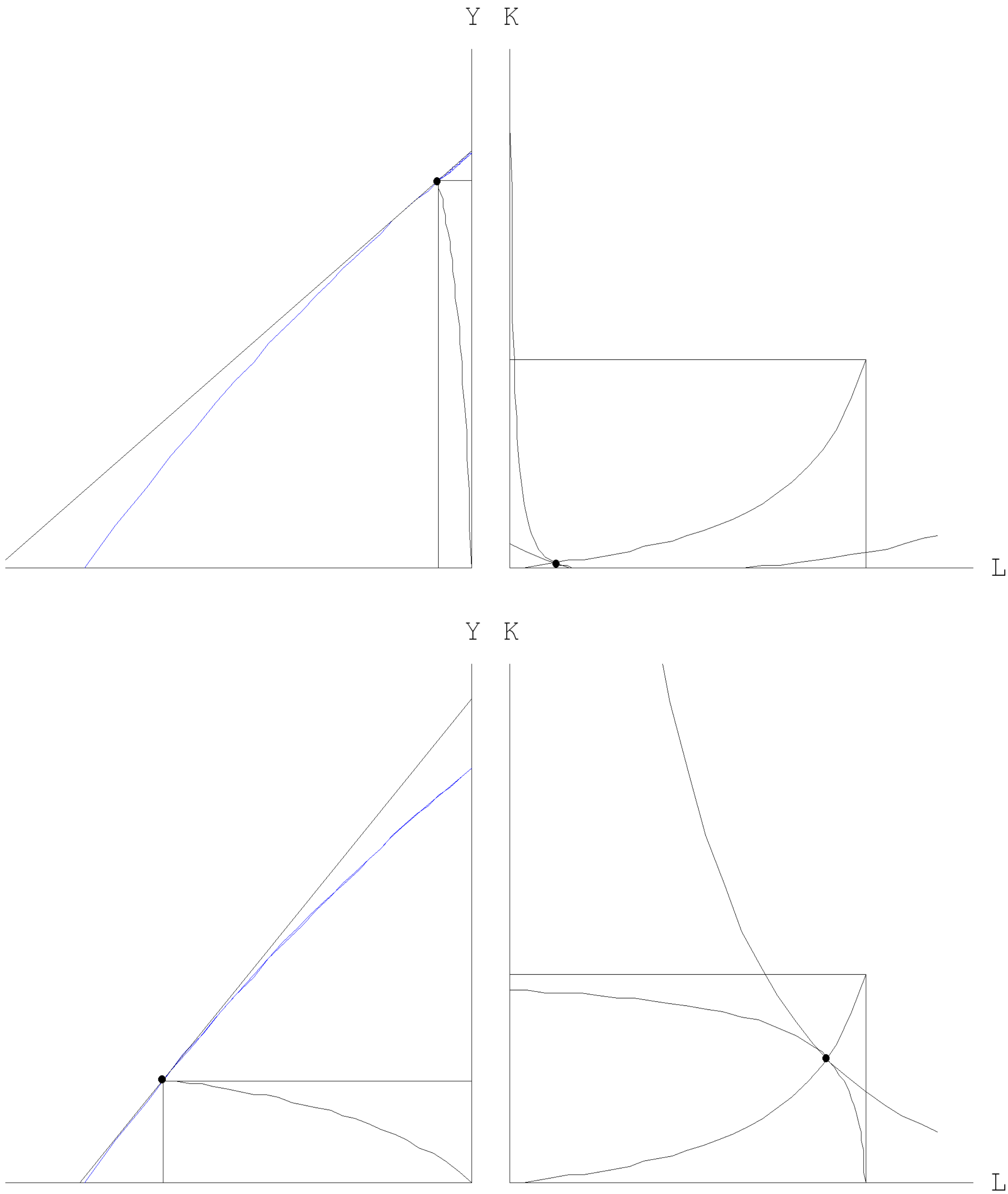
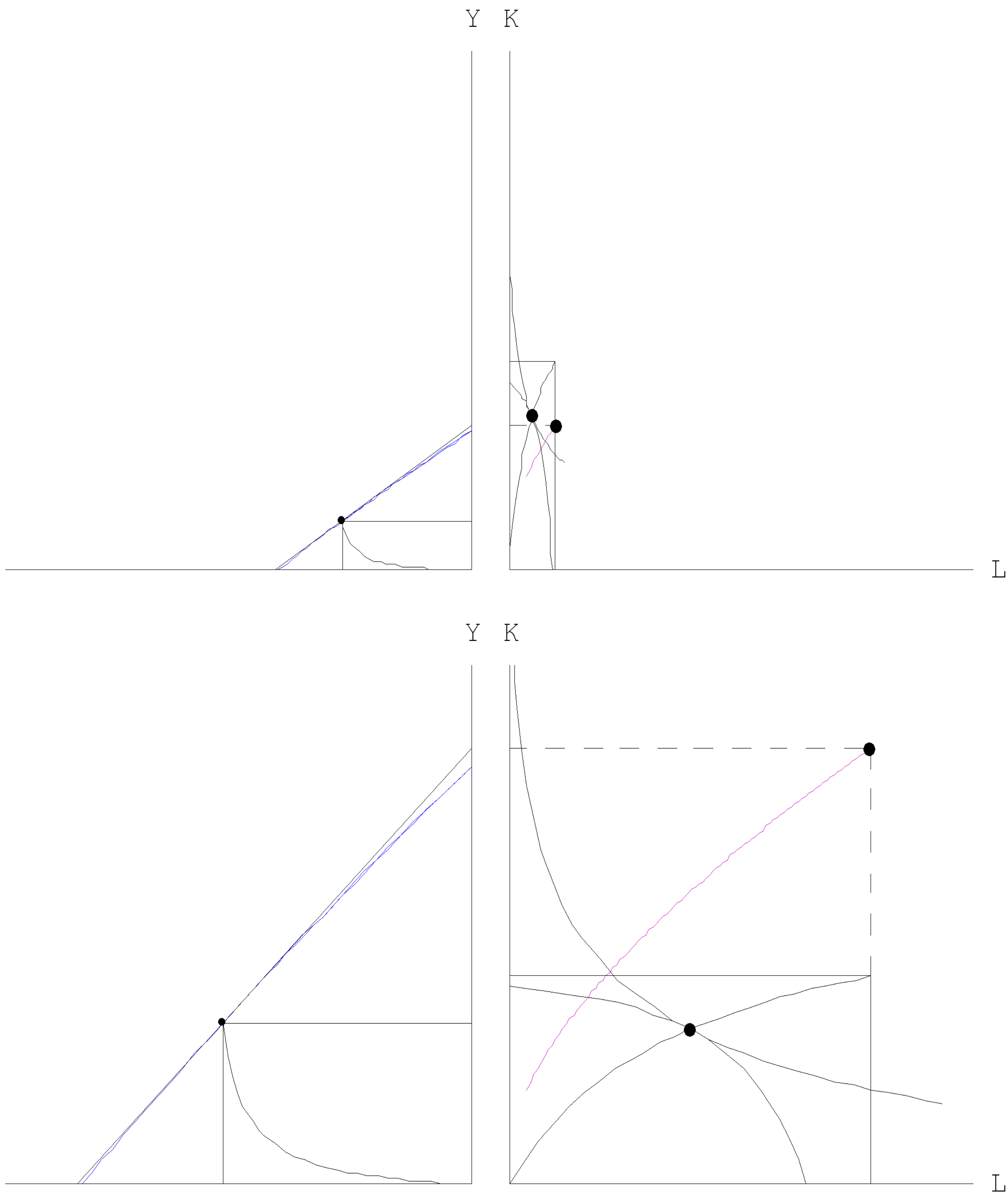


Figure 5 adds the iso-GDP line tangent to the production possibilities frontier. As the relative price of good X changes, the production possibilities frontier and the corresponding exchange boxes are generated. Note that movement along the production possibilities curve results in new exchange boxes, but no change in the production box. There is, of course, a change in the point on the production Pareto optimal points. This means the relative price of the inputs (i.e., the ratio of the Lagrange multipliers) is also changing. An animated version of figure 6 is also provided.

Figure 6



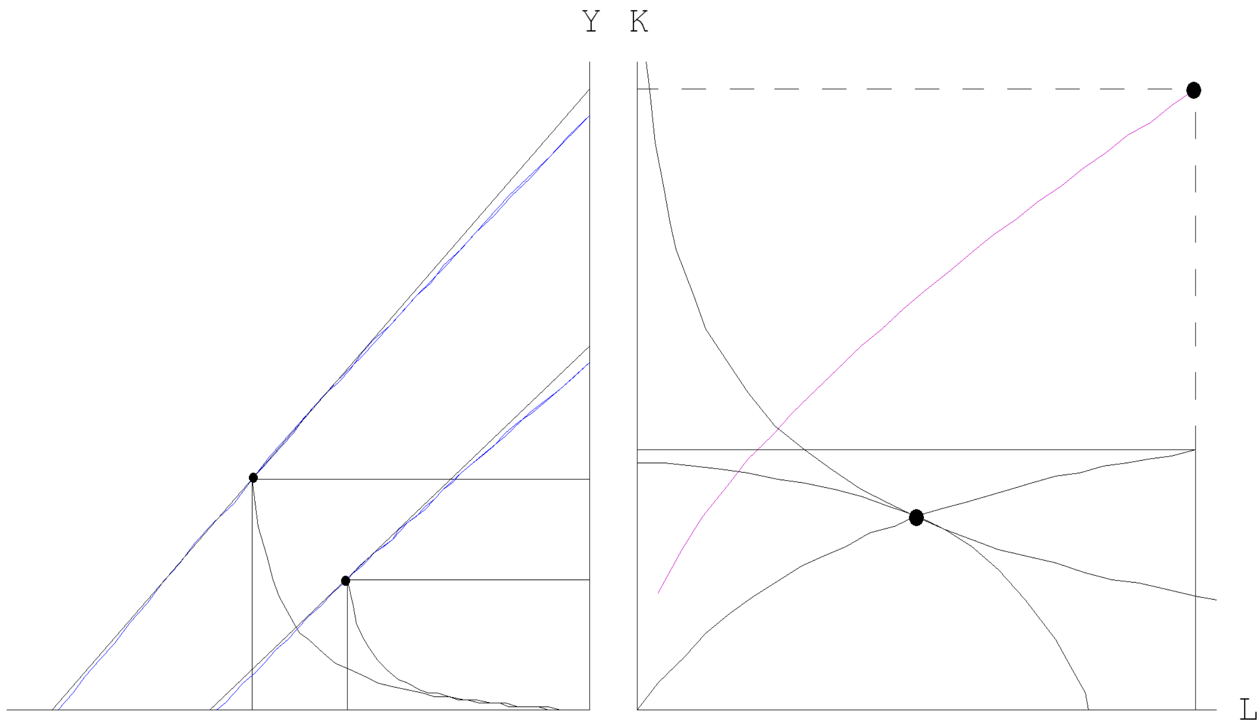
Assume that good Y is the numeraire (i.e., assume $p_y = 1$). The iso-GDP line associated with $\text{GDP} = \text{GDP}_0$ is now

$$\text{GDP}_0 = p_1 X + Y.$$

For $X = 0$ the vertical intercept of the iso-GDP line is GDP_0. This provides a simple way of geometrically deriving a short run production function. When the coordinate of intercept of the left quadrant is combined with the endowment of labor in the right quadrant, a combination of labor and GDP is obtained. Doing this for a set of labor endowments traces out a short run aggregate production function. Figure 6 show this derivation in static and animated forms.

Figure 6 provides one notion of what an aggregate production function might be. As additional labor is added to the economy figure 6 assumes the labor is distributed to the industries in a constant ratio. In order for this to happen the relative price of good X must change. It is not obvious that this is happening in figure 6. Figure 7 makes this change in price more apparent by showing one frame of the left quadrant that remains fixed throughout the animation.

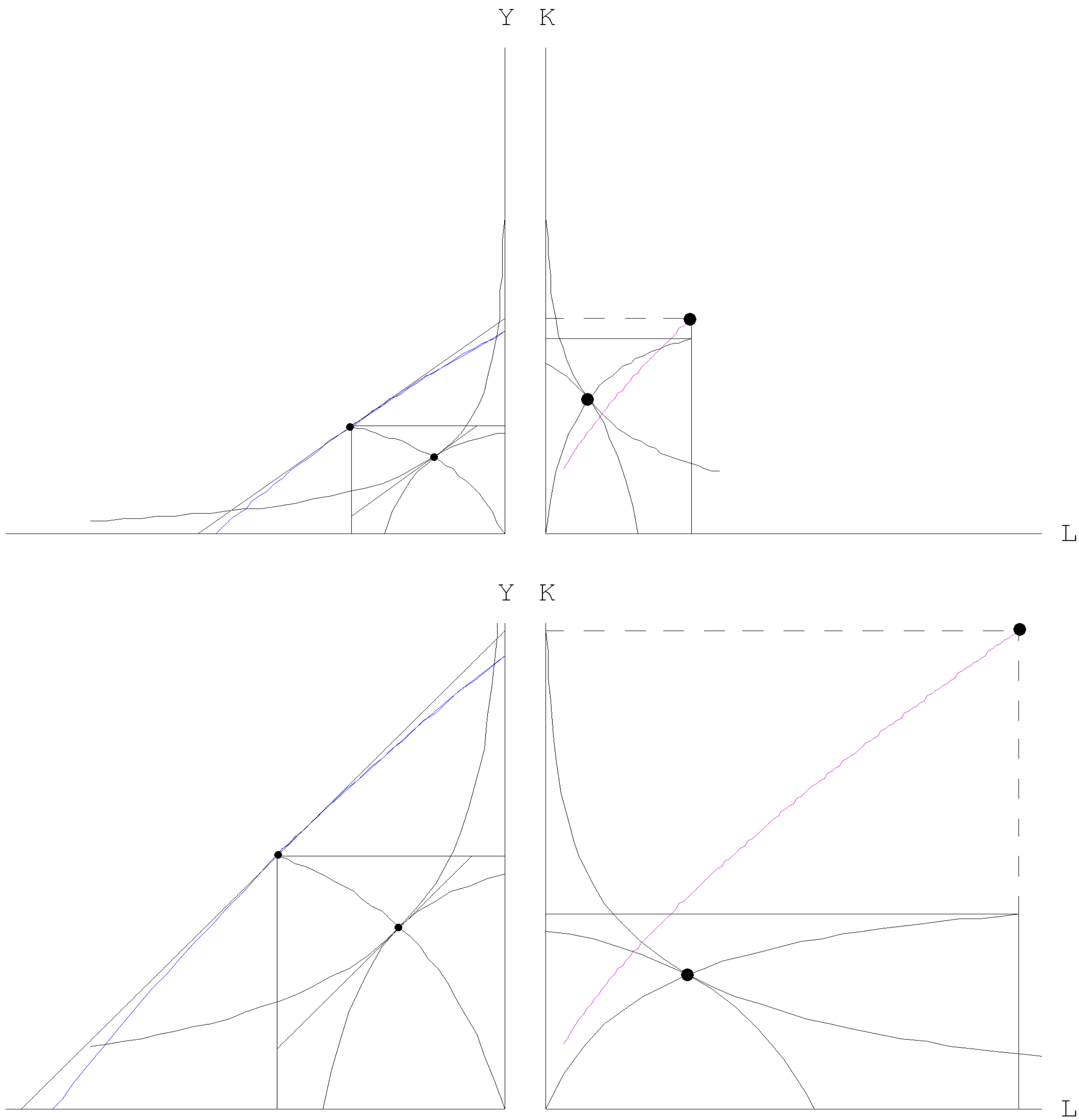
Figure 7



Welfare implications of the notion of aggregate supply can be obtained by taking this model one step further. Recall that each point on the set of production Pareto optimal allocations maps into a point on the production possibilities curve. Each point on the possibilities frontier, in turn, is associated with an Edgeworth exchange box, and each exchange box has a set of allocations of the total amounts of goods X and good Y that are Pareto optimal in exchange. When these Pareto optimal allocations of a given X and Y are mapped back into the utility functions of the two consumers the utility possibilities frontier is obtained. The utility possibilities curve is completely analogous to the production possibilities frontier, but there is a utility possibilities frontier associated with each point on the production possibilities curve. Assume that there is a social welfare function that satisfies the Pareto principle; that is, combinations of utilities that make one person better off without making the other person worse off are socially preferred to combinations of utilities that do not satisfy this property. Any social welfare function satisfying the Pareto principal will consider only allocations on the outer envelope of the utility possibilities frontiers. As the economy moves along a given production possibilities curve–presumably due to a change in the relative price of good X–the outer envelope of the utility possibilities curves is generated. This construct is called the grand utility frontier (see Bator[1]).

While it is straightforward to use similar animated graphics to obtain the utility possibilities frontier and the grand utilities frontier, there is a shortcut that avoids the necessity of this derivation. If the iso-GDP line tangent to the production possibilities curve is projected into the exchange box, the point on the set of Pareto optimal exchanges for which the indifference curves are tangent to this line will map into a point on the grand utility frontier (see Bator[1]). Figure 8 provides a version of figure 7 satisfying this additional property.

Figure 8



3. Changes in Capital Stock and Technology

Section two of this paper derived one notion of a short run aggregate production function. In this setting additional labor is distributed proportionally to the two industries. This necessitates changes in the relative price of good X and one may want to rule this out in alternative versions of aggregate supply. With the assumption of constant proportional distribution of the labor endowment, the effect of changing the endowment of capital can be simulated. Here it is really pointless to try to capture the dynamics with series of static images. The reader is encouraged to view the animated figure 8 to see how an increase in the endowment of capital affects the model. In short, increase in capital makes the Edgeworth production box become taller. As the Production possibilities frontier expands the production function makes the expected upward shift.

The effects of changes in technology on the aggregate production function can also be simulated. The animated figure 9 shows the effects of α_1 falling from 0.8 to 0.2, with $\alpha_2 = 0.6$ and $\rho = 0.25$. This means the industry producing good X (i.e., industry one) is moving to a more capital-intensive technology, while the technology in the industry producing good Y (i.e., industry two) has a constant technology. As the technology changes the set of Pareto optimal points in production shifts upward and the short run production function shifts upward. Part of this shift seems to be associated with an outward shift of the production possibilities frontier. As α_1 moves closer to 0.2 the assumption that the proportion of labor going to each industry remains constant has a marked effect. There is a movement down the production possibilities curve (toward good Y) which necessitates a big increase in the relative price of good X. This makes the iso-GDP line very steep and contributes greatly to the increase in output. This is clearly a case in which a model with constant output prices should also be considered. (The prices also changed when capital was increased, but the effect of the short run production function was much more subtle.)

The animated figure 10 shows the effects of α_2 falling from 0.8 to 0.2, with $\alpha_1 = 0.6$ and $\rho = 0.25$. The production function becomes flatter and intersects the original production function. The interpretation of this diagram is problematic.

4. Discussion

This paper has presented a method for considering aggregate production functions with micro foundations. These simulations of the model show promise for future work in this area. One point that certainly needs to be addressed is the assumption that increases in the labor endowment go to the two industries in constant proportions. This assumption means the relative price of the two goods changes as labor increases–or as other parameters in the model change. One alternative notion of aggregate production might insist that the relative prices of the two goods remain constant as labor increases. This involves is a relatively easy modification of the *Mathematica* code.

Joan Robinson's [15] fulminations over the sad state the aggregate production function include the following statement:

> *Each point on the pseudo-production function is intended to represent a possible position of equilibrium. Time, so to say, is at right angles to the blackboard on which the curve is drawn. At each point, an economy is conceived to be moving from the past into the future with the rate of profit and the technique of production shown at that point.*

One needs to be clear about what is changing and what is not when there is a movement along the short run production function. Certainly labor, output, the production possibilities frontier, the utilities possibilities frontier and the grand utility frontier are all changing at the same time. It seems reasonable also to assume capital, technology (i.e., α_1, α_2 and ρ), and preferences are all held constant. This leaves out the price of good X (since good Y is the numeraire), the prices of the inputs (represented by the two Lagrange multipliers) and the social welfare function.

Suppose one is willing to let the shadow prices of the inputs ($\lambda_L^*(\cdot)$ and $\lambda_K^*(\cdot)$) vary when the endowment of labor changes, but insists that any reasonable notion of a short run aggregate production function must hold the prices of output constant. Start with intial endowment of capital and labor and take as given the initial distribution of labor between the two industries. Solving the optimization problem gives a point on the set of Pareto optimal production points. This combination of inputs maps into a point on the production possibilities curve. Projecting the constant slope of the iso-GDP line into the Edgeworth production box yields a distribution of the produced goods between the two consumers. Mapping this allocation of the goods into the utility function gives a point on the grand utility frontier. Although this combination of utilities need not be the one that maximizes social welfare, suppose it is. Now allow labor to increase, holding the prices of the two goods constant and see what happens. The Edgeworth production box, the production possibilities curve, the utility possibilities frontier and the grand utility frontier all begin to shift outward. The point on the new grand utility frontier that the new iso-GDP line (with the same slope as the original iso-GDP line) picks out will, in general, not be a point that maximizes the social welfare function. The stubborn insistence that output price remain constant requires one to deviate from social welfare maximization, or worse, to be willing to accept a social welfare function whose parameters must change whenever the endowment of labor changes.

This entire *gedankenexperiment* can be run in reverse, and doing so is consistent with modern welfare economics. The constant social welfare function picks out a point on the grand utility frontier. This point maps into a distribution of utility and a point on the production possibilities curve. The point on the production possibilities curve yields an efficient distribution of the inputs between the two industries. Now let labor increase and see what happens. Just as in the previous thought experiment, the Edgeworth production box, the production possibilities curve, the utility possibilities frontier and the grand utility frontier all begin to shift outward. This time the constant social welfare function will pick out a new point on the grand utility frontier. The second theorem of welfare economics indicates that any Pareto optimal allocation can be supported by a competitive equilibrium upon appropriate redistribution of the initial endowment. When the initial endowments of the goods are redistributed the equilibrium prices of output will change. It seems that a constant social welfare function and constant output prices are at odds in this notion of aggregate supply.

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