Optimal Tax-Timing and Asset Allocation when Tax Rebates on Capital Losses are Limited

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Abstract

This article analyzes the optimal dynamic consumption portfolio problem in the presence of capital gains taxes. It explicitly takes limited capital loss deduction and the 3,000 dollar amount that can be offset against other income into account. It generalizes the classical result of Constantinides (1983) that it is optimal to realize capital losses immediately. Compared to tax-systems in which capital losses can only be offset against other income, the investment decision becomes substantially more difficult for two reasons. First, the investor has to make a decision on how to use a loss, i.e. whether to offset it against realized capital gains or to potentially postpone the realization of capital gains and offset it against other income. Second, in our setting it can be optimal to cut capital gains short which prevents investors from getting locked in and helps keeping portfolios diversified. The investor's wealth level has a substantial impact on the optimal investment strategy.

JEL Classification Codes: G11, H21, H24

Key Words: tax-timing, asset allocation, capital losses, tax loss carry-forward, limits on tax rebates, effective tax rate

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1 Introduction

According to the seminal work of Constantinides (1983), it is optimal to realize losses immediately and the tax realization strategy on an individual portfolio is separable from other aspects of portfolio choices under certain conditions. These include: (1) investors do not face any short-selling constraints, (2) wash-sales are permitted,[1] (3) long-term and short-term capital gains are taxed at the same capital gains tax-rate, and (4) there is no limit on tax rebates for incurred capital losses.

There is an extensive literature studying optimal portfolio decisions, relaxing some of these assumptions. Dybvig and Koo (1996) and DeMiguel and Uppal (2005) show that for short-selling constrained investors the optimal asset allocation decision depends on the tax basis of the asset in a complicated way. Dammon et al. (2001) show that for short-selling constrained investors the diversification benefit of reducing a volatile position can significantly outweigh the tax cost of selling an asset with an unrealized capital gain. The results of Dammon et al. (1989) suggest that the value of the option to realize long-term gains in order to regain the opportunity of realizing short-term losses is negatively related to the stocks price volatility.

Stiglitz (1983) suggests selling (or shorting, if necessary) highly correlated assets instead of realizing capital gains to circumvent wash sale rules. Gallmeyer et al. (2006) address this issue in a multi-asset setting.

If short-term capital gains are taxed at a higher tax-rate than long-term capital gains, Constantinides (1984) shows that it can be optimal to sell assets with an unrealized capital gain as soon as they qualify for long-term treatment in order to regain the opportunity of producing short-term losses. Dammon and Spatt (1996) extend the approach of Constantinides (1984) by allowing the number of trading periods before a short term position becomes a long term position to be greater than one. In particular, they show that contrary to intuition, it can be optimal to defer small short-term losses even in the absence of transaction costs. This finding is due to the fact that realizing these losses and repurchasing the asset restarts the short-term holding period and thus the time the investor has to wait until potential future gains qualify for long-term treatment.

[1] A transaction is termed a wash sale if a stock is sold to realize a capital loss and repurchased immediately. Under current US tax-rules wash sales do not qualify for the capital loss deduction if the same stock is repurchased within thirty days before or after the sale. Under current US tax law wash sales are permitted and it is not allowed to short a security in which one has a long position to avoid realizing capital gains. Investors realizing such a "shorting-the-box-strategy" are treated as if they had sold the long position and hence their capital gains are taxed.

This paper relaxes assumption (4) and studies optimal portfolio decisions when amounts of capital losses deductible against other income are limited. To the best of our knowledge, there are only three papers taking the different taxable treatment of capital gains and losses explicitly into account. Gallmeyer and Srivastava (2003) deal with arbitrage concerns and show that under quite mild conditions, the lack of pre-tax arbitrage implies the lack of post-tax arbitrage. Ehling et al. (2007) and Marekwica (2007) deal with optimal investment decisions of private investors in tax-systems where there are no tax-rebate payments. While their studies do not allow for tax rebate payments for incurred capital losses, we take the fact into account that the US-tax code allows for deducting losses of up to $ 3,000 per year from other income.

This paper generalizes a key result of Constantinides (1983) by showing that in tax-systems where capital losses can only be offset against other income and in the one-asset case of tax-systems with limited deduction of capital losses from other income, it remains optimal to realize capital losses immediately. limited capital loss deduction it remains optimal to realize losses immediately. It further extends the approaches of Ehling et al. (2007) and Marekwica (2007) by allowing for deductibility of capital losses from other income. In contrast to their setting and that of Constantinides (1983), it can be optimal to cut unrealized capital gains short which significantly complicates the investment decision. Cutting unrealized capital gains short provides the investor with the opportunity of offsetting future capital losses against other income. Offsetting losses against other income is desirable for two reasons. First, it increases the investor's cash at hand that can be invested and earn profits immediately while offsetting losses only avoids tax-payments when capital gains are realized. Second, other income is usually subject to a higher tax rate than capital gains such that the investor saves higher tax payments when offsetting losses against other income.

In addition, in a tax-system that allows for offsetting losses against other income, the investor has to decide whether to offset loses against realized capital gains or other income. Since losses have to be offset against realized capital gains first, the decision to offset losses against other income requires the investor to limit the realization of capital gains and ties the decision on how to use capital losses to the asset allocation decision.

The remainder of this paper proceeds as follows. Section 2 presents our model and explains which factors driving asset allocation are caused by limited capital loss deduction. Section 3 contains our numerical solution to the investor's life cycle consumption investment problem. Section 4 concludes.

2 The Model

We consider the consumption-portfolio problem in the presence of capital gains taxation and limited capital loss deductibility in discrete time. Our assumptions concerning the security market, the taxable treatment of profits, the optimal tax-timing strategy with unrealized capital losses and the investor's consumption-portfolio problem are outlined below.

2.1 Investment Opportunity Set

The investment opportunity set our investor is facing consists of a risky dividend-paying stock and a risk-free money market account.[2] The stock pays a risk-free constant post-tax dividend rate d, the money market account pays a post-tax return r. The pre-tax capital gains rate of the stock g_t from period t to $t+1$ is lognormally distributed with mean μ and standard deviation σ.

2.2 Taxable Treatment of Profits

We impose assumptions (2) and (3), i.e. in our model wash sales are permitted and long- and short-term capital gains are subject to the same tax rate. Income from interest, ordinary income and dividends is taxed at rate τ_i.[3] Realized capital gains are taxable at rate $\tau_g \leq \tau_i$. The tax basis for equity currently held is the weighted average purchase price of the assets.

The focus of analysis is a feature of the tax-code that – to the best of our knowledge – has not received attention in the portfolio choice literature so far – the limited deductibility of capital losses against ordinary income.

The common assumption in the portfolio choice literature dealing with capital gains taxes is that capital gains and losses are treated symmetrically (see e.g. Constantinides (1983), Dammon et al. (2001, 2004), DeMiguel and Uppal (2005), Gallmeyer et al. (2006), Garlappi et al. (2001), Huang (2007), Hur (2001)).

Definition 2.1 (Symmetric treatment, ST). *A tax-system with symmetric treatment of realized capital gains and losses is a tax-system in which the same tax-rate applies to realized*

[2] We focus on the one-asset case in this paper to keep our problem numerically tractable.

[3] Given the fact that the lower tax-rate applicable to dividend income is only granted until 2010 and from 2011 on it will again rise to the tax-rate on ordinary income, we do not consider different tax-rates on dividends and interest payments here.

capital gains and capital losses. In case the investor realizes a capital loss, there is an immediate tax rebate payment the investor can reinvest.

We consider the ST case as a benchmark in our analysis. The second tax-system we consider as a benchmark is a tax-system in which realized capital losses can only be offset against realized capital gains, but not against other income. Such a tax-system is analyzed in Gallmeyer and Srivastava (2003), Ehling et al. (2007) and Marekwica (2007). Such a taxable treatment of capital gains can e.g. be found in the Canadian or several European tax codes, including those of the UK and Germany for instance.

Definition 2.2 (**No deductions, ND**). *In a tax-system with no deductions, the investor is compensated for incurred capital losses with a tax loss carry-forward that is offset against realized capital gains. An amount not being offset against realized capital gains is carried over indefinitely. A tax loss carry-forward that has not been used until the end of an investor's life is not passed to the investor's heirs.*

Compared to the ST case the compensation for realized capital losses does not come as an immediate reduction of taxes on ordinary income but as a tax loss carry-forward which is a less attractive compensation for two reasons. First, in contrast to the implicit tax rebate payment caused by the lower tax payments on ordinary income, a tax loss carry-forward does not pay any interest. Second, a tax loss carry-forward bears the risk of never being used and thus ending up worthless. This risk is especially important if the investor is old and the expected remaining investment horizon is short. However, if capital losses are partly deductible from ordinary income as under current US tax law, a tax loss carry-forward might be a more attractive compensation than an immediate tax rebate payment at tax rate τ_g as in the ST case. This is due to the fact that the investor's tax-rate on ordinary income τ_i usually exceeds the tax-rate on capital gains τ_g such that offsetting one dollar of tax loss carry-forward from ordinary income decreases the investor's tax payments by a higher amount than offsetting the dollar against realized capital gains.

Definition 2.3 (**Limited deduction, LD**). *In a tax-system with limited tax rebates, an investor is compensated for incurred capital losses with a tax loss carry-forward. This tax loss carry-forward has to be first offset against realized capital gains. Each year, an amount of a potentially remaining tax loss carry-forward not exceeding some finite amount M is offset against ordinary income.*[4] *A tax loss carry-forward remaining after this procedure is carried*

[4]Under current US tax law M is equal to \$ 3,000.

over indefinitely. A tax loss carry-forward that has not been used until the end of an investor's life is not passed to the investor's heirs.

If an investor in the LD case at time t is endowed with an initial tax loss carry-forward $L_{t-1} \leq 0$ from the previous period, the tax loss carry-forward is offset against realized capital gains. The remaining taxable gain T_t is given by

$$T_t = \max\left(G_t + L_{t-1}, 0\right). \tag{1}$$

The remaining tax loss carry-forward RL_t after offsetting it against realized capital gains is given by

$$RL_t = \min\left(G_t + L_{t-1}, 0\right). \tag{2}$$

If this remaining tax loss carry-forward RL_t is non-zero, the lesser of the absolute value of the remaining tax loss carry-forward and some upper bond M is offset against ordinary income. If $M = 0$, the tax-system of the LD type becomes a tax-system of the ND type. The amount deductible D_t is thus given by

$$D_t = \min\left(-RL_t, M\right). \tag{3}$$

That amount of the investor's remaining tax loss carry-forward that cannot be deducted from ordinary income is carried over to the next period as tax loss carry-forward L_t. It is given by

$$L_t = RL_t + D_t. \tag{4}$$

The two key differences between the LD case and the two benchmark cases ST and ND are the tax-timing of unrealized gains and the opportunity to use the tax loss carry-forward in two different ways.

In the ND case the investor can only use a tax loss carry-forward to deduct it from future realized capital gains, i.e. there is no incentive to defer the use of a tax loss carry-forward. In the ST case the investor can never end up with a tax loss carry-forward. Only in the LD case the investor can make a decision on how the tax loss carry-forward shall be used, i.e. whether to offset the tax loss carry-forward from realized capital gains or ordinary income.

Offsetting capital losses from ordinary income has two advantages compared to offsetting them from realized capital gains. First, it increases the investor's total wealth invested which allows to earn profits. Second, ordinary income is usually subject to a higher tax rate than (long-term) capital gains such that the tax advantage from offsetting capital losses from ordinary income outweighs the tax advantage from offsetting it against realized capital gains. Therefore, in contrast to the ND case where it is optimal to deduct the tax loss carry-forward from realized capital gains immediately, investors in our setting have an incentive to postpone the realization of capital gains once they are endowed with a tax loss carry-forward. This incentive tends to leave investors with unbalanced portfolios.[5]

In the ST and ND case, the only motive for selling equity with unrealized capital gains is rebalancing the portfolio. In contrast, in the LD case, the investor has a second motive for realizing capital gains. By cutting capital gains short, she regains the opportunity of offsetting capital losses against other income which is usually subject to a higher tax rate than capital gains.[6] By cutting capital gains short, she pays τ_g dollars per unit of unrealized capital gains, but regains the opportunity of offsetting potential future losses against other income subject to tax rate $\tau_i \geq \tau_g$. Therefore, in contrast to the ST and the ND case, besides a decision on optimal consumption and the desired level of her equity exposure, an investor endowed with unrealized capital gains has to make an informed decision on how much of her unrealized capital gains to cut short.

Consequently, in the LD case, there are two reasons for realizing capital gains. First, the investor might want to rebalance her equity exposure and sell some equity. Second, the investor might want sell equity to regain the opportunity of offsetting potential future losses against other income and immediately repurchase that equity.[7] While the first motive for realizing capital gains only affects the investor's equity exposure, but does not affect her unrealized capital gains per unit of equity, the second motive does not affect her equity exposure, but only her unrealized capital gains per unit of equity.

2.3 Optimal Tax-Timing in the LD Case

Given assumptions (1) to (4), Constantinides (1983) shows that it is optimal to realize capital losses immediately. In fact, his prove also holds without imposing assumption (1) that investors

[5] The higher tax rate applicable to realized capital losses makes volatile assets appealing and can be a factor that helps explaining the high valuation of some risky assets.

[6] The reason for cutting gains short is similar to that in Constantinides (1984). While in his setting the reason is the different taxable treatment of long and short-term capital gains, in our setting the reason is the the different tax rates applicable to capital gains and losses.

[7] Another way of cutting gains per unit of stock short is to first purchase additional units of equity which decreases the average purchase price and then sell the required number of units of the risky asset to end up with the desired equity exposure. Since both ways result in the same equity exposure and the the average purchase price, we do not elaborate this second way of cutting gains in more detail here.

do not face any short-selling constraints and can also be applied for short-selling constrained investors. In this section, we argue that his prove can be generalized to tax-systems of the ND case and the one-asset case of tax-systems of the LD type by additionally dropping assumption (4).

Theorem 2.1. *In tax-systems of the ND type and the one-asset case of tax-systems of the LD type where assumptions (2) and (3) hold, it is optimal to realize capital losses immediately, if* $\tau_i \geq \tau_g$.

A formal proof of theorem 2.1 is given in Appendix A. The economic intuition behind the theorem is as follows: Since a tax loss carry-forward does not pay any interest its value can never be above the maximum amount of wealth the tax loss can be converted into. This maximum amount is equal to the investor's tax-rate on ordinary income in the LD case. The only way to receive compensation at tax-rate τ_i is generating a tax loss carry-forward, i.e. realizing the loss. Even in case the investor cannot offset her entire losses from other income immediately or trades in a tax-system of the ND case, it remains optimal to realize the entire losses due to the higher flexibility of the tax loss carry-forward compared to carrying unrealized capital losses that are tied to the asset and carry a risk of getting lost in case of a capital gain.

However, theorem 2.1 cannot be generalized to the multiple asset case if $\tau_i \neq \tau_g$. In the multiple asset case with $\tau_i > \tau_g$ the investor can end up in a state with one asset being endowed with unrealized capital gains and one asset being endowed with unrealized capital losses. When the investor wants to realize some of the capital gains to rebalance her portfolio, it might be optimal to postpone the realization of the unrealized capital losses to avoid offsetting them against the capital gains in the present period and retain the opportunity of offsetting them against other income in some forthcoming period. Since realized losses and a tax loss carry-forward first have to be offset against realized capital gains, unrealized capital losses bear a timing option – the investor can decide when to realize them. By choosing periods in which no capital gains are realized the investor can offset her losses against other income at a tax rate that is usually above the capital gains tax rate.

In the multiple asset case with $\tau_i < \tau_g$ offsetting losses against other income is subject to a lower tax rate than offsetting losses against realized capital gains. Consequently, it can be optimal not to realize all unrealized losses to avoid offsetting them at tax rate τ_i. However, in tax-systems found around the world, the tax rate on other income is usually not below the tax rate on capital gains.

2.4 A One-Period Example

Before introducing the investor's consumption-portfolio problem over the life cycle, we first turn to the relation between our two benchmark tax-systems ST and ND to the LD tax-system in a one-period example. We consider an investor who is not endowed with an initial tax loss carry-forward and who invests an amount of W_0 dollars in a risky asset from period 0 to period 1. $\chi_{\{g_t \geq 0\}}$ denotes the indicator function which is one, if $g_t \geq 0$ and zero otherwise. The investor's amount invested in the stock at time 1 before trading is then given by

$$W_1 = W_0 \left(1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}}\right)\right) + \min\left(-W_0 g_0 \chi_{\{g_0 < 0\}}, M\right) \tau_i.$$

Dividing by W_0 provides the investor's one-period return

$$\frac{W_1}{W_0} = 1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}}\right) + \min\left(-g_0 \chi_{\{g_0 < 0\}}, \frac{M}{W_0}\right) \tau_i.$$

We first consider the two borderline cases when W_0 goes to infinity and to zero, respectively. It holds that $W_0 \rightarrow \infty \Rightarrow \frac{M}{W_0} \rightarrow 0$, i.e. that

$$\begin{aligned}
\frac{W_1}{W_0} &= 1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}}\right) + \min\left(-g_0 \chi_{\{g_0 < 0\}}, 0\right) \tau_i \\
&= 1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}}\right)
\end{aligned}$$

implying that ceteris paribus the return of an investor with substantial investments converges to the return of an investor in the ND case. For such an investor the opportunity of offsetting a limited amount of losses from ordinary income does not have an impact on the return on equity. For $W_0 \rightarrow 0 \Rightarrow \frac{M}{W_0} \rightarrow \infty$, it holds that

$$\begin{aligned}
\frac{W_1}{W_0} &= 1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}}\right) + \min\left(-g_0 \chi_{\{g_0 < 0\}}, \infty\right) \tau_i \\
&= 1 + d + g_0 \left(1 - \tau_g \chi_{\{g_0 \geq 0\}} - \tau_i \chi_{\{g_0 < 0\}}\right)
\end{aligned}$$

implying that ceteris paribus for an investor with very low wealth and in case that $\tau_g = \tau_i$, the return converges to the returns of an investor in the ST case.

If the investor's tax-rate on ordinary income τ_i exceeds the tax-rate on capital gains τ_g, an investor with low wealth prefers to trade in a tax-system of the LD type to a tax-system of the ST type since realized capital losses qualify for higher savings in the former tax-system.

For W_0 different from zero and finite, the return on equity is a weighted average of the ST and the ND return. If a denotes the weight of the ND return and $1-a$ the weight of the ST return, a is given by

$$a = 1 - \min\left(1, \frac{M}{W_0|g_0|}\right)\frac{\tau_i}{\tau_g}. \tag{5}$$

The derivation of equation (5) can be found in Appendix A. In contrast to the ST and the ND case, in the LD case, the investor's return depends on W_0. The higher W_0, the more similar the risk-return profile ot that of an investor in the ND case. For W_0 very small, $a = 1 - \frac{\tau_i}{\tau_g} < 0$. This is due to the fact that in the ST case the tax-rate applicable to losses is τ_g while in the LD case realized losses can be offset from other income which is subject to tax rate τ_i.

The lower the investor's wealth the more attractive the risk-return profile of the risky asset since in case of a negative return the investor may expect to offset capital losses from ordinary income which are substantial in relation to total wealth. If, however, the investor is endowed with substantial wealth, the risk-return profile of risky assets becomes less attractive since the amount deductible from other income is small relative to total wealth.

2.5 The Life Cycle Model

We consider an economy consisting of short-selling constrained investors living for at most T years, who can only trade at time $t = 0, 1, \ldots, T$. The investor derives utility from the consumption C_t of a single good and bequest. The investor's utility function is of the CRRA-type with parameter of risk-aversion of $\gamma \in [0, \infty)$. The parameter γ represents the investor's willingness to substitute consumption among different states in time. It also represents the elasticity of consumption, which is given by $\frac{1}{\gamma}$. For simplicity, we assume that all income is derived from financial assets. Losses not exceeding a constant amount of M qualify for tax rebate payments and are subject to tax rate τ_i.

By θ_t we denote the fraction of the investor's unrealized capital gains that are realized to cut capital gains short without changing the investor's equity exposure. By P_t we denote the price of the stock at the beginning of period t. By P_t^* we denote the investor's purchase price after trading at time t, q_t denotes the number of stocks the investor holds from time t to $t+1$. The total number N_t of units of the stocks that are sold at time t is then given by

$$N_t = \max(q_{t-1} - q_t, 0) + \min(q_{t-1}, q_t)\,\theta_t. \tag{6}$$

The first summand in equation (6) defines the number of units of stocks sold to reduce the investor's equity exposure after trading. It does not affect the amount of unrealized gains per stock. The second summand denotes the number of stocks sold and immediately repurchased to cut gains short. It affects the amount of unrealized gains per stock, but leaves the investor's equity exposure from time t to $t+1$ unaffected.

If the investor faces unrealized capital losses, it is optimal to realize these losses immediately (theorem 2.1) and repurchase the desired equity exposure. Consequently, her purchase price after trading is equal to the current market price, i.e. $P_t^* = P_t$ if $P_{t-1}^* \geq P_t$.

If, on the other hand, the investor faces unrealized capital losses, her purchase price P_t^* is a weighted average of her historical purchase price and the current market price. The weight assigned to the historical purchase price is given by the number of stocks after realization of capital gains. The weight assigned to the current market price is given by the number of stocks $q_{t-1} - N_t$ after cutting gains short. The number of stocks the investor purchases is given by the sum of the number of stocks $\max(q_t - q_{t-1}, 0)$ the investor purchases to increase her equity exposure and the number of stocks $\min(q_t, q_{t-1})\,\theta_t$ the investor repurchases immediately after having sold them to cut unrealized capital gains short. Consequently,

$$P_t^* = \begin{cases} \frac{[q_{t-1}-\max(q_{t-1}-q_t,0)-\min(q_{t-1},q_t)\theta_t]P_{t-1}^*+[\max(q_t-q_{t-1},0)+\min(q_{t-1},q_t)\theta_t]P_t}{q_t} & \text{if } P_{t-1}^* < P_t \\ P_t & \text{if } P_{t-1}^* \geq P_t. \end{cases} \tag{7}$$

The investor's realized capital gains or losses G_t at time t are given by

$$G_t = \left[\chi_{\{P_t > P_{t-1}^*\}}\Big(\max(q_{t-1} - q_t, 0) + \min(q_{t-1}, q_t)\,\theta_t\Big) + \chi_{\{P_t \leq P_{t-1}^*\}} q_{t-1}\right]\left(P_t - P_{t-1}^*\right) \tag{8}$$

where $\chi_{\{P_{t-1}^* > P_t\}}$ denotes the characteristic function, which is one for $P_{t-1}^* > P_t$ and zero otherwise.

By R we denote the gross after-tax return of the risk-free asset. d is a constant after-tax dividend of equity, b_t is the number of units of the risk-free asset with purchase price one the investor holds from time t to $t+1$. W_t is the investor's beginning-of-period-t-wealth before trading, C_t is the investor's period t consumption. i is a constant inflation rate. It is assumed that the bequeathed wealth is used to purchase an H-period annuity and that this H-period annuity provides the beneficiary with nominal consumption of $A_H W_t (1+i)^{k-t}$ at date k ($t+1 \leq k \leq t+H$), in which $A_H \equiv \frac{r^*(1+r^*)^H}{(1+r^*)^H - 1}$ is the H-period annuity factor, r^* is the after-tax real bond return. $F(t)$ denotes the time 0 probability that the investor is still alive through period t ($t \leq T$). The parameter β represents the investor's utility discount factor.

The investor's optimization problem is then given by

$$\max_{C_t, q_t, \theta_t} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t \Big(F(t) U\left(\frac{C_t}{(1+i)^t}\right) + (F(t-1) - F(t)) \sum_{k=t+1}^{t+H} \beta^{k-t} U\left(\frac{A_H W_t}{(1+i)^t}\right) \Big) \right] \tag{9}$$

s.t.

$$W_t = q_{t-1}(1+d) P_t + b_{t-1}(1+r), \qquad t = 0, \ldots, T \tag{10}$$

$$W_t = \tau_g T_t + q_t P_t + b_t + C_t - \tau_i D_t \qquad t = 0, \ldots, T-1 \tag{11}$$

$$q_t \geq 0, b_t \geq 0 \qquad t = 0, \ldots, T-1 \tag{12}$$

and equations (1) to (4) given the initial holding of bonds b_{-1}, stocks q_{-1}, the initial tax-basis P^*_{-1}, the price of one unit of the stock P_0, the initial wealth W_0 and the initial tax loss carry-forward L_{-1}.

According to equation (9), the investor maximizes discounted expected utility of lifetime consumption and bequest. Equation (10) defines the investor's beginning of period t wealth as the sum of wealth in stocks and wealth in bonds before trading at time t, including the after-tax interest and dividend income, but before any capital gains taxes resulting from trading at time t. Equation (11) is the investor's budget constraint at time t. If the investor trades equity, T_t is subject to the capital gains tax rate τ_g and D_t qualifies for tax rebate payments subject to tax rate τ_i.

By letting X_t denote the vector of the investor's state variables, $V_t(.)$ the investor's value function at time t, $f(t)$ the probability of surviving from period t to $t+1$ given the investor is alive at the beginning of period t, and taking into account that the sum in the last term of the objective function (9) can be simplified by making use of the fact that $\sum_{k=t+1}^{t+H} \beta^{k-t} = \frac{\beta(1-\beta^H)}{1-\beta}$, the Bellmann equation for the optimization problem can be written as

$$\begin{aligned} V_t(X_t) = \max_{C_t, q_t, \theta_t} \Bigg[& f(t) U\left(\frac{C_t}{(1+i)^t}\right) + f(t) \beta \mathbb{E}_t \left[V_{t+1}(X_{t+1}) \right] \\ & + (1 - f(t)) \frac{\beta\left(1-\beta^H\right)}{1-\beta} U\left(\frac{A_H W_t}{(1+i)^t}\right) \Bigg] \end{aligned} \tag{13}$$

for $t = 0, \ldots, T-1$ subject to Equations (1), (4), (7), (8), and (10) to (12) with terminal condition $V_T(X_T) = U\left(\frac{A_H W_T}{(1+i)^T}\right)$. The state variables required to solve the problem at time

t are the investor's beginning-of-period-wealth W_t before trading, the initial tax loss carry-forward L_{t-1}, the price of the stock P_t, its tax basis P^*_{t-1}, and the number of stocks q_{t-1} the investor holds at the beginning of period t before trading. Thus, the vector of state variables X_t at time t can be represented as

$$X_t = [P_t, W_t, L_{t-1}, P^*_{t-1}, q_{t-1}]. \tag{14}$$

We rewrite the optimization problem by normalizing with the investor's beginning-of-period-wealth W_t and use the relation between P^*_{t-1} and P_t as a state variable, which allows us to reduce the number of state variables to four: the investor's basis-price-ratio $p^*_{t-1} \equiv \frac{P^*_t}{P_t}$, her initial equity exposure $s_t \equiv \frac{q_{t-1}P_t}{W_t}$, her initial tax loss carry-forward to wealth ratio $l_{t-1} \equiv \frac{L_{t-1}}{W_t}$ and the fraction $m_t \equiv \frac{M}{W_t}$ of total wealth qualifying for tax rebate payments. We solve the rewritten optimization problem by backward-induction. The technical details can be found in Appendix B.

2.6 Base Case Parameter Values

For the numerical analysis, it is assumed, that annual inflation is $i = 3.5\%$. The tax rate on realized capital gains is assumed to be $\tau_g = 20\%$. The tax rate on interest and dividends is assumed to be $\tau_i = 36\%$.[8] In line with current US tax law we assume that the maximum amount of losses qualifying for tax rebate payments subject to tax rate τ_i is given by $M = 3,000$.

The pre-tax risk-free rate is 6% such that the after-tax risk-free rate is $r = 3.84\%$. The return on equity is lognormally distributed, serially independent, comes with an expected capital gain of $\mu = 7\%$, a standard deviation of $\sigma = 20.7\%$ (which corresponds to a standard deviation of the real return of about 20%) and a constant pre-tax dividend rate of 2% in each period such that the after-tax dividend rate is $d = 1.28\%$. The correct choice of the equity premium has been subject to numerous theoretical and empirical research (see Siegel (2005) for a survey). While the historical risk-premium has been about 6% (Mehra and Prescott (1985)) in the US since 1872, economists doubt whether this will be true in future periods. We follow the current consensus which is about 3% to 4% (see e.g. Cocco et al. (2005), Dammon et al. (2001), Fama and French (2002), Gallmeyer et al. (2006) and Gomes and Michaelides (2005)).

[8] Given that the lower tax-rate applicable to dividend income is only granted until 2010 and from 2011 on it will again rise to the tax-rate on ordinary income, we do not consider different tax-rates on dividends and interest payments here.

We assume the investor makes decisions annually starting at age 20 ($t = 0$). The maximum age the investor can attain is set to 100 years ($T = 80$). It is further assumed that the relative risk-aversion of the investor is $\gamma = 3$ and the annual utility discount factor is $\beta = 0.96$. H is set to $H = 60$ in the bequest function, indicating that the investor wishes to provide the beneficiary with an income stream for the next 60 years. The data for the survival probabilities of our female investor are taken from the 2001 Commissioners Standard Ordinary Mortality Table. Table 1 summarizes our choice for the base-case parameter values.

Table 1 about here

3 Numerical Evidence

Having introduced the taxable treatment of capital gains in the three different types of tax-systems and the investor's optimization problem, we now turn to its numerical solution. We first analyze our base-case scenario and contrast optimal conditional investment strategies in the three different types of tax-systems in section 3.1. Section 3.2 analyzes when it is optimal to cut gains short. The impact of an initial tax loss carry-forward on optimal investment strategies is discussed in section 3.3. In section 3.4, we quantify the effective tax rate that makes an investor indifferent between being compensated for a tax loss carry-forward immediately and keeping the tax loss carry-forward to use it in forthcoming periods. Section 3.5 summarizes the results of a Monte Carlo analysis on the evolution of the investor's optimal consumption investment strategy over the life cycle.

3.1 Optimal Investment Policy without Tax Loss Carry-Forward

We begin the discussion of our numerical results by first considering the optimal investment policy of an investor who is not endowed with an initial tax loss carry-forward. In general, the investor's optimal equity exposure depends on her basis-price-ratio, her initial equity exposure, her initial tax loss carry-forward and her wealth-level. Her basis-price-ratio indicates whether the investor faces an unrealized capital gain (basis-price-ratio less than one) or loss (basis-price-ratio above one). The basis-price-ratio thereby indicates potential tax payments or tax loss carry-forwards granted when selling equity. The investor's initial equity proportion indicates to which extend the investor is affected by the unrealized capital gains or losses per unit of equity.

An initial tax loss carry-forward provides the investor with the opportunity of avoiding capital gains tax payments when offsetting it against realized capital gains or allows the investor to offset it against other income. The investor's wealth level affects the investor's optimal investment decision as it determines which fraction of total wealth can be offset against other income. Since M is a constant amount, the fraction of losses than can be offset against other income is higher for investors with low wealth levels than for investors with high wealth levels. The length of the remaining investment horizon has an impact on the investor's optimal equity exposure due to the fact that a tax loss carry-forward cannot be bequeathed and unrealized capital gains are forgiven at death and thereby escape taxation.

Figure 1 about here

Figure 1 depicts the relation between the optimal equity exposure of an investor at age 30 not being endowed with an initial tax loss carry-forward and the investor's initial basis-price-ratio as well as her initial equity proportion. The upper graphs show her optimal equity exposure in a tax-system of the LD type when being endowed with an initial level of wealth before trading of \$ 3,000 (upper left graph) and \$ 3,000,000 (upper right graph), respectively. The lower graphs depict the investor's optimal equity exposure in a tax-system of the ST type (lower left graph) and the ND type (lower right graph).

The optimal investment policies in the tax-systems of the LD type differ substantially. An investor with an initial wealth-level of \$ 3,000 (left graph) increases her equity exposure monotonically as her basis-price-ratio rises. When the investor is endowed with an initial basis-price-ratio above one, indicating that the investor faces unrealized capital losses, she optimally realizes these losses immediately. This leaves the investor with an immediate tax rebate payment for all incurred capital losses and increases her wealth-level. This increase is the higher, the higher the unrealized capital losses per unit of equity, i.e. the higher the investor's basis-price-ratio, and the higher the investor's initial equity exposure. As we defined the optimal equity exposure as the fraction of the investor's equity after trading relative to her beginning-of-period wealth, the optimal equity exposure increases when the investor's wealth-level after trading increases, which is e.g. the case when she receives tax rebate payments.

When the investor faces unrealized capital gains, she has to decide whether to cut these gains short to regain the opportunity of offsetting potential future capital losses against other income. Cutting gains short is the more desirable, the higher the investor's potential future tax rebate payments relative to total wealth are. For investors with low levels of wealth, the fraction of capital losses that can be offset against future income is substantial. Consequently, an investor with a low wealth-level optimally realizes her capital gains. Due to the tax payments associated with the cutting of her unrealized gains, her wealth level decreases which is why the investor's optimal equity exposure decreases as her initial equity exposure increases and her basis-price-ratio decreases.[9]

For an investor who is endowed with an initial wealth-level of \$ 3,000,000 (upper right graph), the optimal equity exposure is substantially lower. Additionally, the impact of her basis-price-ratio and her initial equity proportion on her optimal equity exposure differs fundamentally from that of the investor with \$ 3,000 initial wealth. The difference in the optimal equity exposure between the two graphs arises from the different fraction of potential losses that can be offset against other income. The investor being endowed with a low wealth-level of only \$ 3,000 can offset all potential losses against other income. This is not true for the investor who is endowed with an initial wealth-level of \$ 3,000,000. who can only offset $\frac{3{,}000}{3{,}000{,}000} = 0.1\%$ such that her investment decision becomes quite similar to that of an investor in a tax-system of the ND type (lower right graph) who cannot offset any capital losses from other income. Both investors in tax-systems of the LD type with high wealth-level and investors in tax-systems of the ND type increase their equity exposures when being endowed with a significant initial equity exposure and either unrealized capital gains or losses.

The reasons for the higher equity exposure with unrealized capital gains and losses, however, are remarkably different. Being endowed with unrealized capital gains, the investor seeks to avoid capital gains tax payments and therefore accepts a higher equity exposure. Especially, if equity has performed well in the past, its fraction relative to the investor's total wealth has been increasing which might result in an unbalanced portfolio. However, selling equity to rebalance the portfolio results in capital gains tax payments. To avoid the capital gains tax payment, the investor might accept a deviation from her otherwise desired equity exposure – such an investor is also referred to as being locked in. This deviation is higher when her basis-price-ratio is lower, i.e. when her unrealized capital gains per unit of equity are higher and thereby invoke higher tax costs for rebalancing her portfolio. Being endowed with an unrealized capital loss the investor optimally realizes that loss immediately which leaves her with a tax loss carry-forward. In tax-systems of the ND type and tax-systems of the LD type where the investor is

[9]We elaborate the question when to optimally cut unrealized capital gains in more detail in section 3.2.

endowed with substantial wealth and can only offset small amounts against other income this tax loss carry-forward allows the investor to earn some future capital gains tax-free. Hence, the risk-return profile of equity becomes more desirable. Consequently, the optimal equity exposure is above the level of an investor who is not endowed with an unrealized capital loss.

The results in the lower graphs confirm the results of recent literature on optimal investment decisions in tax-systems of the ST and the ND type (see Dammon et al. (2001), Ehling et al. (2007) and Marekwica (2007)). Since tax-systems of the ST type (lower left graph) provide the investor with more generous compensation for realized capital losses, it is not surprising, that the optimal equity exposure in such tax-systems is above the optimal equity exposure in tax-systems of the ND type (lower right graph).

The taxable treatment of capital losses in tax-systems of the LD type is more attractive for an investor than in tax-systems of the ND type due to the opportunity of offsetting losses against other income. While this causes investors with low wealth-levels that can offset a substantial fraction of potential losses against other income to increase their equity exposure, this advantage becomes neglectable to investors that are endowed with substantial wealth and can only offset small amounts of potential losses against other income.

While in tax-systems of the ST and the ND type, the homogeneity of the CRRA utility function assures, that the investor's wealth-level does not have an impact on her investment decision, this is not true in tax-systems of the LD type, where the wealth-level affects the fraction of losses that can be offset against other income. Since the tax rate τ_i applicable to tax rebate payments resulting from losses being offset against other income exceeds the tax rate on capital gains τ_g, it can be optimal to cut capital gains short to regain the opportunity of offsetting losses at tax rate τ_i.

3.2 When to Cut Gains Short

Analyzing the differences between the optimal equity exposure for an investor with low and high wealth-level in a tax-system of the LD type, we argued that it might be optimal to cut capital gains short to regain the opportunity of offsetting potential future losses against other income. Furthermore, our results in section 3.1 indicate that the investor's optimal equity exposure depends crucially on her wealth-level.

Figure 2 about here

Figure 2 analyzes this relation between the investor's initial wealth-level and her optimal equity exposure (left graph) as well as the optimal fraction of capital gains to cut short (right graph) for an investor at age 30 who is not endowed with a tax loss carry-forward and whose initial equity exposure is 60%. If the investor faces unrealized capital gains, her optimal equity exposure depends on whether she cuts these gains short or not.

If she does not cut her gains short, each trade has an impact on her basis-price-ratio or her tax payments. If, however, the investor cuts all her capital gains short, she can choose her desired equity exposure without facing any additional tax consequences or changes in her basis-price-ratio.

The right graph in figure 2 shows that the investor optimally realizes all capital gains when her wealth level is small. She does not realize any capital gains in order to cut her basis-price-ratio only when her wealth level is substantial. This dependency of the optimal realization of capital gains and the investor's wealth level is again due to the fact that the investor can only realize a constant amount of capital losses each year. Consequently, if the investor's wealth-level is small, she can offset a substantial fraction of losses against other income. If, however, her initial wealth-level is substantial, the fraction of losses that can be offset against other income is small.

The reason for cutting gains short is the advantage from offsetting capital losses against other income. Since the advantage the investor yields from cutting gains short is substantial when her wealth-level is small and small when her wealth-level is big, she optimally cuts gains short, when her wealth-level is small and does not cut her gains short, when her wealth-level is substantial. The cut-off point is at around \$ 400,000, such that investors with less than these \$ 400,000 tend to cut their gains short and investors with even higher wealth-levels tend not to cut their gains short.

The left graph of figure 2 shows how the investor's optimal equity exposure depends on her basis-price-ratio and her wealth-level. If the investor's wealth-level is substantial and she does not cut gains short, her optimal equity exposure increases as her basis-price-ratio drops below one, indicating that she faces unrealized capital gains in her equity. If, however, the investor is endowed with a low initial wealth-level, she optimally cuts her capital gains short and her optimal equity exposure slightly increases as her wealth-level decreases, i.e. as the fraction of losses that can be offset against other income increases.

3.3 Investment with Initial Tax Loss Carry-Forward

So far, we have considered the optimal investment strategy of an investor, who is not endowed with an initial tax loss carry-forward. An investor who is endowed with an initial tax loss carry-forward has to make an informed decision on whether to realize her capital gains and to offset the tax loss carry-forward against these gains or to postpone the realization of capital gains and to offset the tax loss carry-forward against other income.

Figure 3 about here.

Figure 3 depicts the optimal equity exposure (left graph) and the optimal fraction of gains to cut short (right graph) for an investor at age 30, who is endowed with an initial wealth-level of \$ 3,000 and a tax loss carry-forward of 30% of her initial wealth (i.e. a tax loss carry-forward of \$ 900). If the investor faces substantial unrealized capital gains, which is the case if the investor's basis-price-ratio is small and her initial equity proportion is high, the investor optimally realizes her capital gains immediately and uses her tax loss carry-forward to offset it against these realized capital gains. Even though her tax rate on capital gains is substantially below her tax rate on other income, which she could earn by postponing the realization of capital gains by one period, she realizes her capital gains immediately.

In total, cutting capital gains short has three effects. First, the investor can reduce her initial equity proportion to her desired level of equity exposure. Second, the investor can offset future capital losses against other income. And third, the investor has to offset her present tax loss carry-forward against her realized capital gains first.

While the third factor suggests that the investor should postpone the realization of her capital gains, the first two factors suggest that the investor should realize her capital gains immediately. The first factor is crucial, if the investor's initial equity proportion deviates substantially from her desired equity exposure. The second factor is the more important, the higher her unrealized capital gains per unit of equity are. If the investor is only endowed with very small capital gains, she can at least offset that part of potential future losses from other income that exceed her unrealized capital gains. Consequently, for investors with low unrealized capital gains, the advantage from cutting her unrealized capital gains short immediately is small, which is why the investor prefers to offset her tax loss carry-forward against other income in that case. As a result, the investor's optimal equity exposure is substantially higher with small unrealized capital gains than with big amounts of unrealized capital gains.

3.4 Effective Tax Rate on Tax Loss Carry-Forward

In this section, we analyze the effective tax rate τ_e applicable to the investor's tax loss carry-forward that would make the investor indifferent between immediately receiving a tax rebate payment and keeping the tax loss carry-forward to offset it from other income or realized capital gains in forthcoming periods.

Since in tax-systems of the LD type each dollar of tax loss carry-forward allows the investor to decrease tax-payments by not more than τ_i dollars, one unit of tax loss carry-forward cannot be worth more than these τ_i dollars. However, if the investor is endowed with a high level of wealth and she faces a significant tax loss carry-forward, her effective tax rate might be worth less than τ_i dollars for three reasons. First, she might not make use of her entire tax loss carry-forward in her life, implying that the potential value of the tax loss carry-forward never turns into wealth that can be consumed or bequeathed. This type of risk is most important for old investors facing high mortality rates. Second, even if the investor can make use of her entire tax loss carry-forward, it might take several periods until her entire tax loss carry-forward is converted to wealth and she can earn profits from it. Third, she might want to offset parts of her tax loss carry-forward against realized capital gains. Consequently $\tau_e \leq \tau_i$.

In tax-systems of the ND type, each dollar of tax loss carry-forward cannot be worth more than τ_g dollars since the investor can only offset losses against realized capital gains which are subject to a tax rate of τ_g. Since the tax loss carry-forward does not pay any interest, whereas tax rebate payments can be reinvested and do yield profits, in tax-systems of the ND type one unit of a big tax loss carry-forward should be worth less than one unit of a small tax loss carry-forward. As a result, the effective tax rate should be decreasing as the level of the investor's tax loss carry-forward increases.

This relation does not hold true in tax-systems of the LD type. In these tax-systems the value of the tax loss carry-forward depends on whether it is offset against other income or realized capital gains.

Figure 4 about here

Figure 4 depicts the relation between the investor's effective tax rate and our state variables. The upper left graph shows the impact of the investor's initial equity exposure and the level of her tax loss carry-forward for an investor at age 30 in a tax-system of the ND type, the upper right graph for an investor in a tax-system of the LD type who is endowed with an initial wealth-level of \$ 3,000. The lower left graphs depict the impact of the investor's age (lower left graph) and the investor's wealth-level (lower right graph) in a tax-system of the LD type.

The upper left graph shows that in tax-systems of the ND type the effective tax rate is decreasing in the initial equity proportion and the and increasing in the level of the tax loss carry-forward (in absolute value) for an investor being endowed with unrealized capital gains. Being endowed with unrealized capital gains and a substantial initial equity proportion, the investor tends to make use of her tax loss carry-forward earlier than an investor being endowed with a small initial equity proportion. Consequently, the average waiting time until the tax loss carry-forward is used and provides the investor with the opportunity of earning interest is shorter which is why the effective tax increases as the investor's initial equity proportion does.

The effective tax rate decreases in the level of the tax loss carry-forward since a high level of the tax loss carry-forward carries a lower probability of using the entire tax loss carry-forward. Even in case it is entirely used, the average waiting time until its usage is longer. As a result, the investor tends to earn profits from the tax advantage due to the tax loss carry-forward. Consequently, the compensation the investor asks for one dollar of tax loss carry-forward to make her indifferent between receiving that compensation immediately and keeping her tax loss carry-forward for future periods, is decreasing as her tax loss carry-forward increases.

If the investor's initial equity proportion is substantial and her tax loss carry-forward is small, the effective tax rate reaches its maximum value of τ_g, indicating that the investor makes use of her entire tax loss carry-forward immediately to reduce her equity exposure and rebalance her portfolio.

For an investor in a tax-system of the LD type being endowed with an initial wealth-level of \$ 3,000 (upper right graph) the relation between the investor's effective tax rate and her initial equity proportion and her initial tax loss carry-forward looks entirely different. In contrast to the tax-system of the ND type, the effective tax rate decreases as the investor's initial equity exposure increases and it increases as the investor's initial tax loss carry-forward increases (in absolute value). In contrast to tax-systems of the ND type, in tax-systems of the LD type the investor can use her tax loss carry-forward in two different ways. The investor can offset the tax loss carry-forward against realized capital gains or she can offset it against other income. The effective tax rate depends heavily on how the investor uses her tax loss carry-forward.

For high levels of the initial equity proportion, the diversification motive and the desire not to get locked in outweighs the incentive to postpone the realization of capital gains to offset losses against other income. Hence, with low levels of her initial tax loss carry-forward and high levels of her initial equity proportion, the investor tends to cut her capital gains short which forces her to offset her tax loss carry-forward against these realized capital gains such that the effective tax rate is equal to the tax rate on capital gains. As the level of the investor's initial tax loss carry-forward increases in absolute value, she ends up at some point where it is no longer optimal to cut capital gains short. As the tax rate on other income is higher than the tax rate on capital gains, this causes the effective tax rate to increase substantially. However, since the investor still wants to rebalance her portfolio, the investor has to offset some part of her tax loss carry-forward against capital gains. The fraction of her tax loss carry-forward that is offset against capital gains increases as her initial equity proportion increases, which is why her effective tax rate decreases as her initial equity proportion increases.

The lower left graph shows how the investor's age and her initial tax loss carry-forward determine the effective tax rate for an investor who is endowed with an initial equity exposure of 60% and an initial wealth-level of $ 3,000 in a tax-system of the LD type. In line with our finding in the upper right graph, the investor's effective tax rate increases as the level of her tax loss carry-forward increases (in absolute value). As the investor's initial loss carry-forward exceeds a certain level, the investor postpones cutting her losses short to offset her tax loss carry-forward against other income. Hence, at that point the level of her initial tax loss carry-forward increases substantially. At age 85 the investor stops cutting her gains short as from that age on, the impact of the forgiveness of capital gains when being bequeathed outweighs the diversification concern and the desire to offset losses against other income. As the investor's age increases further, her effective tax rate declines. This is due to the reason that at the investor's time of death an unused tax loss carry-forward is forfeited. Consequently, the effective tax rate decreases as the investor's mortality rates rise.

The lower right graph depicts the impact of the investor's wealth-level and her tax loss carry-forward on her effective tax rate for an investor at age 30 with an initial equity proportion of 60% in a tax-system of the LD type. It shows that for significant levels of the investor's initial tax loss carry-forward, her effective tax rate decreases as her wealth-level increases. This is caused by the fact that with increasing wealth-level, the fraction of the investor's losses that can be offset against other income decreases. Consequently, she will offset a higher fraction of her losses against capital gains. When the investor's wealth-level is very high, the relation between her initial tax loss carry-forward and her effective tax rate becomes very similar to that of an investor in a tax-system of the ND type.

3.5 Unconditional Strategies

Having analyzed the investor's optimal investment policy given specific values of the state variables, we next turn to the investor's optimal unconditional investment policy over the life cycle. While the graphs in figures 1 to 3 provide a good intuition about the impact of the state variables on the investor's optimal equity exposure and the different tax-effects that drive these results, they do not reveal how likely the investor ends up in which state. An investor who cuts capital gains short each period is e.g. very unlikely to end up in a state with substantial unrealized capital gains.

To analyze the investor's optimal investment strategy over the life cycle we run 50,000 simulations on our optimal grids in tax-systems of the LD, the ND and the ST case. We consider an investor who enters the market at age 20, who neither faces unrealized capital gains or losses, who is not endowed with an initial tax loss carry-forward and whose initial wealth is \$ 10,000. In the LD case we additionally run a simulation with an initial wealth-level of \$ 100,000 to explore the impact of wealth on optimal life cycle investment strategies.

Throughout our paper, we analyze three tax-systems that do not only differ in their treatment of realized capital losses, but also in the state variables optimal decisions depend on and in the number of decisions the investor has to make itself.

In tax-systems of the ST type the investor has to make informed decisions on her consumption-wealth ratio c_t and her optimal equity exposure α_t at each point in time t. The state variables required to make such a decision are the investor's basis-price-ratio p^*_{t-1} and her initial equity exposure s_t. In tax-systems of the ND type the consumption investment decision additionally depends on the investor's initial tax loss carry-forward l_{t-1}.

In tax-systems of the LD type the optimal consumption investment decision is even more complicated. First, in addition to her optimal consumption-wealth ratio c_t and her optimal equity exposure α_t, the investor has to decide which fraction θ_t of unrealized capital gains per unit of equity to cut short. Second, this consumption investment decision also depends on the fraction m_t of total wealth qualifying for tax rebate payments. Table 2 summarizes the decision and state variables for the three types of tax-systems.

Table 2 about here

In tax-systems with tax-timing option, there are two reasons why an investor might choose a high equity exposure. First, equity has an appealing risk-return profile. In tax-systems of the LD type the risk-return profile of the risky asset depends on the investor's wealth-level which determines the fraction of losses that can be offset against other income. Besides human capital and the flexibility of labor supply (Bodie et al. (1992)), information costs (Haliassos and Bertaut (1995)), changing risk aversion with age (Ballente and Green (2004)) and cointegration of stock and labor markets (Benzoni et al. (2007)), the lower fraction of losses that can be offset against other income with increasing wealth-level is another reason why private investors might want to decrease their equity exposure over the life cycle.

Second, the investor might be locked in and wants to avoid the tax payments she is confronted with when selling equity. Especially when the investor is old and faces high mortality rates, this motive is very important since the step up in tax-basis for assets bequeathed allows the investor to entirely escape the taxation of her capital gains. In tax-systems of the LD type, there is even a third reason: The investor might want to offset a tax loss carry-forward against other income and therefore wants to postpone the realization of capital gains.

We first present the results of our simulations in the base case setting in subsection 3.5.1. In subsection 3.5.2, we consider the impact of the forgiveness of capital gains when being bequeathed and consider a tax-system in which unrealized capital gains are subject to taxation when being bequeathed.

3.5.1 Base-Case Setting

We first consider the investor's optimal life cycle optimization problem in our base-case setting.

Table 3 about here

In table 3, we summarize the evolution of the investor's state variables and her optimal investment decisions over the life cycle from 50,000 simulations on the optimal path for an investor trading in a tax-system of the ST, the ND or the LD type. We used the same realizations of the stochastic capital gains of the risky asset in all simulations to make sure that our results can be compared with each other easily. Panel A contains our results for an investor at age 30, panel B for an investor at age 60, and panel C for an investor at age 90. We show the mean, the standard deviation and percentiles of the distribution of the investor's optimal equity exposure α_t and her basis-price-ratio before trading p^*_{t-1} for tax-systems of all three types. We further show the level of her initial tax loss carry-forward l_{t-1} for tax-systems of the LD and the ND type and and the optimal realization of capital gains θ_t for the tax-system of the LD type.

As argued above, the investor's wealth level does not have an impact on her optimal investment decision in tax-systems of the ST and the ND type due to the homogeneity of the CRRA utility function. However, it significantly affects optimal investment decisions in tax-systems of the LD type as shown in figure 1. The columns marked LD^4 refer to an investor whose initial wealth-level at age 20 is \$ $10,000 = 10^4$, columns marked LD^5 refer to an investor whose initial wealth-level at age 20 is \$ $100,000 = 10^5$.

Table 3 confirms that an investor trading in the LD tax-system holds substantially more equity when being endowed with a low initial wealth-level of \$ 10,000 than when being endowed with a higher initial wealth-level of \$ 100,000. When the investor is young, her investment decision is mainly driven by the first factor. The risk-return profile of the risky asset is more appealing when being endowed with a lower wealth-level since a higher fraction of potential losses can be offset against other income qualifying for substantial tax rebate payments. Consequently, her initial tax loss carry-forward is substantially below that of an investor with a higher wealth-level and her equity exposure at young age is higher. Besides the higher equity exposure of an investor in the LD^4 case, the higher fraction of losses qualifying for tax rebate payments is a second factor explaining, why the wealth-level tends to be more than one tenth of the wealth-level of the LD^5 investor even though the initial wealth-level at age 20 of the later was chosen to be ten times as big as that of the former.[10]

As the opportunity of offsetting potential unrealized capital gains against other income is very appealing, the investor realizes all her unrealized capital gains when being young or middle-aged and being endowed with a low initial wealth-level. Being endowed with a higher wealth-level the investor decreases the fraction of capital gains being cut short. Consequently, the LD^5 investor faces higher unrealized capital which can be seen from the evolution of the

[10]When the investor is old, this finding is no longer true for the highest percentiles of the distribution of wealth. This is due to the fact that the investor in the LD^5 case tends to become locked in earlier than the investor in the LD^4 case. Thus, her equity exposure is higher. In case of positive realizations of the stochastic equity return, they face a higher growth in their wealth. However, the distribution of their wealth is subject to a higher volatility.

investor's basis-price-ratio. Hence, she tends to become locked inearlier than the LD^4 investor.

In the LD^4 case the average fraction of losses that is offset against other income at tax rate τ_i over the life cycle is 93.8%, its standard deviation is 6.6%. In the LD^5 case an average fraction of only 61.8% at a standard deviation of 10.6% is offset against other income, indicating that even in the LD^5 case the investor makes substantial efforts for not getting locked in.

At the age of 90, both an LD^4 and an LD^5 investor do not cut their capital gains. This result is caused by the reset provision of the tax code according to which unrealized capital gains are forgiven when being bequeathed. Hence, the high level of the investor's equity exposure is driven by the high mortality rates and the desire to postpone the realization of capital gains to espace the capital gains tax.

Investing in a tax-system of the LD type is ceteris paribus more attractive than investing in a tax-system of the ND type due to the opportunity of getting tax rebate payments for realized capital losses. Consequently, at young age, the investor's equity exposure in the LD tax-system is higher than in the ND tax-system. The difference in the investor's equity exposure is the higher, the higher the fraction of losses qualifying for tax rebate payments, i.e. the lower the investor's wealth level. Since the investor never cuts unrealized capital gains short in tax-systems of the ND type, she tends to become locked in significantly earlier, which can be seen by comparing the evolution of the investor's basis-price-ratio over the life cycle. Consequently, when the investor gets older, her equity exposure in tax-systems of the ND type increases faster than in tax-systems of the LD type where diversification can be achieved with lower tax payments.

Whether investing in a tax-system of the LD type or a tax-system of the ST type is more desirable for an investor depends crucially on her wealth-level. As argued in section 2.4, for an investor with a very small wealth-level, investing in the tax-system of the LD type is more desirable due to the higher tax rebate payments on realized capital losses. For an investor with a very high wealth-level, however, investing in tax-systems of the ST type is more desirable, since such a tax-system does not limit the amount of losses that can be offset against other income. In the cases LD^4 and LD^5 analyzed here, the advantage of the higher tax rebate payments for realized capital losses in tax-systems of the LD type outweighs the advantage of unlimited tax rebate payments in tax-systems of the ST type for young investors. As the investor ages, this result does no longer hold true.

While the investor in the LD^4 case still chooses a higher equity exposure at the age of 60, the investor in the tax-system of the LD^5 case no longer does. This finding can be attributed to two causes. First, in the course of time the investor's wealth-level increases and consequently, the fraction of losses than qualify for tax rebate payments decreases. As a result, the risk-return-profile of the risky asset becomes less desirable. Second, the investor in the tax-system of the ST type already tends to become locked in, which can be seen from the distribution of her basis-price-ratio. For the same reasons the investor's equity exposure in the tax-system of the ST type tends to be higher than in the LD^4 and LD^5 case at the age of 90.

Investing in a tax-system of the ST type is ceteris paribus more desirable than investing in a tax-system of the ND type due to the tax rebate payments for realized capital losses. While at young age, this causes the investor to choose a slightly higher equity exposure, we confirm the finding of Ehling et al. (2007) that the differences in the investor's investment strategies and her basis-price-ratio become neglectable once the investor is locked in. As soon as the investor is locked in, she has a strong incentive not to realize her capital gains to save the tax rebate payments. Consequently, the investment decisions, investors in tax-systems of the ST and ND type are facing once they are locked in, are very similar, which is why the evolution of their investment strategies and state variables does not differ substantially.

The key difference between tax-systems of the LD type on the one hand and tax-systems of the ND and ST type on the other hand is the fact that cutting unrealized capital gains short is not desirable in tax-systems of the ND and the ST type, but can be optimal in tax-systems of the LD type to regain the opportunity of offsetting capital losses against other income qualifying for higher tax rebate payments. In tax-systems of the ND and ST type there is no such incentive to cut realized capital gains short, which is why investors in such types of tax-systems tend to become locked in quite early. The LD^5 case shows that even with a substantial initial wealth-level there is an incentive to cut gains short. In the LD^4 case the investor even tends to realize all capital gains when being young such that the distribution of her initial basis-price ratio at age 30 and age 60 is almost the same. Consequently, the opportunity of offsetting losses against other income is a strong incentive to cut capital gains short which leaves private investor's with well-diversified portfolios. In contrast, investors in tax-systems of the ND or the ST type not facing an incentive to cut capital gains short, tend to become locked in, which leaves them with unbalanced portfolios. The opportunity of offsetting losses against other income therefore causes optimal portfolios of US-American investors trading in a tax-system of the LD type to be well-diversified, while Canadian or European investors trading in tax-systems of the ND type do not have an incentive to cut gains short and therefore tend to optimally hold less diversified portfolios.

Our results in the base-case setting suggest, that the forgiveness of capital gains at death has a substantial impact on optimal tax-timing strategies – especially for old investors that are facing higher mortality rates. To explore the impact of this special feature of the tax code, we next turn to tax-systems in which unrealized capital gains that are passed to the investor's heirs are treated as being realized and are subject to an immediate capital gains taxation. Such a taxable treatment of capital gains at death is e.g. found in several European tax codes.

3.5.2 Mandatory Realization of Capital Gains when Bequeathed

Having analyzed the investor's optimal life cycle consumption investment problem and the evolution of her unrealized capital gains, her tax loss carry-forward and her wealth-level in the base-case setting, we explore the impact of the taxable treatment of unrealized capital gains when being bequeathed in this section.

Table 4 about here

Table 4 shows the investor's optimal investment strategy and evolution of state-variables in such a tax-system. Our results in this section only differ from those in the previous section by the different taxable treatment of unrealized capital gains when being passed to the investor's heirs.

Our results in table 4 show that the taxation of unrealized capital gains at death changes the investor's optimal investment strategy substantially – especially when the investor is old and faces high mortality rates. Dammon et al. (2001) argue that for short-selling constrained investors there is a tradeoff between diversification concerns and the motive to postpone the realization of capital gains to defer the tax-payment. Due to the fact that for ending up with substantial unrealized capital gains and a badly diversified portfolio, it takes some time, at young age the investor's portfolios are very similar to those of an investor in our base case setting. As the investor gets older, our results indicate, that due to the increasing importance of the diversification motive, the investor tends to hold a lower equity exposure in all three types of tax-systems. The evolution of the investor's basis-price-ratio further indicates, that the investor tends to realize a substantially higher fraction of her capital gains to rebalance her portfolio. In line with this finding, the desire to cut unrealized capital gains short in tax-systems of the LD type, is substantially higher when capital gains are subject to taxation when being bequeathed. In the LD^4 tax-system the average fraction of capital losses that are offset against other income is 91.79% compared to 93.8% in the base case setting. In the LD^5 tax-system the average fraction of capital losses offset against other income decreases from 61.8% to 55.3%, confirming our finding that the investor's diversification motive is substantially stronger when capital gains are subject to taxation when being passed to the investor's heirs.

Due to the higher motive to cut unrealized capital gains short in tax-systems of the LD type, the investor is more likely to end up with an initial tax loss carry-forward. While being locked in, a negative return on equity only decreases the investor's unrealized capital gains, a negative capital gain causes an investor who is not endowed with an unrealized capital gain to end up with a tax loss carry-forward. At high age the investor tends to be endowed with a wealth-level that does not allow her to earn tax rebate payments for all her losses, which is why she ends up with an initial tax loss carry-forward in the forthcoming period.

The lower equity exposure and the higher diversification motive also affect the investor's wealth level. At young age the equity exposure is not significantly different from our base case scenario and the investor's wealth-level does not differ much, accordingly. At higher age, the investor's equity exposure is significantly lower such that her average wealth-level is significantly lower, too. However, due to the higher diversification concern, the investor's wealth-level is subject to a substantially lower volatility.

In total, the taxation of capital gains when being passed to the investor's heirs weakens the desire to postpone the taxation of capital gains by not realizing them and strengthens the diversification motive. These effects are most important when the investor is old and faces high mortality rates.

4 Conclusion

This article analyzes the optimal dynamic consumption portfolio problem in the presence of capital gains taxes. It explicitly takes limited capital loss deduction and the 3,000 dollar amount that can be offset against other income into account. It generalizes the classical result of Constantinides (1983) that it is optimal to realize capital losses immediately to tax-systems where capital losses can only be offset against realized capital gains as well as the one-asset case of tax-systems where capital losses can also be offset against other income. The article shows that in tax-systems that allow for offsetting limited amounts of capital losses against other income cause investors to hold more diversified portfolios, especially when their total wealth invested is small.

Compared to tax-systems in which capital losses can only be offset against other income, the investment decision becomes substantially more difficult in the setting analyzed here for two reasons. First, the investor has to make a decision on how to use a loss, i.e. whether to offset it against realized capital gains or to potentially postpone the realization of capital gains and offset it against other income. Second, in our setting it can be optimal to cut more capital gains short than are required for rebalancing the portfolio. In contrast to tax-systems where capital gains and losses are subject to the same taxable treatment and tax-systems where losses can only be offset against capital gains, the investor's wealth-level has a substantial impact on her optimal investment strategy. Investors with low wealth-levels tend to cut unrealized capital gains short to regain the opportunity of offsetting losses against other income. This causes optimal portfolios of US-American investors to be well-diversified, while optimal portfolios of Canadian or European investors trading in tax-systems where losses can only be offset against realized capital gains are subject to the risk of getting locked in. Consequently, to postpone the payment of capital gains taxes their optimal portfolios tend to be less well-diversified. However, at old age the step up in the tax basis – which is not known in many European tax codes – prevents the portfolios of US-American investors from being well-diversified.

In order to keep the optimization problem numerically tractable, the model in our paper restricts the number of risky assets to one. It would be interesting to explore optimal tax-timing strategies in the multi-asset case. In particular, analyzing how investors would optimally realize losses in the multi-asset case where it is no longer optimal to realize losses immediately is a fruitful field for further research. Despite the use of a super computer the one-asset case is already challenging from a numerical perspective. We therefore leave the two-asset case to further research.

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A Appendix A – Generalization of Constantinides (1983)

A.1 The One-Asset Case

A.1.1 Wealth, Unrealized Gains and tax loss carry-forward

In tax-systems of the ND and LD type, optimal asset allocation depends on total wealth W_t before trading, the initial tax loss carry-forward L_{t-1}, unrealized capital gains U_t before trading, and the length of the remaining investment horizon. The key to understanding optimal tax-timing in such a tax-system is understanding the relation between W_t, U_t and L_{t-1}. We show that in the one-asset case the result of Constantinides (1983) that it is optimal to realize capital losses immediately can be generalized to tax-systems of the LD and ND type. We first turn to tax-systems of the LD type.

A tax loss carry-forward of one dollar can be used in two ways. First, it can be subtracted from a realized capital gain to reduce capital gains taxes. Second, in the absence of a realized capital gain, the tax loss carry-forward can be offset against other income if $M \geq 1$. Thus, one dollar of tax loss carry-forward can be shifted to $\tau_i \geq \tau_g$ dollars of wealth if $M \geq 1$. Shifting the tax loss carry-forward to wealth by offsetting it against other income is a dominating strategy, since one dollar of tax loss carry-forward can reduce future tax burden by not more than τ_i dollars. Furthermore, in contrast with the tax loss carry-forward, the τ_i dollars of tax rebate can be reinvested and earn profits. By investing them in the risk-free asset, their value is always at least as high as the future tax burden of the unrealized capital gain.

Thus, if two investment strategies result in the same unrealized capital gains before trading, but one of them results in a higher pre-tax wealth W_t before trading and the other in a higher tax loss carry-forward L_{t-1} (in absolute value), the strategy with the higher pre-tax wealth is at least as good as the strategy with the higher tax loss carry-forward, if for every τ_i extra dollars of wealth W_t of the first strategy, the second strategy does not have more than one dollar of extra tax loss carry-forward L_{t-1}. If $A \succeq B$ denotes "A is at least as good as B", then this finding can also be expressed as

$$\begin{pmatrix} W_t = \tau_i \\ L_{t-1} = 0 \end{pmatrix} \succeq \begin{pmatrix} W_t = 0 \\ L_{t-1} = -1 \end{pmatrix}. \tag{A.1}$$

An investor endowed with one dollar of unrealized capital gains $U_t = 1$ at the beginning of period t before trading and one dollar of tax loss carry-forward $L_{t-1} = -1$ can use the tax loss carry-forward in two ways. It can either be used to realize the capital gain without having to pay the capital gains tax or it can be used to generate a net capital loss at time t and thus to earn a tax rebate of τ_i dollars if $M \geq 1$. As argued above, the value of the tax rebate is at least as high as the future tax burden due to the unrealized capital gain when invested in the risk-free asset. Accordingly, realizing the net capital loss to increase W_t and leaving U_t unrealized is a dominating tax-timing strategy if the investor does not want to decrease her equity exposure immediately.

An investor who is neither endowed with that dollar of unrealized capital gain nor that dollar of tax loss carry-forward can be considered an investor who has realized the capital gain and used the tax loss carry-forward to avoid the capital gains tax payment. However, the investor then lacks the desirable opportunity of offsetting the tax loss carry-forward from other income. Hence:

$$\begin{pmatrix} U_t = 1 \\ L_{t-1} = -1 \end{pmatrix} \succeq \begin{pmatrix} U_t = 0 \\ L_{t-1} = 0 \end{pmatrix}. \tag{A.2}$$

The unrealized capital gain U_t is the product of the number of units q_{t-1} of the risky asset and the unrealized capital gain $P_t - P^*_{t-1}$ per unit of the risky asset. Then U_t is given by

$$U_t = q_{t-1} \cdot \left(P_t - P^*_{t-1}\right). \tag{A.3}$$

Equation (A.2) only depends on U_t. In particular, it is independent from the composition of U_t, i.e. whether a given capital gain U_t results from a high equity exposure with a small capital gain or a small equity exposure with a high capital gain.

Each dollar of unrealized capital gains results in a tax burden of τ_g dollars when realizing them. When $U_t = 1$ and $W_t = \tau_g$, the τ_g dollars of wealth allow for earning profits on these τ_g dollars. By investing the τ_g dollars in the risk-free asset, its value is always at least as high as the present unrealized capital gain. Consequently:

$$\begin{pmatrix} W_t = \tau_g \\ U_t = 1 \end{pmatrix} \succeq \begin{pmatrix} W_t = 0 \\ U_t = 0 \end{pmatrix}. \tag{A.4}$$

A.1.2 The Optimal Tax-Timing Strategy

In the following, the investment decision of an investor endowed with an initial tax loss carry-forward of L_{t-1} is considered. We assume that the return on the risky asset consists only of capital gains, i.e. the asset does not pay any dividend or interest.[11] If the investor does not trade the risky asset, the purchase price of the risky asset does not change and $P_t^* = P_{t-1}^*$. If the investor purchases the asset at price P_t, its purchase price is given by $P_t^* = P_t$.

If that loss does not exceed M in absolute value, that is, if $-\left(P_t - P_{t-1}^*\right) \leq M$, the classical result of Constantinides (1983) applies and the investor should sell the asset to realize that loss. If, however, the net capital loss exceeds M, i.e. $P_t - P_{t-1}^* < -M$, the preconditions under which the result of Constantinides (1983) is derived are no longer full-filled.

In the following it is shown that it remains optimal to realize an unrealized loss immediately even though a potential tax loss carry-forward is a less attractive compensation than a tax refund, and the purchase price P_t^* is increased from P_{t-1}^* to P_t, thereby increasing the risk of getting "locked in" in forthcoming periods.

To prove that the optimal tax-timing strategy is to realize losses immediately, we consider three strategies of an investor who is initially endowed with one unit of the risky asset at time t acquired at price P_{t-1}^* who wants to hold one unit of the risky asset from time t to $t+1$.[12] Since all other strategies are linear combinations of these three strategies, it suffices to show that one of these strategies is at least as good as the two other strategies.[13] First, the investor can sell the risky asset to realize the unrealized net capital loss, and immediately repurchase it (strategy one). Second, the investor can avoid transactions (strategy two). Third, the investor can sell just enough of the risky asset to realize the maximum loss M that can be offset against other income and repurchase the sold amount of the risky asset immediately (strategy three). In case the tax loss carry-forward L_{t-1} exceeds the upper limit M qualifying for tax rebates or $M = 0$, the investor does not even have to sell any assets to realize the desired capital loss and strategies two and three coincide.

[11] We will show later in this section that the optimal tax-timing strategy is not affected by this assumption and does not differ from the optimal tax-timing strategy with an asset that pays dividend or interest.

[12] It suffices to consider an investor who does not change the number of risky assets in her portfolio. An investor who wants to increase the number of risky assets in her portfolio faces the same tax-timing decision (with potentially different purchase prices after trading in period t) as an investor who does not change the number of risky assets in her portfolio. An investor who decreases the number of risky assets in her portfolio faces a given minimum realized net capital loss which is equivalent to a higher given initial tax loss carry-forward.

[13] To derive the optimal tax-timing strategy of an investor who additionally holds some risk-free bonds from time t to $t+1$, it suffices to analyze the case of an investor who holds only one unit of the risky asset since the return on the risk-free bonds do not have an impact on optimal tax-timing.

All other tax-timing strategies are linear combinations of these three strategies. Any strategy selling a fraction of the risky asset which is greater than that of strategy three, but less than that of strategy one results in a portfolio and a tax loss carry-forward that is a linear combination of those of strategy one and three. Accordingly, any strategy selling some fraction of the risky asset which is less that that of strategy three, but more that that of strategy two results in a portfolio and a tax loss carry-forward that is a linear combination of those of strategies two and three. To prove that strategy one is an optimal tax-timing strategy, it thus suffices to show that strategy one performs at least as good as strategies two and three.

The three strategies only differ in their purchase price of the risky asset P_t^*, the tax loss carry-forward L_t, the unrealized capital gain U_{t+1}, and the investor's wealth W_{t+1} at the beginning of period $t+1$ before trading.

When the investor follows strategy one and sells the risky asset, a net capital loss of $P_t - P_{t-1}^*$ is realized and the purchase price decreases to $P_t^* = P_t$. As $P_t - P_{t-1}^* < -M \Rightarrow P_t - P_{t-1}^* + L_{t-1} < -M$, the deductible net capital loss is

$$D_t^{(1)} = \max\left(P_t - P_{t-1}^* + L_{t-1}; -M\right) = -M. \tag{A.5}$$

Thus, the tax refund is $M\tau_i$ dollars. The remaining tax loss carry-forward is given by

$$L_t^{(1)} = P_t - P_{t-1}^* + M + L_{t-1}. \tag{A.6}$$

If the investor follows strategy two and does not do any transactions in period t, the purchase price remains at $P_t^* = P_{t-1}^*$, the deductible net capital loss is

$$D_t^{(2)} = \max\left(L_{t-1}; -M\right). \tag{A.7}$$

Thus, the tax refund is $\max\left(L_{t-1}; -M\right)\tau_i$ and the remaining tax loss carry-forward is

$$L_t^{(2)} = L_{t-1} - \max\left(L_{t-1}; -M\right). \tag{A.8}$$

If the investor follows strategy three, an investment strategy is chosen such that the net deductible capital loss is given by

$$D_t^{(3)} = -M \tag{A.9}$$

and accordingly, the tax refund under strategy three is $M\tau_i$. The remaining tax loss carry-forward is

$$L_t^{(3)} = 0. \tag{A.10}$$

Let $W_t^{(i)}$ denote the pre-tax wealth in period t of strategy i ($i \in \mathbb{N}_3 \equiv \{n \in \mathbb{N} | n \leq 3\}$) before trading. Then

$$W_{t+1}^{(1)} = P_{t+1} + M\tau_i (1 + r) \tag{A.11}$$

$$W_{t+1}^{(2)} = P_{t+1} - \max (L_{t-1}; -M) \tau_i (1 + r) \tag{A.12}$$

$$W_{t+1}^{(3)} = P_{t+1} + M\tau_i (1 + r). \tag{A.13}$$

If the investor follows tax-timing strategy three, two cases have to be distinguished concerning the amount of the risky asset to be sold. First, if $\max (L_{t-1}; -M) = -M$, then the tax loss carry-forward L_{t-1} from period $t-1$ suffices to realize the desired net capital loss in period t. In this case, the investor does not have to do any transactions, and strategies two and three coincide. For case three, it thus suffices to consider the case that $L_{t-1} > -M$ in which the investor still has to sell some fraction of the risky assets. The amount of the risky assets the investor has to sell is then equivalent to a fraction f of the risky asset, such that $-M = f\left(P_t - P_{t-1}^*\right) + L_{t-1} \Leftrightarrow f = \frac{-M - L_{t-1}}{P_t - P_{t-1}^*}$.

Let $U_t^{(i)}$ denote the unrealized capital gains (or losses) in period t of strategy i ($i \in \mathbb{N}_3$) before trading. Then

$$U_{t+1}^{(1)} = P_{t+1} - P_t \tag{A.14}$$

$$U_{t+1}^{(2)} = P_{t+1} - P_{t-1}^* \tag{A.15}$$

$$U_{t+1}^{(3)} = P_{t+1} - P_{t-1}^* + L_{t-1} + M. \tag{A.16}$$

Table 5 summarizes the properties of the three tax-timing strategies.

Table 5 about here

With equation (A.2), it holds in case that $\max (L_{t-1}; -M) = L_{t-1}$ for the relation between strategies one and three that

$$\begin{pmatrix} W_{t+1}^{(1)} \\ U_{t+1}^{(1)} \\ L_t^{(1)} \end{pmatrix} = \begin{pmatrix} P_{t+1} + M\tau_i\,(1+r) \\ P_{t+1} - P_t \\ P_t - P_{t-1}^* + M + L_{t-1} \end{pmatrix} \succeq \begin{pmatrix} P_{t+1} + M\tau_i\,(1+r) \\ P_{t+1} - P_{t-1}^* + M + L_{t-1} \\ 0 \end{pmatrix} \succeq \begin{pmatrix} W_{t+1}^{(3)} \\ U_{t+1}^{(3)} \\ L_t^{(3)} \end{pmatrix}. \quad \text{(A.17)}$$

Thus, strategy one is at least as good as strategy three if $\max(L_{t-1}; -M) = L_{t-1}$. The economic reason for this finding is that the tax loss carry-forward of strategy one can be more easily converted to wealth and earn profits than the lower unrealized capital gain of strategy three. In case that $\max(L_{t-1}; -M) = -M$ strategies two and three coincide. To show that strategy one is an optimal tax-timing strategy it remains to show that strategy one is at least as good as strategy two.

For the relation between strategies one and two, we distinguish two cases. First, if $M + L_{t+1} \leq 0 \Leftrightarrow \max(L_{t-1}; -M) = -M$, it holds with Equation (A.1) that

$$\begin{aligned} \begin{pmatrix} W_{t+1}^{(1)} \\ U_{t+1}^{(1)} \\ L_t^{(1)} \end{pmatrix} &= \begin{pmatrix} P_{t+1} + M\tau_i\,(1+r) \\ P_{t+1} - P_t \\ P_t - P_{t-1}^* + M + L_{t-1} \end{pmatrix} \succeq \begin{pmatrix} P_{t+1} + M\tau_i\,(1+r) \\ P_{t+1} - P_{t-1}^* \\ L_{t-1} + M \end{pmatrix} \\ &= \begin{pmatrix} P_{t+1} - \max(L_{t-1}; -M)\,\tau_i\,(1+r) \\ P_{t+1} - P_{t-1}^* \\ L_{t-1} - \max(L_{t-1}; -M) \end{pmatrix} \succeq \begin{pmatrix} W_{t+1}^{(2)} \\ U_{t+1}^{(2)} \\ L_t^{(2)} \end{pmatrix}. \end{aligned} \quad \text{(A.18)}$$

Second, if $M + L_{t-1} > 0 \Leftrightarrow \max(L_{t-1}; -M) = L_{t-1}$ an argument similar to that of equation (A.18) applies. Thus, strategy one is at least as good as strategy two, which shows that independent from the realization of P_{t+1}, strategy one always performs at least as good as strategies two and three. Furthermore, strategy one sometimes results in higher wealth than strategy two by allowing to earn the risk-free interest rate on the tax rebates. Hence, strategy one is an optimal tax-timing strategy and unrealized capital losses should be realized immediately.

So far it has been assumed that the risky asset does not pay any dividend. If, however, the risky asset does pay some dividend, all strategies are affected from these payments in the same way, since under all three strategies, the investor holds one unit of the risky asset and thereby receives the same amount of dividend. Hence, the results derived above also hold for risky assets whose returns consist of both capital gains and dividend payments.

In a tax-system of the ND type the tax loss carry-forward can only be offset against realized capital gains and can thus never be worth more than τ_g dollars. The proof for the LD case applies by replacing τ_i by τ_g and considering the special case with $M = 0$.

A.2 The ND Multiple-Asset Case

In tax-systems of the ND type it remains an optimal tax-timing strategy to realize losses immediately in the multiple-asset case. This is due to the fact that the value of one extra unit of tax loss carry-forward is always at least as big as one unit of lower unrealized capital gains since the tax loss carry-forward can always be used to offset these capital gains. In contrast to unrealized capital gains a tax loss carry-forward can not only be used to offset unrealized capital gains from the unit of stock the tax loss carry-forward has been generated from, but can also be offset from other realized capital gains.

To illustrate this point, we consider an investor who is endowed with 10 units of a risk asset with purchase price of 20 and current market price of 10. We further assume the market price of the asset to increase to 40 next period in which – for whatsoever reason – the investor wants to sell 5 units of the stock. In case the investor does not realize the loss, her taxable capital gain in the next period is $5 \cdot (40 - 20) = 100$ her remaining tax loss carry-forward is zero, and the purchase price of the remaining units of the stock is 20. In case the investor realizes the loss, she is endowed with a tax loss carry-forward of 100 and her purchase price decreases to 10. In the next period she can make use of the tax loss carry-forward such that her taxable capital gains are $5 \cdot (40 - 10) - 100 = 50$. Realizing the loss provides the investor with the opportunity of offsetting the losses from 10 units of the stock from the capital gains of 5 units of the stock, while not realizing the loss is equivalent to restricting the offsetting of losses of one unit of the stock to one and the same unit of the stock.

B Appendix B - Rewritten Optimization Problem

For the numerical solution of the optimization problem (9) to (12) we normalize with beginning-of-period-wealth W_t. Let $s_t \equiv \frac{q_{t-1}P_t}{W_t}$ denote the fraction of the investor's beginning-of-period-wealth before trading invested into equity, $\alpha_t \equiv \frac{q_t P_t}{W_t}$ the investor's fraction of beginning-of-period-wealth allocated to equity after trading, $b_t' \equiv \frac{b_t}{W_t}$ the fraction of the beginning-of-period-wealth allocated to risk-free bonds after trading, $c_t \equiv \frac{C_t}{W_t}$ the consumption-wealth-ratio, $p_{t-1}^* \equiv \frac{P_{t-1}^*}{P_t}$ the investors basis-price-ratio, $t_t \equiv \frac{T_t}{W_t}$ the fraction of the investor's beginning-of-period-wealth that is taxable at the capital gains tax rate, $l_{t-1} \equiv \frac{L_{t-1}}{W_t}$ the fraction of the investor's tax loss carry-forward to beginning-of-period-wealth, $d_t \equiv \frac{D_t}{W_t}$ the amount deductible to beginning-of-period-wealth, $g_t \equiv \frac{P_{t+1}}{P_t} - 1$ the capital gain on the stock in period t, and

$$R_t \equiv \frac{\alpha_t (1+d)(1+g_t) + b_t' R}{\alpha_t + b_t'} \tag{B.1}$$

the gross nominal return on the investor's portfolio after trading in period t and payment of taxes on dividends and interest, but before payment of capital gains taxes. Defining $v_t(x_t) \equiv \frac{V_t(X_t)}{W_t^{1-\gamma}}$ to be the normalized value function and $\rho_t \equiv \frac{W_{t+1}}{W_t(1+i)}$ to be the investor's real growth of wealth before capital gains taxes, the investor's optimization problem can be rewritten as

$$v_t(x_t) = \max_{c_t, \alpha_t . \theta_t} \left[f(t)U(c_t) + f(t)\beta \mathbb{E}\left[v_{t+1}(x_{t+1}) \rho_t^{1-\gamma} \right] + (1 - f(t)) \frac{\beta \left(1 - \beta^H\right)}{1-\beta} U(A_H) \right] \tag{B.2}$$

s.t.

$$1 = \tau_g t_t + \alpha_t + b_t' + c_t - \tau_i d_t \qquad t = 0, \dots, T-1 \tag{B.3}$$

$$\rho_t = \frac{(1 - \tau_g t_t + \tau_i d_t - c_t) R_t}{1+i} \qquad t = 0, \dots, T-1 \tag{B.4}$$

$$\alpha_t, b_t' \geq 0 \qquad t = 0, \dots, T-1 \tag{B.5}$$

in which t_t and d_t are given by

$$t_t = \max(\delta_t + l_{t-1}; 0) \tag{B.6}$$

$$d_t = \min\left(-\min(\delta_t + l_{t-1}, 0), \frac{M}{W_t} \right). \tag{B.7}$$

The fraction of realized gains to beginning-of-period-wealth δ_t and l_t are given by

$$\begin{aligned} \delta_t \equiv \frac{G_t}{W_t} &= \left(\chi_{\{1>p^*_{t-1}\}} \left(\max\left(s_t - \alpha_t, 0\right) + \min\left(s_t, \alpha_t\right) \theta_t\right) + \chi_{\{1 \le p^*_{t-1}\}} s_t\right) \left(1 - p^*_{t-1}\right) & \text{(B.8)} \\ l_t &= \min\left(\delta_t + l_{t-1}, 0\right) + d_t & \text{(B.9)} \end{aligned}$$

and p^*_t is given by

$$p^*_t = \begin{cases} \frac{(s_t - \max(s_t - \alpha_t, 0) - \min(s_t, \alpha_t)\theta_t) p^*_{t-1} + \max(\alpha_t - s_t, 0) + \min(s_t, \alpha_t)\theta_t}{\alpha_t (g_t + 1)} & \text{if } p^*_{t-1} < 1 \\ \frac{1}{g_t + 1} & \text{if } p^*_{t-1} \geq 1. \end{cases} \tag{B.10}$$

At time T the investor's normalized value function takes the value

$$v_T = \frac{\beta\left(1 - \beta^H\right)}{1 - \beta} U\left(A_H\right) \tag{B.11}$$

in all states due to the forgiveness of capital gains when being bequeathed. The vector x_t of state variables at time t of the normalized optimization problem is given by

$$x_t = [p^*_{t-1}, s_t, l_{t-1}, m_t] \tag{B.12}$$

in which $m_t \equiv \frac{M}{W_t}$. For values of the state-variables that are not on the grid, we perform cubic spline interpolation. The integral in the expectation of the investor's utility is computed using Gaussian quadrature.

Base-Case Parameter Values

Description	Parameter	Value
Risk-aversion	γ	3
Length of investment horizon	T	80
Number of years annuity beneficiary	H	60
Utility discount factor	β	0.96
Post-tax dividend rate	d	1.28%
Expected pre-tax capital gains rate stock	μ	7%
Standard deviation of capital gains rate stock	σ	20.7%
Post-tax interest payment of bond	r	3.84%
Inflation rate	i	3.5%
Tax rate on interest and dividend income	τ_i	36%
Tax rate on realized capital gains	τ_g	20%

Table 1: This table reports our parameter values used in the base-case.

Comparison of Tax-Systems

	ST	ND	LD
Decision variables	c_t, α_t	c_t, α_t	c_t, α_t, θ_t
State variables	s_t, p^*_{t-1}	s_t, p^*_{t-1}, l_{t-1}	$s_t, p^*_{t-1}, l_{t-1}, m_t$

Table 2: This table shows the state-variables an optimal consumption investment decision depends on and the decision variables the investor has to choose in order to determine her consumption investment decision. c_t denotes the investor's consumption-wealth ratio, α_t denotes the investors equity exposure relative to her beginning-of-period wealth, θ_t denotes the fraction of capital gains per unit of equity the investor cuts short, s_t is the investor's initial equity exposure, p^*_{t-1} is the investor's initial basis-price-ratio, l_{t-1} is the investor's initial tax loss carry-forward before trading in period t, m_t is the fraction of the investor's losses relative to her beginning-of-period-wealth qualifying for tax rebate payments.

Simulation Analysis - Base Case Setting

Percentile	α_t in % LD[4]	α_t in % LD[5]	α_t in % ND	α_t in % ST	θ_t in % LD[4]	θ_t in % LD[5]	p^*_{t-1} LD[4]	p^*_{t-1} LD[5]	p^*_{t-1} ND	p^*_{t-1} ST	l_{t-1} in % LD[4]	l_{t-1} in % LD[5]	l_{t-1} in % ND	W_t in thousand LD[4]	W_t in thousand LD[5]	W_t in thousand ND	W_t in thousand ST
Panel A - Age 30																	
1	45.2	32.7	26.9	30.9	100	39	0.552	0.552	0.162	0.162	0	-20.2	-27.4	10	86	9	9
10	49.5	35.8	29.3	33.6	100	65	0.736	0.733	0.289	0.289	0	-9.2	-10.4	12	108	10	10
50	55.8	41.3	34.0	38.4	100	88	0.955	0.952	0.533	0.533	0	0	0	17	143	13	13
90	60.9	44.7	44.3	44.3	100	98	1.238	1.237	0.925	0.912	0	0	0	23	189	17	18
99	63.1	48.1	47.2	47.0	100	98	1.651	1.651	1.238	1.238	0	0	0	28	236	22	24
Mean	53.5	40.9	35.7	38.5	100	84	0.977	0.934	0.580	0.580	0	-2.8	-3.0	17	147	13	14
Std	4.2	3.4	5.4	4.1	1	15	0.183	0.183	0.254	0.254	0	4.7	5.7	4	32	3	3
Panel B - Age 60																	
1	39.0	29.6	26.0	26.7	8	0	0.552	0.061	0.009	0.009	-6.4	-17.8	-17.6	12	117	12	12
10	41.8	33.8	30.4	32.6	100	0	0.736	0.180	0.031	0.031	-0.9	-4.8	0	20	172	17	17
50	46.1	39.7	43.0	43.5	100	0	0.955	0.638	0.129	0.129	0	0	0	34	292	28	30
90	52.8	46.8	51.1	51.0	100	30	1.238	1.238	0.955	0.955	0	0	0	61	571	56	59
99	58.6	50.7	52.6	52.5	100	81	1.651	1.238	0.955	0.955	0	0	0	94	1,046	104	109
Mean	46.7	39.8	41.7	42.5	96	9	0.969	0.615	0.209	0.208	-0.4	-1.3	-0.6	38	342	33	35
Std	4.3	4.7	7.7	6.9	17	17	0.182	0.317	0.217	0.215	1.4	3.6	3	17	187	19	20
Panel C - Age 90																	
1	38.2	32.9	32.2	33.0	0	0	0.113	0.006	0.001	0.001	-3.9	-3.2	-5.4	18	180	17	17
10	41.5	38.8	38.3	40.5	0	0	0.251	0.025	0.005	0.005	0	0	0	36	315	28	29
50	47.7	53.8	60.4	61.0	0	0	0.558	0.133	0.036	0.036	0	0	0	76	761	65	68
90	58.2	67.4	76.7	76.8	0	0	0.955	0.518	0.292	0.288	0	0	0	179	2,115	181	190
99	66.9	73.7	80.9	80.9	0	0	1.238	0.955	0.831	0.821	0	0	0	387	5,156	441	462
Mean	49.0	53.5	58.9	59.7	0	0	0.581	0.211	0.104	0.103	-0.1	-0.1	-0.2	97	1,053	91	96
Std	6.5	10.5	13.8	15.2	0	0	0.266	0.221	0.168	0.165	0.8	1.1	1.6	77	1,053	90	94

Table 3: This table shows the evolution of the investor's optimal investment strategy (her optimal equity exposure α_t as well as her optimal cutting of gains θ_t in tax-systems of the LD type) and her basis-price-ratio before trading p^*_{t-1}, her tax loss carry-forward before trading l_{t-1} and her wealth-level W_t in thousand dollars over the life cycle in tax-systems of the LD, the ND and the ST case. The values presented here, are the results of 50,000 simulations on the optimal paths. Since the investor's initial wealth has an impact on optimal investment decisions in tax-systems of the LD type, we run simulations for both an investor who is endowed with an initial wealth-level of \$ 10,000 (columns LD^4) and an investor who is endowed with an initial wealth-level of \$ 100,000 (columns LD^5).

Simulation Analysis - Mandatory Realization of Capital Gains when Bequeathed

	α_t in %				θ_t in %		p^*_{t-1}				l_{t-1} in %			W_t in thousand			
Percentile	LD^4	LD^5	ND	ST	LD^4	LD^5	LD^4	LD^5	ND	ST	LD^4	LD^5	ND	LD^4	LD^5	ND	ST
Panel A - Age 30																	
1	41.0	32.2	27.2	31.4	100	20	0.552	0.552	0.162	0.162	0	-19.7	-27.5	10	87	9	9
10	46.4	35.2	29.7	33.7	100	50	0.736	0.732	0.289	0.289	0	-9.2	-10.4	13	109	10	10
50	53.5	40.4	34.1	38.6	100	80	0.955	0.947	0.533	0.533	0	0	0	17	145	13	13
90	59.4	44.5	44.6	44.5	100	99	1.238	1.237	0.927	0.912	0	0	0	22	190	17	18
99	62.5	47.4	47.1	46.9	100	100	1.651	1.651	1.238	1.238	0	0	0	27	238	23	24
Mean	53.2	40.2	35.9	38.6	100	79	0.973	0.948	0.580	0.580	0	-2.7	-3.0	17	148	13	14
Std	4.9	3.5	5.3	4.0	0	19	0.183	0.183	0.254	0.254	0	4.5	5.8	4	32	3	3
Panel B - Age 60																	
1	38.2	29.9	26.7	27.6	31	0	0.552	0.155	0.009	0.009	-4.7	-21.1	-18.4	13	121	12	12
10	39.5	33.1	30.7	32.8	100	0	0.736	0.502	0.031	0.031	0	-8.0	0	20	178	17	17
50	41.2	37.7	41.7	42.0	100	38	0.955	0.863	0.131	0.131	0	0	0	33	287	29	30
90	50.0	40.5	47.4	47.4	100	92	1.238	1.175	0.518	0.524	0	0	0	57	505	56	59
99	58.1	46.6	48.4	48.4	100	100	1.651	1.238	0.955	0.955	0	0	0	89	921	102	107
Mean	43.1	37.2	40.3	41.1	98	40	0.969	0.830	0.213	0.214	-0.2	-2.2	-0.6	36	324	34	35
Std	4.5	3.1	6.5	5.7	11	31	0.182	0.521	0.220	0.221	1.0	4.5	3.1	16	157	18	19
Panel C - Age 90																	
1	27.2	28.0	28.1	32.2	0	0	0.552	0.015	0.001	0.001	-9.5	-11.5	-6.9	19	176	17	18
10	35.5	32.0	33.1	33.8	23	0	0.710	0.072	0.006	0.006	-2.0	-1.3	0	34	299	29	31
50	40.2	39.5	40.6	40.7	71	0	0.939	0.347	0.059	0.061	0	0	0	67	606	63	67
90	43.1	44.3	44.7	44.6	100	0	1.238	0.848	0.380	0.384	0	0	0	128	1,400	151	159
99	49.0	45.6	45.0	45.0	100	74	1.494	1.238	0.878	0.897	0	0	0	226	2,941	320	336
Mean	39.8	38.5	39.8	40.1	65	2	0.937	0.408	0.137	0.139	-0.7	-0.6	-0.2	76	764	81	85
Std	3.7	4.6	4.5	4.0	31	11	0.185	0.295	0.188	0.190	1.8	2.3	1.6	43	566	63	66

Table 4: This table shows the evolution of the investor's optimal investment strategy (her optimal equity exposure α_t as well as her optimal cutting of gains θ_t in tax-systems of the LD type) and her basis-price-ratio before trading p^*_{t-1}, her tax loss carry-forward before trading l_{t-1} and her wealth-level W_t in thousand dollars over the life cycle in tax-systems of the LD, the ND and the ST case when unrealized are taxable when being bequeathed. The values presented here, are the results of 50,000 simulations on the optimal paths. Since the investor's initial wealth has an impact on optimal investment decisions in tax-systems of the LD type, we run simulations for both an investor who is endowed with an initial wealth-level of \$ 10,000 (columns LD^4) and an investor who is endowed with an initial wealth-level of \$ 100,000 (columns LD^5).

Comparison of Investment Strategies

	strategy one	strategy two	strategy three
W_{t+1}	$P_{t+1} + M\tau_i\,(1+r)$	$P_{t+1} - \max\,(L_{t-1}; -M)\,\tau_i\,(1+r)$	$P_{t+1} + M\tau_i\,(1+r)$
U_{t+1}	$P_{t+1} - P_t$	$P_{t+1} - P^*_{t-1}$	$P_{t+1} - P^*_{t-1} + M + L_{t-1}$
L_t	$P_t - P^*_{t-1} + M + L_{t-1}$	$L_{t-1} - \max\,(L_{t-1}; -M)$	0

Table 5: This table shows the investor's total wealth W_{t+1}, her unrealized capital gains U_{t+1} and her tax loss carry-forward L_t when following strategy one, two or three.

Optimal Investment Policy

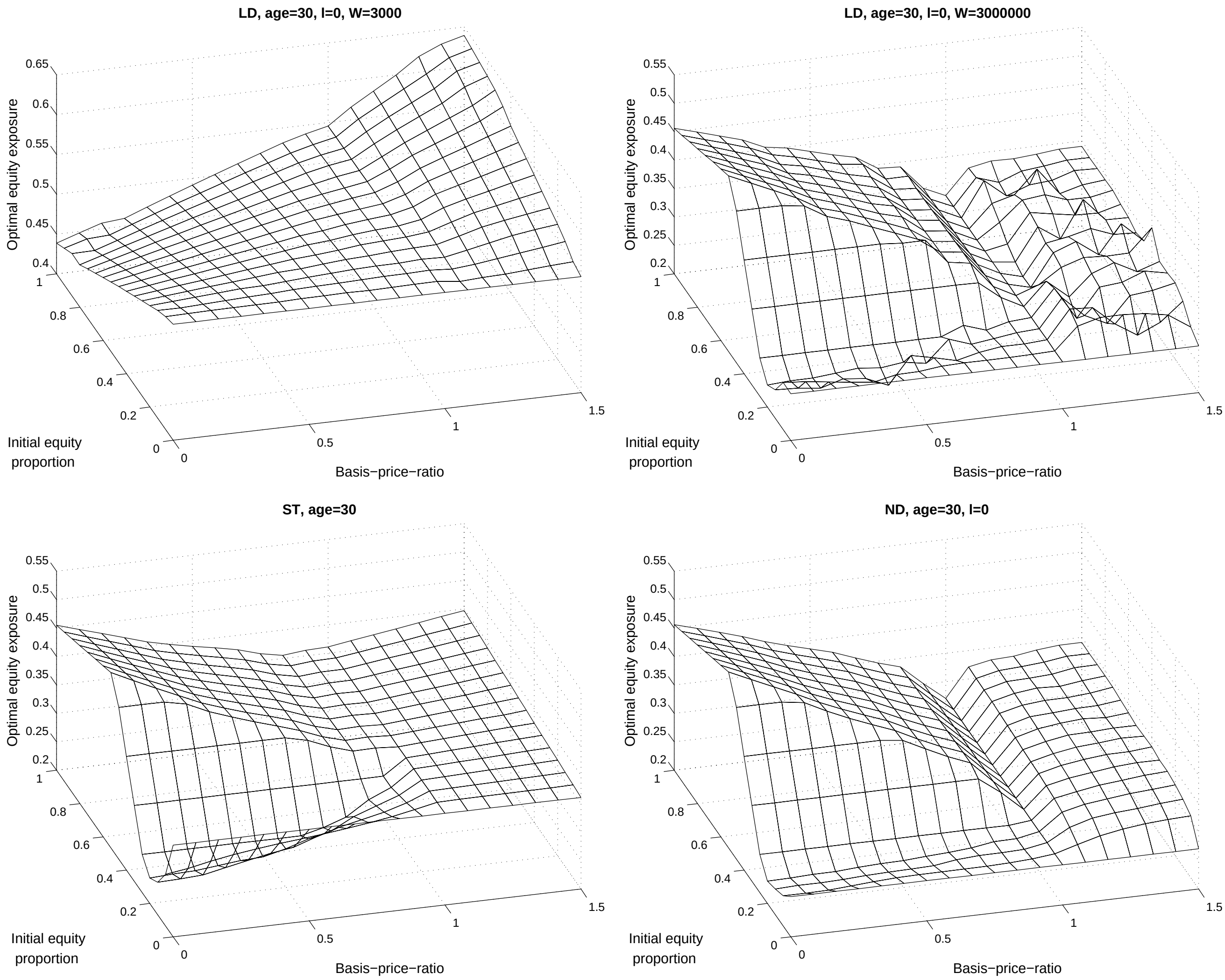

Figure 1: This figure shows the relation between the investor's optimal equity exposure and her initial equity exposure as well as her initial basis-price-ratio for an investor at age 30 for an investor who is not endowed with an initial tax loss carry-forward. The upper left graph shows the optimal equity exposure of an investor in a tax-system of the LD type who is endowed with an initial wealth of \$ 3,000, the upper right graph for an investor endowed with an initial wealth of \$ 3,000,000. The lower left graph shows the optimal equity exposure of an investor in a tax-system of the ST type, the lower right graph depicts the optimal equity exposure for an investor trading in a tax-system of the ND type.

Optimal Investment Policy and Wealth-Level

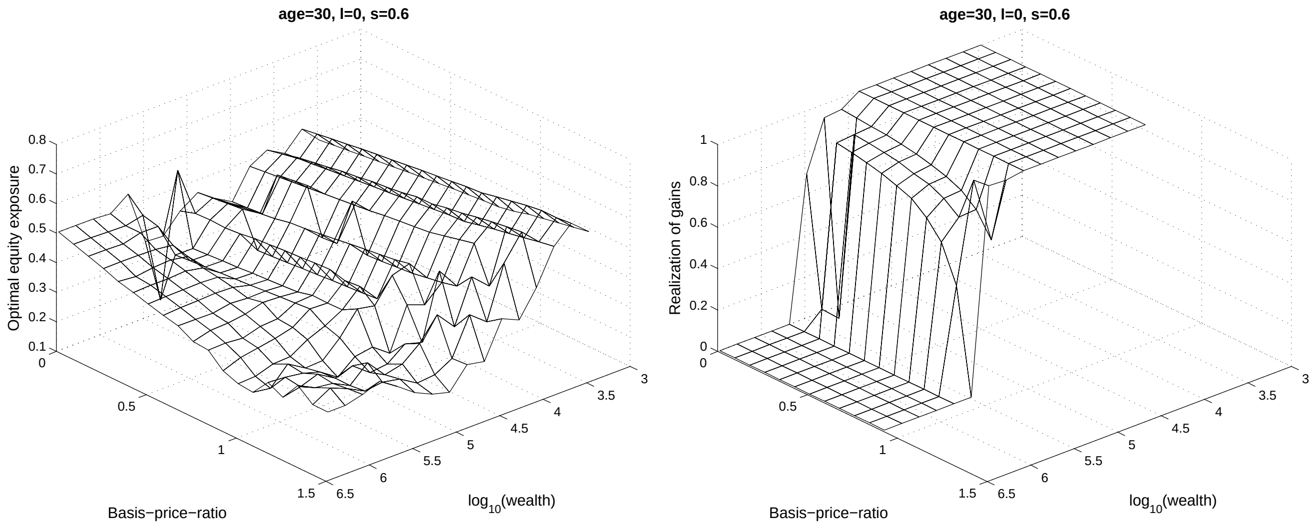

Figure 2: This figure depicts how the investor's optimal equity exposure (left graph) and her optimal cutting of unrealized capital gains (right graph) depends on her basis-price-ratio and her initial wealth level. We consider an investor who is not endowed with an initial tax loss carry-forward $l = 0$ and whose initial equity exposure is $s = 60\%$.

Optimal Investment Policy with Tax Loss Carry-Forward

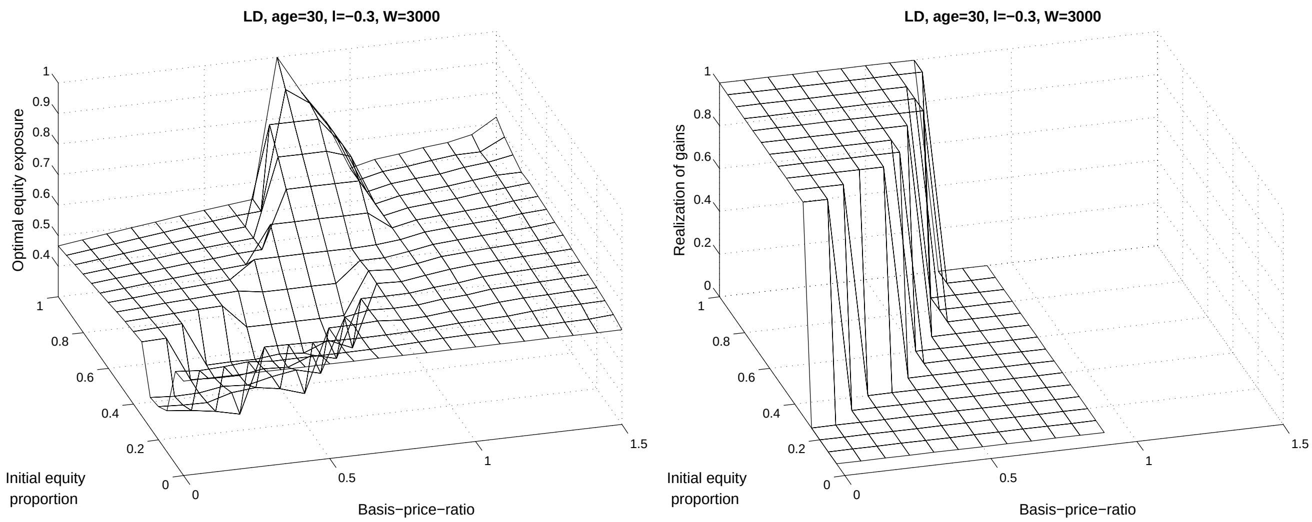

Figure 3: This figure depicts the relation between the investor's equity exposure (left graph) as well as her optimal cutting of unrealized capital gains (right graph) depending on her initial equity exposure and her initial basis-price-ratio for an investor at age 30, who is endowed with an initial tax loss carry-forward of $l = -30\%$ of her initial wealth and a total wealth of \$ 3,000.

Effective Tax Rate for Value of Tax Loss Carry-Forward

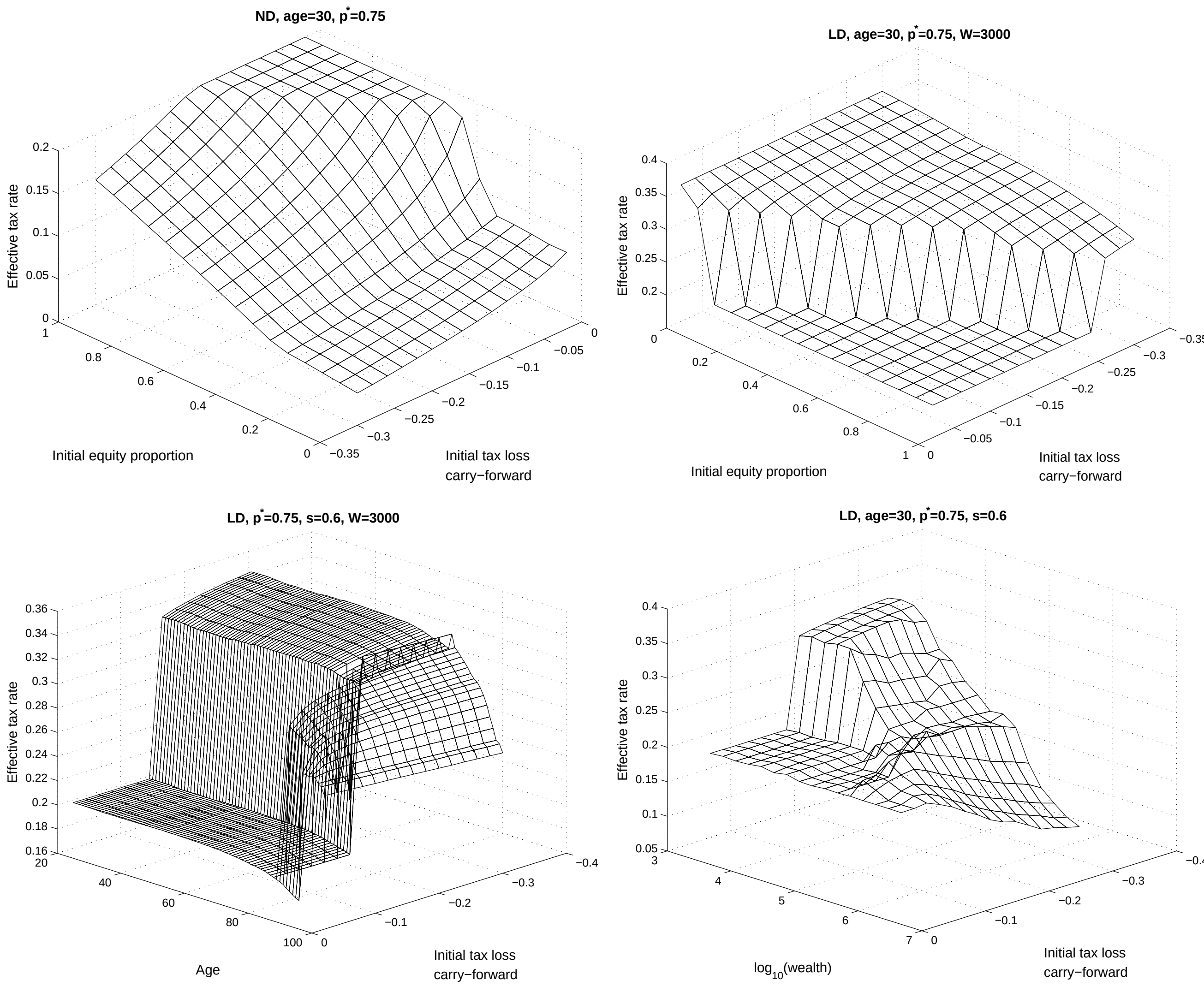

Figure 4: This figure depicts the relation between the effective tax rate applicable to a tax loss carry-forward that makes an investor indifferent between receiving an immediate tax rebate payment at that tax rate and keeping the tax loss carry-forward to offset it from future realized capital gains or future realized capital gains for an investor being endowed with an initial basis-price-ratio of $p^* = 0.75$, indicating that the investor is endowed with unrealized capital gains. The upper left graph shows the impact of the investor's initial equity exposure and the level of her tax loss carry-forward on her effective tax rate at age 30 in a tax-system of the ND type, the upper right graph in a tax-system of the LD type for an investor being endowed with an initial wealth-level of \$ 3,000. The lower left graph depicts the impact of the investor's age and her tax loss carry-forward for an investor with an initial equity exposure of 60% and an initial wealth-level of \$ 3,000 in a tax-system of the LD type. the lower right graph contains information on the relation between the investor's wealth-level and her initial tax loss carry-forward for an investor at age 30 with an initial equity exposure of 60% in a tax-system of the LD type.