UNOBSERVABLE HETEROGENEITY IN DIRECTED SEARCH

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ABSTRACT. This paper provides a directed search model designed to explain the residual part of wage variation left over after the impact of education and other observable worker characteristics has been removed. Workers have private information about their characteristics at the time they apply for jobs. Firms can observe these characteristics once workers apply, and hire the worker with the characteristic that they like. The paper focuses on the case in which firms aren't able to condition their wage offers on these characteristics. The paper shows how to extend directed search arguments to deal with arbitrary distributions of worker and firm types. The paper then illustrates how data on the relationship between exit wage and unemployment duration can be used to identify the unobserved distributions of worker and firm types. The model also has testable predictions. For example, certain easily checked properties of the offer distribution of wages imply that workers who are hired by the highest wage firms should also be the workers who have the shortest unemployment duration. This is in strict contrast to the usual directed search story in which high wages are always accompanied by higher probability of unemployment.

1. Introduction

One shortcoming of most directed search models compared to models with purely random matching is the fact that they are based on strong symmetry assumptions. Even papers that are explicitly designed to allow for differences among traders (for example Shimer (2005) and Shi (2002)) restrict themselves to distributions of wages and types with finite support. At a minimum, this makes it difficult for them to match with econometric data that tends to involve continuous distributions. Furthermore, the empirical content of these models applies to the relationship between observables. For example, workers with more education will receive higher wages. Apart from differences in these observables, all traders are assumed to be the same, leaving the models mute about the large part of the variation in wages that is apparently unrelated to observables.

The purpose of this paper is to provide an alternative version of the directed search model that can be used to explain this residual variation. In the model, this variation is due to unobservable worker characteristics, and to the (equally unobservable) value of these characteristics to firms. The empirical content of the model comes from the relationship it establishes between the offer distribution of wages and the relationship between wages at which workers leave unemployment and their average search duration.[1] Addison, Centeno, and Portugal (2004) present evidence to suggest that exit wages and unemployment duration are negatively correlated. In a more standard directed search terminology, this means that employment probability seems to be positively correlated with wage. The evidence is not strong, but it is striking that it provides no support at all for the classic prediction of directed search - that workers who apply at high wage firms will have a lower probability of employment.

It is shown in the model below that any systematic tendency in this data is tied to the properties of the wage offer distribution. The relationship between exit wage and unemployment duration is driven by two considerations. The first is completely intuitive - higher wage firms will tend to hire workers whose quality is higher, and these workers will tend to be more likely to find jobs no matter where they apply. This is confounded by the possibility that higher quality workers will tend to apply where there are a lot of other high quality workers. This is where the directed search model plays a role since it ties down the application strategy for workers of different qualities. The characterization of the equilibrium application strategy provided below makes it possible to provide a readily checked property of the offer distribution that will determine the relationship between exit wage and duration. Assuming that the wage offer distribution is has the usual skewed (log normal) shape, it will support an inverse relationship between exit wage and duration provided the density doesn't decline too rapidly to the right of its peak.

A second phenomena that the model in this paper can be used to address is duration dependence (Machin and Manning (1999) or Addison, Centeno, and Portugal (2004)). Workers who have been unemployed for a long time tend to wait longer for a job offer than workers who

[1]The model in the paper is static, so the actual relationship is between the probability of employment and the wage at which a worker is hired. In the steady state of such a model, the expected duration of unemployment is just the reciprocal of the probability of employment.

are newly unemployed. Directed search models typically assume that all workers are the same or differ only in ways that are observable to an outsider. Workers apply to high wage firms only because their equilibrium mixed strategies require them to do so with some positive probability. When workers' application behavior is driven by an underlying characteristic, workers who adopt a risky application strategy in one period will tend to persist in this behavior. Workers who have been unemployed for a long time are more likely to be workers whose types support a risky application strategy. Duration dependence is then simply a consequence of the workers underlying characteristic and does not reflect any kind of discouragement effect. Again, the model below captures this phenomenon in a simple way.

Finally, the model below illustrates how data on the wage distribution and the relationship between exit wage and unemployment duration can be used to attribute wage variation to either variation in the unobservable characteristics of workers, or to variation in firms valuation of these characteristics.

Most of the paper is concerned with the case where there is a continuum of workers and firms. However, the basic logic of directed search involves mixed application strategies where workers apply with higher probability at higher wage firms. It isn't at all clear what mixed application strategies mean when there are a continuum of different firms to choose from all offering different wages. To get some insight into this process the paper starts with the analysis of the equilibrium of the application sub-game that occurs after a finite number of firms have posted their wages for a finite number of workers. By taking limits of the equilibrium payoffs as the number of firms and workers grow large, this approach defines payoffs in a continuum model in which workers and firms best reply to a distribution of wage offers, and a *reservation wage* application strategy for workers. The payoff functions defined by these limits define a large game that captures the logic of directed search when there is a continuum of different traders on each side of the market. The construction of this large game is one of the central contributions of the paper. The equilibrium arguments are based on an adaptation of the argument in Peters and Severinov (1997), and resemble the mixed equilibrium that were characterized in Shi (2002) and Shimer (2005), albeit under much different assumptions. There is a unique symmetric equilibrium in which workers randomize over the wages at which they submit applications. In any finite directed search game, this application strategy is conceptually straightforward, but complex since it involves a potentially large number of different application probabilities for each of the different wages being offered in equilibrium. The paper shows the sense in which this application strategy converges to a simple reservation wage rule similar to the one in Shimer (2005) as the number of traders becomes large.

One other paper that deals with a continuum of types is Lang and Manove (2006). They describe an equilibrium with two workers and two firms in which the workers' types are drawn from a continuous distribution as they are here. They show that firms offer the same wage provided their output does not depend on the type of the worker that they hire. They point out the distinction between this result and the discrimination results when there are two types as in Lang, Manove, and Dickens (1999).

The paper begins with a description of search equilibrium with a finite number of firms and workers. This section is an attempt to motivate the reservation wage strategy that workers use in the large game. The paper then presents a basic set of limit theorems that are used to define the payoff functions in the large search game. The equilibrium in the large search game is characterized in Section 4, and the relationship between wage offer distributions and search duration is analyzed. The final section concludes. Detailed proofs of the limit theorems are contained in an appendix.

2. Fundamentals

A labor market consists of M and N firms and workers respectively. To begin, assume these sets are finite and consist of m firms and n workers with $n = \tau m$. Each worker has a characteristic y contained in a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^+$. These characteristics are observable to firms once workers apply, but initially, they are private information. When M and N are finite, it will be assumed that each worker's characteristic is independently drawn from a distribution F. The distribution F is assumed to be differentiable and monotonically increasing and to satisfy the property that $\frac{F'(y)}{1-F(y)}$ is uniformly bounded.[2] It is assumed impossible for firms to reward this characteristic directly. A worker's payoff is simply the wage he receives. Workers are risk neutral.

Firms characteristics are drawn from a set $X = [\underline{x}, \overline{x}]$. The distribution of firms' characteristics will be assumed to be given by a smooth monotonically increasing function H. Each firm has a single job that

[2]One necessary condition for this to hold is that the density of F at the highest wage in the support of the distribution must be zero. This rules out, for example, a uniform distribution, or a distribution in which all workers have an identical characteristic that is commonly know.

it wants to fill. It chooses the wage that it wishes to pay the worker who fills this job. Each firm's wage is chosen from a compact interval $W \subset \mathbb{R}^+$. Payoffs for firms depend on the wage they offer and on the characteristic of the worker they hire, and. of course, on their own characteristic. The payoff for every firm is $v : W \times Y \times X \to \mathbb{R}$. It is assumed that v is jointly continuous, that the family $v(\cdot, y, \cdot)$ is an equicontinuous family of functions from Y into $\mathbb{R}$, and that the derivative of v with respect to y is bounded for all w.

The Bayesian game that determines wages and matches starts with firms simultaneously choosing their wages. After observing the wage offers each worker applies to one and only one firm. Once applications are made to the firms, each firm chooses to hire the worker who applies to it who has the highest characteristic. Since all workers are in some sense equally well qualified for the jobs that firms offer, we assume that the firm does not have the option of refusing to hire once it observes the characteristics of the workers who apply.

3. Equilibrium of the Worker Application Sub-game

It is the market with a continuum of workers and firms that is ultimately of interest in this paper. However, it isn't obvious how to model payoffs in this continuum. Workers actions can't naturally be modeled as a distribution of actions, and it isn't clear why the essential friction in directed search doesn't disappear in the continuum because of simple assortative matching. To clarify these things, this paper describes payoffs in the continuum as limits of payoffs from sequences of large finite matching markets. This section then digresses in order to study the continuation equilibrium in a finite matching problem. This digression helps to explain the intuition for the limit game, which otherwise might appear quite ad hoc. The main limit theorem is presented at the beginning of Section 4, and summarizes the results of this section.

A strategy for worker i in the finite application sub-game is a function $\pi^i : W^N \times Y \to S^{m-1}$, where $S^{m-1} = \left\{\pi \in \mathbb{R}^m_+ : \sum_{i=1}^m \pi_i = 1\right\}$.[3] This section analyzes symmetric equilibria in which every worker uses an application strategy that is a common function of his or her type. The idea that is fundamental to directed search is that these application strategies depend on the array of wages on offer. For the purposes of characterizing the equilibrium in the application sub-game associated with a fixed set of wages, the notation that captures this will be

[3] We ignore the possibility that a worker might not apply to any firm since that is a strictly dominated strategy given the assumptions about payoffs.

suppressed and we write $\pi_j(y)$ to be the probability with which each worker whose type is y applies to firm j.

Since firms always hire the worker with the highest type who applies, worker i will match with firm j in equilibrium so long as every other worker in the market either has a lower type than he does, or applies to some other firm. So the probability that a worker is hired if he applies to firm j is given by

$$\left[1 - \int_y^{\overline{y}} \pi_j\left(y'\right) dF\left(y'\right)\right]^{n-1}$$

The payoff to the worker is equal to this probability multiplied by the wage that the firm offers. It will simplify the argument in this section to assume that wages are ordered in such a way that $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_m$.

The unique (symmetric) equilibrium for the application sub-game is given by the following Lemma.

Lemma 3.1. *For any array of wages $w_1, \ldots w_m$ offered by firms for which $w_1 > 0$, there is an array $\{y_K, \ldots, y_m\}$ with $K \geqslant 1$ and a set $\{\pi_j^k\}_{k \geqslant K; j \geqslant k}$ of probabilities satisfying $\pi_j^k > 0$ and $\sum_{j=k}^m \pi_j^k = 1$ for each k and such that the strategy*

$$\pi_j(y) = \begin{cases} \pi_j^k & \text{if } j \geqslant k; y \in [y_k, y_{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

is almost everywhere a unique continuation equilibrium application strategy. The probabilities π_j^i satisfy

$$\left(\frac{\pi_j^i}{\pi_i^i}\right)^{n-1} = \frac{w_i}{w_j} \tag{1}$$

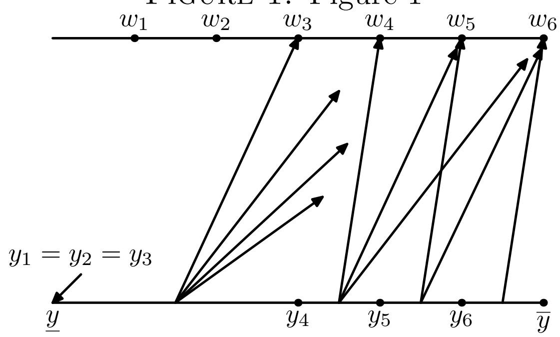
for each $j > i$.

Furthermore, the numbers $\{y_k\}$ and $\{\pi_j^k\}$ depend continuously on the wages offered by firms.

The proof is included in the appendix. The theorem is hard to state because there are many different probabilities that have to be described. However, the content of the theorem can be explained with the help of Figure 1.

The lower line in the figure represents the set of possible types that workers might have, from $\underline{y}$ to $\overline{y}$. The upper line represents the array of wages on offer. There are six firms in this example, each offers a distinct wage. The theorem says that the set of types can be partitioned into $m - K + 1$ different subsets. In the picture there are four such subsets with cutoff points given by $y_4, \ldots y_6$. The interpretation of the interval is that workers whose types are in $(y_j, y_{j+1}]$ all behave in the same way.

Figure 1. Figure 1



They randomize their applications over firms whose wages are w_j or higher using exactly the same mixed strategy. The arrows in the figure indicate the wages at which the corresponding worker type will apply. Note that lower type workers apply with positive probability to more firms. This is intuitive. The highest type workers expect to be hired no matter where they apply, so they should restrict their applications to the highest wage firms.

Observe that workers whose type is in the interval $(y_j, y_{j+1}]$ don't apply to firms whose wage is below w_j. They apply to all firms offering wages higher than w_j with strictly positive probability. Following Shimer (2005) we refer to w_j as worker j's *reservation wage.* Then we say that the worker applies to every wage above his reservation wage with strictly positive probability. Observe that by (*1*), each worker applies with the highest probability to the firm that offers his or her reservation wage. The larger the wage is relative to the worker's reservation wage, the lower the probability with which the worker applies. This is quite different from standard directed search. The tradeoff that workers face in this model is that they face stiffer competition at higher wage firms, instead of simply more of it. The equation (*1*) also indicates that this effect is smaller the more workers there are. As the limit theorem below argues, probabilities won't vary across firms in the limit and workers will apply with equal probability at every wage above their reservation wage.

In the Figure, wages w_1 and w_2 are low enough that they are below the reservation wage of every worker type so that they don't receive applications. All workers receive a higher expected payoff by applying at wage w_3 and taking a chance at being hired, that they get from applying at wage w_1 or w_2 and being hired for sure. Normally one would expect all wages that are offered to receive applications from some worker types. In this case, the partition of worker types would be characterized by an $m-1$ tuple $(y_2, \ldots, y_m)$. To simplify the notation in what follows we restrict attention to this case. The type y_2 will play an important role in what follows. This worker type will be hired for sure if she applies at the lowest wage w_1 because higher type workers won't apply at this wage. On the other hand workers whose types are below y_2 will face competition at the wage w_1.

Finally, it is worthwhile to give some intuition about why the equilibrium works out the way it does. For example, an alternative that might seem reasonable is that workers and firms match assortatively with the highest quality workers applying to the highest wage firms, with lower quality workers applying exclusively at lower wages. To see why this won't work, consider the lowest worker type y_j who applies at wage w_j. If workers application strategies involve sorting, worker y_j will be hired for sure if he applies at any wage below w_j and will be hired at the wage w_j if no higher type workers apply. Any worker who has type $y_k < y_j$ is less likely than worker y_j to be hired at a firm offering a wage $w' < w_j$, but has exactly the same chance as worker y_j of being hired at wage w_j. So workers with types below y_j would strictly prefer to apply at wage w_j than at any lower wage. To prevent this, there must be some chance that workers with types below y_j also apply at wage w_j in equilibrium.

All the outcomes depend on the number of workers and firms. To avoid adding notation, everything is indexed by n and recall that the ratio of the number of workers to firms is constant and equal to τ. Fix an array of wages. Let $\{\pi_1(\cdot), \ldots, \pi_m(\cdot)\}$ be the continuation equilibrium associated with this array of wages. Observe that by Lemma 3.1, each function $\pi_j(y)$ is a step function with jumps at the points y_j. Define the function

$$\omega_n(y) = \min\{w_j : \pi_j(y) > 0\}$$

The function $\omega_n(y)$ gives the reservation wage of a worker of type y. This function is a step function whose 'steps' occur at the critical points y_j identified by Lemma 3.1. The limit from the left of $\omega_n(y_j)$ at y_j is w_{j-1}, while the right limit of $\omega_n(y_j) = w_j$. Denote its 'inverse' function by

$$y_n^*(w) = \sup\{y' : \omega_n(y') \leq w\}$$

In words, the inverse function gives the highest type who chooses a firm offering wage w with strictly positive probability. Despite the fact that the notation suppresses this, bear in mind that the functions y_n^* and ω_n both depend on the array of wages on offer.

3.1. **Wages.** Consider the firm who offers the wage w_j (i.e., the j^{th} lowest wage). The probability with which a worker drawn randomly both comes to firm j and has a type at least y is

$$\int_{y}^{y_n^*(w_j)} \pi_j(y')dF(y')$$

Let $\tilde{j}(y) = \{j' : \omega_n(y) = w_{j'}\}$ be the index of the lowest wage to which a worker of type y applies. Using this the integral above can be written

$$\sum_{j'=\tilde{j}(y)}^{j} \pi_j^{j'} \left[F(y_{j'+1}) - \max\left[F(y_{j'}), F(y)\right]\right]$$

if $\omega_n(y) \leq w_j$. The integral is zero otherwise. Then from firm j's point of view, it looks exactly as if n worker types are being independently drawn from the probability distribution

$$\phi_j(y) \equiv 1 - \pi_j^{\tilde{j}(y)}[F(y_{\tilde{j}(y)+1}) - F(y)] - \sum_{j'=\tilde{j}(y)+1}^{j} \pi_j^{j'}[F(y_{j'+1}) - F(y_{j'})]$$

The distribution function $\phi_j(y)$ has an atom of size

$$1 - \sum_{j'=1}^{j} \pi_j^{j'}[F(y_{j'+1}) - F(y_{j'})]$$

at $\underline{y}$.

The firm will always hires the worker who has the highest type. The probability distribution for the type hired by the firm is then the probability distribution of the highest order statistic from this distribution. This gives the expected payoff for firm j as

$$\int_{\underline{y}}^{\overline{y}} v(w_j, y, x_j)d\phi_j^n(y) \tag{2}$$

By Lemma 3.1, the distribution function $\phi^n(y)$ is continuous at each point y in firm j's wage. So $\phi^n(\cdot)$ varies continuously in the weak topology with firm j's wage. As the family of functions $v(\cdot, y, x_j)$ is equi-continuous, the integral is a continuous function of the wage w_j that the firm offers. The existence of a mixed strategy equilibrium in firms' wages then follows from standard theorems.

4. The Limit Values of Payoffs

Despite the conceptual simplicity of firms' payoff implied by (2), it is difficult to provide much in the way of characterization of the Nash equilibrium of the firms' part of the game. It is tempting to jump to a continuum of workers and firms to see if this helps the characterization. A significant complication in this regard arises from the fact that in a large game, payoffs should be defined for every feasible action against every possible distribution of the actions of the others. For firms this is at least conceptually straightforward since wage distributions are well understood. For workers, the application 'strategy' is not a well defined object in the continuum. The distribution of such things is then a moot point.

To get around this difficulty, this section provides a theorem that shows that limit payoffs of all traders depend on their own actions, on the distribution of wages, and on a single reservation wage function for workers. This result suggests a natural definition of equilibrium for the continuum game.

Let G be a distribution of wages. To approximate the distribution of wages, let G_n be a sequence of step functions that converges weakly to G. Let $\{w_1^n, \ldots, w_m^n\}$ be the finite array of wages whose distribution is G_n. We will use the convention that $w_1^n = w_0$in each approximation, where w_0 is the lowest wage in the support of G. To define payoffs in the limit game, G_n has to be arbitrary in the sense that payoffs in the large game must be defined off the equilibrium path in order to show what the equilibrium is.

Now fix a firm type x and a wage w to be offered by that firm. Suppose that in this approximation, firm x has the $j^{n^{th}}$ highest wage. For any distribution function, G_n, let $G_n^-(w)$ be the left limit of G_n, so that $(1 - G_n^-(w))m$ is the number of firms whose wage offer is *at least* w. Recall that the non-decreasing function $\omega_n : Y \to W$ represents the lowest wage at which worker of type y will apply, and that $y_n(w) = \sup_y \omega_n(y) \leq w$ is the highest worker type who will apply to a firm who offers wage w.

Theorem 4.1. *Let G be a distribution of wages, w an arbitrary wage offered by a firm of type x, and w^-, the largest wage in the support of G that is less than or equal to w. Let G_n be a sequence of distributions that converges weakly to G. Let j_n be the corresponding sequence of indices of firm x's wage (i.e., such that w is the j_n^{th} lowest wage in the distribution associated with G_n). There is a non-decreasing right continuous function $\omega(y)$ and a non-decreasing right continuous function $y^*(w)$ (both of which depend on G) such that for almost every $y \in [\underline{y}, \overline{y}]$,*

$$(3) \qquad \lim_{n\to\infty} \left[1 - \int_{\underline{y}}^{y_n^*(w)} \pi_{j_n}^n(y) dF(y)\right]^{n-1} = \frac{w^-}{w} e^{-\int_{\underline{y}}^{y^*(w^-)} k(y') dF(y')}$$

and

$$\lim_{n\to\infty}\int_{\underline{y}}^{\overline{y}} v(w,y,x)d\phi_{j_n}^n(y) =$$

(4)

$$\frac{w^-}{w}\int_{\underline{y}}^{y^*(w^-)} k(y)v(w,y,x)e^{-\int_y^{y^*(w^-)} k(y')dF(y')}F'(y)dy + v(w,y^*(w^-),x)\left(1-\frac{w^-}{w}\right)$$

whenever $w \geq w_0$*, and*

$$\lim_{n\to\infty}\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y)dF(y)\right]^{n-1} = \min\left[1, \frac{w_0}{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')}\right]$$

and

$$\lim_{n\to\infty}\int_{\underline{y}}^{\overline{y}} v(w,y,x)d\phi_{j_n}^n(y) =$$

$$\int_{\underline{y}}^{y(w)} \tau v(w,y,x)\frac{w_0}{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')}F'(y)dy$$

otherwise. In these expressions, $y\,(w)$*is the solution to*

$$\frac{w_0}{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')} = 1$$

and

$$k(y) = \frac{\tau}{1-G^-(\omega(y))}$$

Furthermore $y^*(w) = \sup\{y : \omega(y) \leq w\}$*.*

The proof of the theorem is, again, included in the appendix. A number of more descriptive comments are in order here.

First, the formulas above differ depending on whether the wage offer a firm makes, or the wage at which a worker applies, is in the support of the existing distribution G of wages. This makes the statement of the Theorem complicated. When the wage is inside the support, the payoffs are actually fairly simple. The probability with which a worker is hired when he or she applies to a firm offering wage w is then given by the formula

$$e^{-\int_y^{y^*(w)} k(y')dF(y')}$$

while the profit for the firm who offers a wage in the support of G is

$$\int_{\underline{y}}^{y^*(w)} k(y)v(w,y,x)e^{-\int_y^{y^*(w)} k(y')dF(y')}F'(y)dy$$

At first glance, these formulas seem independent of the distribution G. Recall however, that $k(y) = \frac{\tau}{1-G^-(\omega(y))}$. The function $\omega(y)$ is the common reservation wage function used by all workers, and $y^*(w)$ is it's 'inverse'. So all traders' payoffs are determined by their own actions (offer or apply at wage w), the distribution of wages G and the reservation wage function ω.

The exponential function is a familiar one in directed search. To get some idea how it works, suppose that all firms offer the same wage w. Then the reservation wage of every type is w and $1-G^-(w) = 1$ (recall that the left limit $G^-(w)$ can be interpreted as the proportion of firms who have wages *below* w). So the probability that a worker of type y trades with a firm offering w is $e^{-\tau(1-F(y))}$. The power in the exponent is the ratio of the number of workers whose type is as large as y to the number of firms.[4]

Payoffs associated with deviations outside the support of the distribution will determine the location of the distribution. We discuss this in more detail below. However, one result that deserves some comment is that the theorem provides payoffs for *almost all* types. If there is a deviation above the support of G, the theorem doesn't provide the right payoff for the highest type since he or she must be hired for sure no matter where they apply. In the computation of the firms' payoff what happens to this type doesn't make any difference since no worker has this type with positive probability. The formula applies to every other worker type, no matter how close it is to $\overline{y}$.

To get some sense of how this works, let w be the deviation that lies strictly above the support of G. Workers of type $y < \overline{y}$ must still have reservation wage w^-. The reason for this is that if there were some interval of types who apply to the deviator with probability 1, the the lowest type in the interval would necessarily trade with the deviator with probability zero once the number of workers was large enough. The probability with which such a worker trades when she applies at wage w^- is $e^{-\int_y^{\overline{y}} k(y')dF(y')}$. Since the expected payoff the worker gets by applying to the deviator must be the same as it is when she applies at wage w^-, and this gives the formula in the theorem.

The theorem defines payoffs for *all* possible wage offers and application decisions that depend, apart from exogenous stuff, on the distribution of wages G, and a single reservation wage rule $\omega(y)$. These formulas provide a natural definition of equilibrium in the large game.

[4]The usual formula for the trading probability differs from this because a worker is hired in the standard model if he is lucky enough to be drawn from the pool of workers who apply. Here the worker is hired if he is the highest quality worker who applies.

First, if every firm chooses a wage that is a best reply to the distribution G and the function ω according to the limiting payoff function given in the theorem, then the associated distribution of best replies should be equal to G. Secondly, each worker type should find that his or her expected payoff (again, according to the limiting payoff function given in the theorem) is the same when they apply to any wage at or above their reservation wage $\omega(y)$, and this payoff should be at least as large as it is when they apply at any wage below their reservation wage.

This is the notion of equilibrium that is adopted in the rest of the paper. Before proceeding, it is useful to clarify somewhat the relationship between this approach and the right approach, which is to compute the limits of sub-game perfect Nash equilibria as the number of workers and firms grow. A brief digression on this follows.

Let H be a distribution of firms' types and $H^n$the distribution of firms' types in a finite approximation consisting of n workers and m firms, where $\frac{n}{m} = \tau$. Imagine constructing the sequence of approximations so that H^n converges weakly to H. Since firms' first stage payoff functions are continuous when there are finitely many workers and firms according to Lemma 3.1, an equilibrium for the entire game must exist in which firms use mixed strategies to set their wages in the first stage. Suppose that G_n is a sequence of realizations of wages that occur in the finite game with n workers when the m firms all use their equilibrium mixed strategies. These random sequences converge almost surely to some distribution function G as n becomes large.[5] Then the formulas in Theorem 4.1 give the limit values of payoffs to workers of each type facing the Bayesian equilibrium limit distribution G of wages. The Theorem also provides the limit value of the payoff for any wage w played against the equilibrium wages of all the other firms. If G is not an equilibrium distribution in the limit game (in the sense described above, that G is a best reply to itself), then some type of firm does strictly better in the limit game than he could do by playing the limit of his equilibrium pure strategy. Convergence means that he would do strictly better in the finite approximations as well when n is large enough.[6] So the weak (or probability) limits of equilibrium wage distributions from large finite games must be equilibrium distributions in the limit game. This is the sense in which we can use the payoff

[5] In this sentence, convergence means uniform convergence and the target distribution is just an unweighted average of the mixed strategies used by all the firms. See Shorak and Wellner (1986).

[6] This statement would be modified to read that he would do strictly better with high probability if equilibria with finite players involved mixed strategies.

functions from the limit game to approximate what happens in large finite approximations.

The converse isn't necessarily true. As is usually the case, large games may have more equilibria than their finite counterparts. However, this isn't much of a problem as the methodology will identify the limits of equilibria of large finite games. Furthermore, though there might be more than one equilibrium in the large game, the number of equilibria is 'small'.

5. Properties of Equilibrium

We now proceed to describe equilibrium and to show how it can be characterized for the general case. We follow with a discussion of equilibrium when firms are the same and add an example to verify conditions from the general case. The definition that follows states the equilibrium conditions that follow from Lemma 4.1. Let $\rho(w, x, G, \omega)$be the profit function described by the Lemma. In other words

$$\frac{w^-}{w}\int_{\underline{y}}^{y^*(w^-)} k(y)v(w,y,x)e^{-\int_y^{y^*(w^-)} k(y')dF(y')}F'(y)dy+v(w,y^*(w^-),x)\left(1-\frac{w^-}{w}\right)$$

if $w \geq w_0$ (where w_0 is the lowest wage in the support of G) and

$$\int_{\underline{y}}^{y(w)} \tau v(w,y,x)\frac{w_0}{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')}F'(y)dy$$

otherwise, where $y(w)$ satisfies $\frac{w_0}{w}e^{-\int_y^{y^*(w_0)} \tau dF(y')} = 1$.

Definition 5.1. An *equilibrium* for the large directed search game is a wage offer distribution G and a reservation wage rule ω satisfying

(1) for every y, worker y's payoff $we^{-\int_y^{y^*(w)} k(y')dF(y')}$ is constant for every wage w exceeding his or her reservation wage $\omega(y)$;
(2) for every w the measure of the set

$$\left\{x : \arg\max_{w'}\left(\rho\left(w', x, G, \omega\right)\right) \leq w\right\}$$

is equal to $G(w)$.

Though the limit theorem defines payoffs for all distributions, including degenerate distributions where all firms offer the same wage, it is of some interest to analyze equilibria in which the distribution of wages is differentiable. Indeed, it is exactly this situation that the limit theorem is designed to address.

We begin with a restriction on the reservation wage rule. Observe that

$$we^{-\int_{y}^{y^*(w)} k(y')dF(y')} = \text{constant} \tag{5}$$

for each $w \geq \omega(y)$. For each worker type y, the derivative of this expression with respect to w should then be zero at every wage above $\omega(y)$. That is

$$we^{-\int_{y}^{y^*(w)} k(y')dF(y')} k(y^*(w))F'(y^*(w))\frac{dy^*(w)}{dw} = e^{-\int_{y}^{y^*(w)} k(y')dF(y')}$$

giving

$$w\frac{\tau}{1-G(w)}F'(y^*(w)) = \frac{1}{dy^*(w)/dw} \tag{6}$$

Since $y^*(w)$ is the inverse function of $\omega(y)$ at each point in the support of G,

$$\omega(y)\frac{\tau}{1-G(\omega(y))}F'(y) = \frac{d\omega(y)}{dy} \tag{7}$$

This system involves two unknown functions G and ω. If w_0 is the lowest wage in the support of G, let y_0 be the highest type for which $\omega(y) = w_0$.[7] Then (7) must hold on the interval $[y_0, \overline{y}]$ with $\omega(y_0) = w_0$ and $\omega(\overline{y})$ equal to the highest wage in the support of G.

When ω is strictly increasing (as it must be to satisfy 7) then a worker of type y who applies to a firm offering his reservation wage $\omega(y)$ will be offered a job with probability 1. The reason is that no worker with a higher type would actually apply at the wage $\omega(y)$. So the constant on the right hand side of (5) is equal to $\omega(y)$. The function $\omega(y)$ doubles as the market payoff function for every worker type above y_0.

It is convenient to extend the market payoff function to all types by observing that since workers are always indifferent about firms whose wage is above their reservation wage, and since workers whose types are in the interval $[\underline{y}, y_0]$ all have reservation wage equal to w_0, the market payoff for these lower types is their expected payoff when they apply to the lowest wage w_0. Overloading notation slightly, we will define

$$\omega(y) = w_0 e^{-\int_{y}^{y^*(w_0)} \tau dF(y')} \tag{8}$$

for worker types below y_0.

[7]In the finite version of the model, the lowest wage was w_1 and the highest type who had w_1 as a reservation wage was called y_2. Hopefully this change in notation will make the argument in the continuum more transparent.

This makes it possible to simplify the firms' profit function. Using this extended market payoff function, for any wage above worker y's reservation wage,

$$we^{-\int_{y}^{y^*(w)} k(y')dF(y')} = \omega(y). \tag{9}$$

Substituting this for the trading probability in the firms profit function when it sets wage $w \geq \omega(y_0)$ gives

$$\int_{\underline{y}}^{y^*(w)} k(y)v(w,y,x)\frac{\omega(y)}{w}F'(y)dy$$

Now since (7) holds for every wage above $\omega(y_0)$, and (8) holds at $\omega(y)$ when $y < y_0$, the firms' profit has the following simpler form[8]

$$\int_{\underline{y}}^{y^*(w)} v(w,y,x)\frac{\omega'(y)}{w}dy$$

In fact, instead of thinking of the firm as choosing a wage w then receiving applications from all workers whose types are less than $y^*(w)$, it is more convenient to think about the firm choosing the highest worker type it wants to attract, then setting its wage equal to this worker type's reservation wage. This makes the firms profit

$$\int_{\underline{y}}^{y} v(\omega(y), y', x)\frac{\omega'(y')}{\omega(y)}dy' \tag{10}$$

Maximizing (10) with respect to y gives the first order condition

$$v[\omega(y), y, x] = -\int_{\underline{y}}^{y}\left[v_w(\omega(y), y', x) - \frac{v(\omega(y), y', x)}{\omega(y)}\right]\omega'(y')dy'$$

The term on the left represents the gain to the firm when it chooses to raise the quality of its best applicant. The term on the right measures the cost to the firm of paying the reservation wage of this higher quality applicant to all the other types of workers who it attracts.

To continue with the analysis, one other concept is needed. A worker of type y applies with equal probability at all firms whose wage is above $\omega(y)$. Nonetheless, the firm who offers the wage $\omega(y)$ is focal for the worker. So define a 'matching function' $h : Y \to X$ which describes for each worker characteristic, the characteristic of the firm who offers that worker type his or her market payoff. The wage distribution can be reconstructed using $\omega(y)$ and $h(y)$ since $G(\omega(y)) = H(h(y))$.

[8]When $y < y_0$ this formula comes from taking the derivative of (8) with respect to y and substituting using the fact that $k(y) = \tau$ when $y < y_0$.

An equilibrium will consist of a pair of non-decreasing functions $\omega(y)$ and $h(y)$ and an interval $[y_0, \overline{y}]$ such that $h(y_0) = \underline{x}$, $h(\overline{y}) = \overline{x}$, the functional equations

$$\omega(y) = \omega(y_0)\, e^{-\tau[F(y_0)-F(y)]}$$

for each $y < y_0$;

$$\omega(y) \frac{\tau}{1 - H(h(y))} F'(y) = \omega'(y) \tag{11}$$

for each $y \geq y_0$;

(12)

$$v[\omega(y), y, h(y)] = -\int_{\underline{y}}^{y} \left[v_w(\omega(y), y', h(y)) - \frac{v(\omega(y), y', h(y))}{\omega(y)} \right] \omega'(y')\, dy'$$

for each $y_0 \leq y \leq \overline{y}$; and

$$\int_{\underline{y}}^{y} \frac{v(\omega(y), y', x)}{\omega(y)} \omega'(y')\, dy' \leq \int_{\underline{y}}^{y_0} \frac{v(\omega(y_0), y', x)}{\omega(y_0)} \omega'(y')\, dy'$$

for each $y < y_0$ and $x \in X$.

The equilibrium conditions described above don't impose any restriction on wage offers above $\omega(\overline{y})$. To see why, observe that a firm who offers a wage above $\omega(\overline{y})$ receives payoff

$$\frac{\omega(\overline{y})}{w} \int_{\underline{y}}^{\overline{y}} v(w, y, x) \frac{\omega'(y)}{\omega(\overline{y})} dy + v(w, \overline{y}, x) \left(1 - \frac{\omega(\overline{y})}{w}\right) =$$

$$\int_{\underline{y}}^{\overline{y}} v(w, y, x) \frac{\omega'(y)}{w} dy + v(w, \overline{y}, x) \left(1 - \frac{\omega(\overline{y})}{w}\right).$$

From this expression, it should be apparent that the derivative of this expression with respect to w will be zero exactly when (12) holds at the point $\overline{y}$. So (12) already guarantees that deviations above the support of the existing distribution won't be profitable (at least local deviations won't be profitable - global conditions require restrictions on v as always).

5.1. **Identical Firms.** The functional equations (11) and (12) are amenable to numerical solution. To get more insight into the way the equations resolve, it helps to think about the case where all firms are the same. We illustrate how to characterize equilibria with wage distributions here.

When firms are the same, the constant profit condition reduces to

$$v[\omega(y), y] = -\int_{\underline{y}}^{y} \left[v_w(\omega(y), y') - \frac{v(\omega(y), y')}{\omega(y)} \right] \omega'(y')\, dy' \tag{13}$$

Since this condition must hold for every $y \geq y_0$, it must hold at y_0. Since the payoff function $\omega(\cdot)$ below y_0 is defined by the wage $\omega(y_0)$, any candidate solution for ω must have initial value (y_0, w_0) satisfying (14)

$$v[w_0, y_0] = -\int_{\underline{y}}^{y_0} \left[v_w(w_0, y') - \frac{v(w_0, y')}{w_0} \right] \tau w_0 e^{-\tau[F(y_0) - F(y')]} F'(y')\, dy'.$$

This condition ensures that firms cannot increase profits by offering a wage slightly below the distribution G.[9]

On the interval $[y_0, \overline{y}]$, the function $\omega(\cdot)$ must be chosen to satisfy (13). If this solution has the property that $\omega'(y) > 0$, then we can proceed to try to define the wage distribution. In order for this curve to represent the payoff to workers of different types, the distribution of wages G must be adjusted so that

$$\omega(y) \frac{\tau}{1 - G(\omega(y))} F'(y) = \omega'(y) \tag{15}$$

is satisfied for each $y \geq y_0$. Using the fact that the solution to (15) derived above is monotonic, the distribution is given by

$$G(w) = 1 - \frac{w\tau F'(\omega^{-1}(w))}{\omega'(\omega^{-1}(w))}. \tag{16}$$

Among the collections of solutions to (13), some subset will satisfy the property that the solution to (16) is non-decreasing in w. Finally from this subset, we need to choose one that has slope $w_0 \tau F'(y_0)$ at its starting point (so that it extends the market payoff function below y_0).

The Figures below illustrate how to construct an equilibrium distribution. To begin we find the family of solutions to (13). These are the dashed lines in Figure 2. These are all market payoffs function ω that hold firms profits constant no matter which wage they offer. These market payoff functions have to coincide with the market payoff functions that are defined for types below y_0. In particular, they have to have the same slope at the bottom of the equilibrium distribution (y_0, w_0). On each of the solution curves ω in this figure, we locate a point on the curve such that $w\tau F'(y) = \omega'(y)$, and collect all these point together to form the curve TT in that Figure. Wherever the equilibrium distribution starts, it must be somewhere on this locus TT.

Now we use the idea that the bottom point of the equilibrium distribution (y_0, w_0) fully determines the profit function for a firm who

[9]Restrictions on F and v will be needed to ensure that local optimality conditions are global. For this general case, we ignore these. They are verified in the example below.

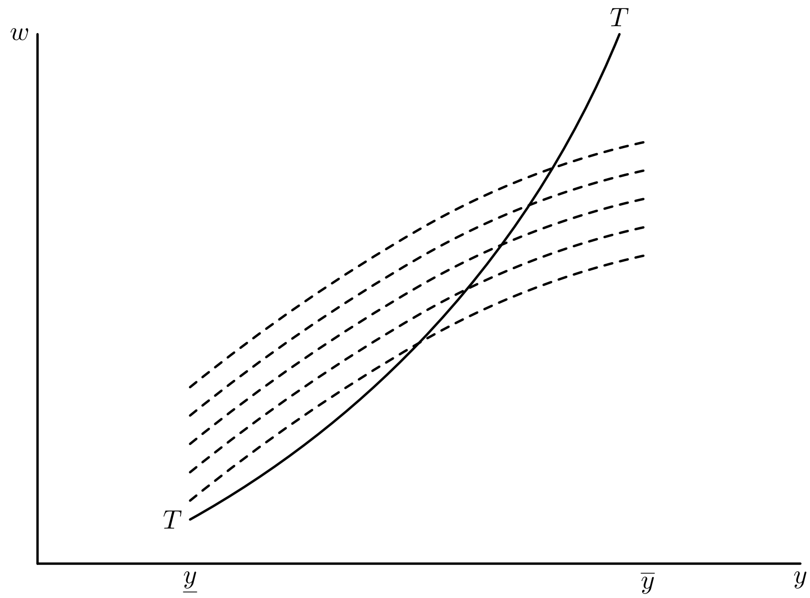


Figure 2. Constant Profit Condition within the Support

decides to offer a wage below the support of the distribution. This simply means finding the locus of all pairs (y, w) such that the first order condition that guarantees that these downward wage cuts are locally unprofitable. We collect all these points together into the locus SS in Figure 3. Formally, SS is the collection of all solutions to (14). From each point on this locus, there is a downward sloping market payoff function for types below the corresponding y_0. These payoff functions are given by the dashed lines in the figure.

The only point that qualifies as a starting point for the distribution G is the point given by the intersection of SS and TT. This intersection point determines a payoff function for worker types below y_0 and a reservation wage rule from the collection of solutions to (13). In Figure 4 we put the two previous curves together to get a candidate solution.

Finally, we use (16) to generate the distribution of wages. At this point there are a number of things that need to be checked. For instance, it isn't immediate that the curves have the shapes shown in the picture, that the G just calculated is increasing, or that the appropriate second order conditions hold at the edges of the distribution. In the appendix, these details are checked for an example in which $v(w, y) = y - w$ and $F(y) = y(2 - y)$. There is nothing innately interesting about these details, so we do not discuss them here. Instead, the discussion takes the properties of the Figures for granted.

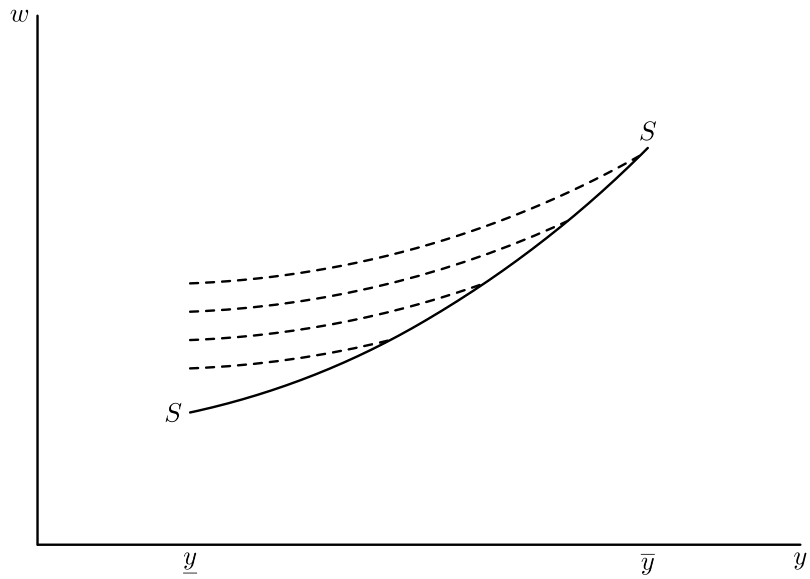


FIGURE 3. First Order Condition at the Bottom

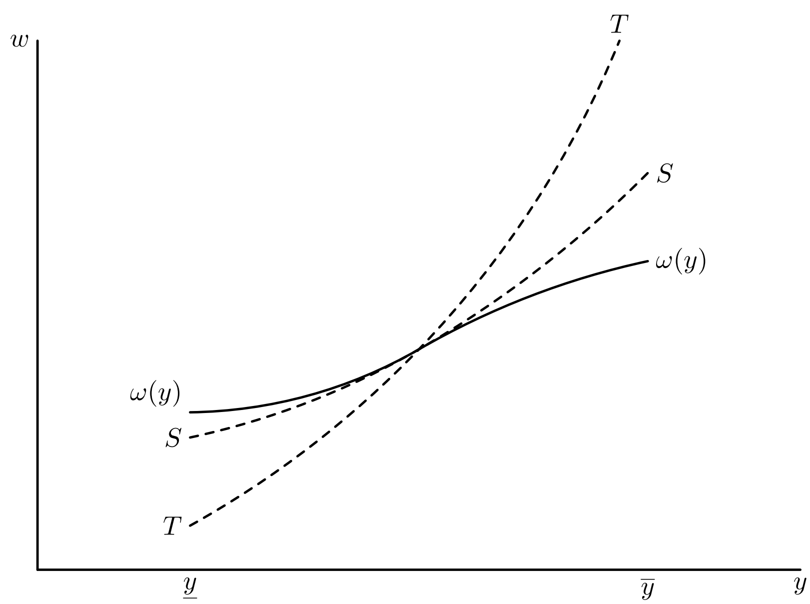


FIGURE 4. The Equilibrium Distribution

It should be emphasized that properties of the equilibrium distribution depend on assumptions about v and F. An equilibrium distribution may not exist. The example in the appendix makes it straightforward to verify that if the distribution F is uniform, every solution to (16) will be decreasing. What that means is that a non-degenerate distribution of wages can't be supported with a uniform distribution. Similarly, if v does not depend on y[10], it is impossible to support an equilibrium distribution. The example in the appendix uses a distribution with a linear density. If there is no equilibrium distribution, there will still be an equilibrium in which all firms offer the same wage. The equilibrium wage is given at the point where the set SS of solutions to (14) intersects the vertical line through $\overline{y}$. This follows from the arguments above since every point on this locus makes downward deviations unprofitable. That upward deviations are unprofitable follows from differentiability of profits. The possible non-existence of an equilibrium distribution is not an issue here, since the primary motivation for this model is to deal with the case where firms differ.

The differences between the dispersed wage equilibrium and the single wage equilibrium deserve some comment. Keep in mind that the properties described in the picture follow under 'reasonable' assumptions as the appendix shows.[11] All workers earn higher wages in the single wage equilibrium than they do in the dispersed wage equilibrium. This is intuitive since low wage firms don't attract high type workers at all. The dispersion then limits the number of firms competing for the high type workers. This should bring down their wages. The distributional arguments illustrate how this effect propagates down to the lower wage workers. This result is similar to the result in Lang, Manove, and Dickens (1999) who show that discrimination against workers of a certain type drives down wages of a second type in a directed search model assuming worker type has no direct effect on productivity.

In the symmetric firm case the dispersed wage equilibrium certainly won't be efficient. The concentration of applications of the higher type workers with the high wage firms creates no sorting gain, but causes more of the high type workers to be rationed. An allocation of workers that maximizes expected output would spread out applications of the high type workers over all firms in the fashion of the symmetric equilibrium. This isn't particularly interesting since there is no gain to sorting workers when firms are identical. Even if there is, the equilibrium won't be efficient. To see why, contrast the results in Shimer (2005) where the distribution of wages is efficient. There a low quality worker who applies to a high quality firm has a limited benefit since the firm can pay the low quality worker a lower wage than it does to

[10] An example like this was analyzed in a different way by Lang and Manove (2006). They come to the same conclusion that a distribution of wages cannot be supported if firm profits don't depend on y.

[11] We have also verified these properties numerically with a linear profit function assuming F is a β distribution.

the high quality worker after it hires the worker. Here the low quality workers earn just as much once hired as the high quality workers do. So they have a much higher incentive here to apply to high wage firms.

Finally the analysis above is meant to illustrate how the equilibrium wage distribution might be computed more generally. The model seems most useful when firms differ. The loci TT and SS still define the starting point for the equilibrium distribution, with both curves computed using the profit function for the firm with the lowest type. All of the reasoning given above applies to the lowest type firm in this case. Once the lower bound of the distribution is determined in this way, the problem is to find a pair of functions (h, ω) that satisfy the two equations (11) and (12) with starting values $\omega(y_0) = w_0$ and $h(y_0) = \underline{x}$. Since this seems to require numerical solution, we don't pursue this her.

5.2. **Offer Distributions and Duration.** Practically, the distributions of firm and worker types will be unknown. Any distribution G can be supported for an appropriate choice for the distribution of firm types. The theory does impose a restriction on the relationship between the wage distribution and unemployment duration. In particular, labor force surveys provide information about the wages at which workers leave unemployment, and the amount of time they spent searching for a job. Averaging unemployment duration over all workers who exit unemployment at different wages gives some information about this relationship. Duration is just the inverse of the matching probability which is the variable of interest in the theory described so far.

So for this section, fix a distribution of wages. Let $\overline{w}_G$ be the maximum point in the support of the distribution of wages. The probability that a worker of type y is hired by *some* firm, using the reasoning above, is given by

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} e^{-\int_y^{y^*(w)} k(y')dF(y')} \frac{G'(w)}{1 - G(\omega(y))} dw$$

Since the expected wage is constant for a worker of type y at every wage above $\omega(y)$, this can be written as

$$Q(y) = \int_{\omega(y)}^{\overline{w}_G} \frac{\omega(y)}{w} \frac{G'(w)}{1 - G(\omega(y))} dw$$

The expected duration of unemployment for a worker of type y is $\frac{1}{Q(y)}$. Of concern below is how this matching probability varies with the worker's type.

The complication in all this is that higher types have higher reservation wages. So despite the fact that they are more likely to be hired at any particular firm than low type workers, they tend to apply at firms where there is a lot of high type competition. So whether or not this probability is increasing in type depends on the function

$$\psi(w) = \int_{w}^{\overline{w}_G} \frac{w}{w'} \frac{G'(w')}{1-G(w)} dw'$$

depends on the wage w. This function represents the expectation of the ratio of any wage to the harmonic mean of higher wages in the distribution G. This function isn't particularly simple conceptually. Nor is it easy to deduce distributions for the unobservables that will support this property. However it is readily checked numerically.

For example, we have done a number of numeric simulations when F is given by different β distributions and firms are symmetric. In each case the simulated function $\psi(w)$ is monotonically increasing. Since this applies to the less interesting case where firms are symmetric, we leave out details. Another approach is simply to ask whether the data appears to support the presumption that ψ is increasing. For this case, we have estimated ψ using data from the 2000 US census controlling for various levels of education. Whether the estimation procedure is parametric or not, the function ψ appears to be increasing. I can't claim enough econometric skill to pretend that these estimates would stand up to any kind of scrutiny, so details aren't included. They do seem to indicate that it is reasonable to consider the case where ψ is increasing, so that is the focus of the rest of this section.

If $\psi'(w) > 0$, for example, then the ex ante probability with which a worker of type y trades is an increasing function of y because of the monotonicity of $\omega(y)$.

Again, y is unobservable. What is observable is the actual duration of workers hired at different wages. From (4), the probability that a worker hired by the firm who offers wage w has a type less than or equal to y_0 is given by

$$\frac{\int_{\underline{y}}^{y_0} k(y) e^{-\int_{y}^{y^*(w)} k(y')dF(y')} F'(y)dy}{\int_{\underline{y}}^{y^*(w)} k(y) e^{-\int_{y}^{y^*(w)} k(y')dF(y')} F'(y)dy} =$$

$$\frac{\int_{\underline{y}}^{y_0} \frac{\tau}{1-G(\omega(y))} \frac{\omega(y)}{w} F'(y)dy}{\int_{\underline{y}}^{y^*(w)} \frac{\tau}{1-G(\omega(y))} \frac{\omega(y)}{w} F'(y)dy}$$

Note that this probability is conditional on some worker being hired by the firm, which explains the denominator. The equality follows by substituting for $k(y)$ and using (9). Substituting (7) gives an even simpler formulation

$$\frac{\int_{\underline{y}}^{y_0} \omega'(y)dy}{w - \omega(\underline{y})} \tag{17}$$

This expression is readily seen to be declining in w. The interpretation is that an increase in the wage moves the distribution function for the type hired by the firm to one that first order stochastically dominates the original distribution.

Now use this distribution to take expectations of $Q(y)$ to get the following:

Proposition 5.2. *If $\psi(w)$ is monotonically increasing then the expected duration of unemployment for a worker hired by a firm is an decreasing function of the wage offered by the firm.*

When $\psi(w)$ is increasing, workers who are hired at high wage firms will tend on average to have spent less time searching for jobs than workers who are hired by low wage firms. This is quite unlike standard directed search where high wages and long duration must go together. This prediction is not a particularly strong test of the model, since the function $\psi(w)$ may not be monotonic. Notice however, that it is a testable consequence of the model that does not rely on any knowledge about the distributions of the unobservables.

The expected duration for a worker hired by a firm offering a wage w is given by the reciprocal of

$$\int_{\underline{y}}^{y^*(w)} \frac{\omega'(y)}{w - \omega(\underline{y})} \psi(\omega(y))dy$$

using the expression for the density of the type of worker hired by the firm that was derived above. It is apparent from this expression that when $\psi(w)$ is non-monotonic, then there will be no systematic relationship between the wage at which a worker is hired and his probability of matching measured as his expected duration. Even in this dimension, the result is quite different from the standard directed search model where wage and employment probability must be inversely related.

Let $\Phi(w)$ be the expected duration of unemployment for workers hired at wage w, which could be estimated from existing data. This function, along with the actual wage distribution constitute the observables in this problem. Fix a set of types, say $[0, 1]$. Using the last expression, the functional equation

$$\int_{\underline{y}}^{y^*(w)} \frac{\omega'(y)}{w - \omega(\underline{y})} \psi(\omega(y))dy = \frac{1}{\Phi(w)}$$

can be solved to recover the function $\omega(y)$. The distribution of worker types is then recovered by solving (7). The distribution of firm types must then be chosen to support the observed distribution G when firms best reply to G and the symmetric strategy $\omega(\cdot)$ used by workers. This makes it possible to recover the productivity of the unobserved distribution of types.

The point of this last argument is simply to show how the model can be used to decompose the wage variation into variation in workers' and firms' types. We leave the analysis of this for future work.

6. Conclusion

This paper illustrates how a directed search model can be used to account for the residual part of wage variation. Part of this involves adjusting the directed search model to allow for rich variation in the types of workers and firms. This improves on existing models that use extensive symmetry assumptions that force the models to behave in counter-factual ways. In the variant proposed here, rich distributions of firm and worker characteristics can be incorporated.

The directed search model does impose some structure on the data. Surprisingly it restricts the relationship between the wage distribution and the function relating unemployment duration and exit wage. Some wage distributions (the uniform being an example) have the property that workers who leave unemployment at high wages must also have shorter unemployment duration. This prediction is distinctly different from standard directed search models where unemployment duration and wage must be positively related.

The driving force in the model presented here is the equilibrium of the workers' application sub-game. Contrary to what one might expect, low quality workers do not restrict their applications to low wage firms. On the contrary, low quality workers make applications at all kinds of different wages. The higher the unobservable quality of the worker, the more discriminating the worker is in the wages at which he applies. It is this property that breaks the strong relationship between wage and unemployment probability. Higher quality workers are more likely, everything else constant, to be hired by firms. High quality workers also apply to higher wage firms on average. In this sense high wages and short duration should be related. This relationship is not unambiguous however. As a workers quality rises, he is more likely to be hired at any given firm, but he will also restrict his applications to firms whose wages are higher. This by itself reduces the probability of employment because high wage firms have bigger queues - the usual directed search story.

Finally, the paper illustrates how observable data on wages and duration can be used to recover the unobserved distributions of firms' and workers' types.

7. Appendix

7.1. Proof of Lemma 3.1.

Proof. The proof is inductive.

Evidently a worker with the highest type will be hired with probability one where ever he applies, so every equilibrium strategy must have the highest type worker apply to one of the firms who offer the highest wage. If $w_{m-1} = w_m$ set $y_m = 1$ and $\pi_m^m = 1$. In this case observe that a worker of type y_m is just indifferent between applying to firm m and $m-1$.

Otherwise, fix an open interval $(y_m, \overline{y})$. The expected payoff to worker i with a type in this interval who applies to firm m is

$$\left[1 - \int_y^{\overline{y}} \pi_m\left(y'\right) dF\left(y'\right)\right]^{n-1} w_m$$

The expected payoff to applying to any firm j whose wage is $w_j < w_m$ is

$$\left[1 - \int_y^{\overline{y}} \pi_j(y') dF\left(y'\right)\right]^{n-1} w_j$$

Now observe that for y_m close enough to $\overline{y}$, workers will strictly prefer applying to firm m than applying to firm j, even if all the workers whose types are higher apply to firm m with probability 1. In other words, for workers whose type is close enough to $\overline{y}$, applying to one of the firms whose wage is highest strictly dominates any other choice. Thus there is some interval near $\overline{y}$ such that workers whose types are in this interval apply to firm m with probability 1 in every Bayesian equilibrium. The lowest type for which this is true is the type y_m such that

$$\left[1 - \int_{y_m}^{\overline{y}} dF\left(y'\right)\right]^{n-1} w_m = w_{m-1}$$

or the type y that satisfies,

$$[F\left(y\right)]^{n-1} w_m = w_{m-1} \tag{18}$$

Then $\pi_m^i(y) = 1 \equiv \pi_m^m$ for every i and for every $y \in (y_m, \overline{y}]\}$ must be true in every Bayesian equilibrium of this sub-game.

Note that y_m is a continuous function of w_m and w_{m-1} and that $y_m \to 1$ as $w_{m-1} \to w_m$. Since π_m^m is constant, it is trivially a continuous function of w_m and w_{m-1}. Furthermore, note that a worker of type y_m gets the same payoff from every firm whose index is greater than or equal to $m-1$.

Now suppose that we have defined cutoff valuations $\{y_{k+1}, \ldots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k+1, \ldots m$ and $j' \geqslant k'$, satisfying $\sum_{j' \geqslant k'} \pi_{j'}^{k'} = 1$ for each k'. Suppose that these satisfy the following conditions:

- (C.1) - $\pi_{j'}(y) = \pi_{j'}^{k'}$ for each $y \in (y_{k'}, y_{k'+1})$ and $\pi_{j'} = 0$ otherwise, in every symmetric Bayesian equilibrium;
- (C.2) - a worker of type $y_{k'}$ where $y_{k'} \in \{y_{k+1}, \ldots, y_m\}$, gets the same payoff from every firm whose index is at least $k'-1$;
- (C.3) - each of these numbers is a continuous function of wages $w_k, \ldots w_m$.

If $y_{k+1} = \underline{y}$, then we have shown that the Bayesian continuation equilibrium for this sub-game is almost everywhere uniquely defined (the exceptions are the cutoff values y_k). So suppose $y_{k+1} > \underline{y}$. We now show that properties (C.1) to (C.3) can be extended to some interval $[y_k, y_{k+1})$ which will be non-degenerate provided $w_k < w_{k-1}$.

If $w_k = w_{k+1}$, or $w_{k-1} = w_k$, set $y_k = y_{k+1}$, $\pi_k^k = 0$ and $\pi_j^k = \pi_j^{k+1}$ for each $j > k$. It is straightforward that valuations $\{y_k, \ldots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k, \ldots m$ satisfy conditions (C.1) to (C.3) of the induction hypothesis.

Otherwise either $w_{k-1} < w_k < w_{k+1}$ or $k = 1$. Each of these cases can be analyzed the same way. In the former case, observe that in this construction, worker types larger than y_{k+1} will never apply to firm k. Thus for y close enough to y_{k+1} applying to *any* firm with wage rate below w_k will be strictly dominated by applying to firm k no matter what workers with types in the interval (y, y_{k+1}) choose to do. In the case where $k = 1$ firm k is already the lowest wage firm. In either case, we conclude that there is an interval of types (y_k, y_{k+1}), with y_k possibly equal to $\underline{y}$, such that workers with types in this interval will apply with positive probability only to firms with wages at least w_k in every Bayesian equilibrium.

By the induction hypothesis, a worker of type y_{k+1} will receive the same payoff from each firm $k+1$ through m. This payoff is given by

$$\left[1 - \sum_{i=1}^{j} \pi_{k+j}^{k+i} [F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j}$$

when this worker applies to firm $k + j$. By the induction hypothesis, this payoff is equal to w_k for each $j \geqslant 1$. Notice that this payoff is independent of what workers whose types are in the interval (y_k, y_{k+1}) choose to do. A worker i of type $y \in (y_k, y_{k+1})$ who applies to firm $k + j$ receives payoff

(19)
$$\left[1 - \int_y^{y_{k+1}} \pi_{k+j}(y)dF(y) - \sum_{i=1}^{j} \pi_{k+j}^{k+i}[F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j}$$

while the same worker who applies to firm k gets

$$\left[1 - \int_y^{y_{k+1}} \pi_k(y)dF(y)\right]^{n-1} w_k \tag{20}$$

The function described in (20) is non-decreasing in y and has a limit from the left at y_{k+1} equal to w_k. Since applying to firms whose wages are lower than w_k is a strictly dominated strategy of a worker of type y close enough to y_{k+1}, it must be the case that for every i, $\int_y^{y_{k+1}} \pi_{k+j}^i(y)dy$ is strictly positive for some j. Then from (19) and (20), $\int_y^{y_{k+1}} \pi_{k+j}(y)dy$ must be strictly positive for all j.

The payoff must be the same at firm k and $k + j$ for each $j > 0$ and for every $y \in (y_k, y_{k+1})$. This requires that (19) and (20) must be equal identically in y. Differentiating this identity repeatedly gives

$$\left(\frac{\pi_{k+j}^k(y)}{\pi_k^k(y)}\right)^{n-1} = \frac{w_k}{w_{k+j}} \tag{21}$$

implying that π_{k+j}^k are constant.

They can all be determined from the condition

$$\sum_{j=0}^{m-k} \pi_{k+j}^k = 1 \tag{22}$$

Notice that by the induction hypothesis, the limits from the left of (19) and (20) at y_{k+1} must both be equal to w_k. Thus (21) and (22) are also sufficient for identity of the payoffs.

Having found the value for π_k^k we can determine the lower bound y_k. Since workers with higher types and higher investments only apply to firms whose wages are at least w_k, this worker is sure to be hired if he applies to the $k - 1^{st}$ firm, assuming that there is one. On the other hand, since he is lowest type who applies to the k^{th} firm, he will be hired by the k^{th} firm only if no other worker with a higher type applies.

Then define y_k as follows: if $k = 1$, then $y_k = y_1 = \underline{y}$; otherwise if

$$\left[1 - \pi_k^k \left(F(y_{k+1}) - F(y)\right)\right]^{n-1} w_k = w_{k-1} \tag{23}$$

has a solution that exceeds $\underline{y}$, set y_k equal to this solution; otherwise set $y_k = \underline{y}$.

This argument extends conditions (C.1) and (C.2) by construction. Property (C.3) is readily verified using, for example, the maximum theorem since $w_{k+j} > 0$ by assumption. □

7.2. **A preliminary Result.**

Lemma 7.1. *For any sequence G_n there is a sub-sequence such that $\omega_n(y)$ converges weakly to a right continuous non-decreasing function $\omega(y)$. Along this sequence, define $y_n^*(w) = \sup\{y : \omega_n(y) \le w\}$. The sequence $y_n^*(\cdot)$ converges weakly to a right continuous non-decreasing function $y^*(\cdot)$.*

Proof. By construction each $\omega_n(y)$ is right continuous and non-decreasing, and for each n $\int_{\underline{w}}^{\overline{w}} d\omega_n(y) \le \overline{w}_G - \underline{w}_G$ where $\overline{w}_G$ and $\underline{w}_G$ are the maximum and minimum points in the support of G respectively. Hence by Helly Compactness Theorem, $\omega_n(y)$ has a subsequence that converges weakly to an non-decreasing right continuous function. Let $y_n^*(\cdot)$ be the sequence associated with $\omega_n(y)$. It is also non-decreasing and right continuous, and so there is a subsequence such that it has a weak limit $y^*(\cdot)$ by the same reasoning. Since $\omega_n(y)$ converges weakly, it converges weakly on any subsequence. So there is a sequence along which both ω_n and its inverse y_n converge weakly. □

7.3. **Proof of Lemma 7.3.**

Define w^s as either the largest wage in the support of G that is less than or equal to w, or if no such wage exists, let w^s be the smallest wage in the support of G that is at least as large as w. For convenience, choose the approximations G_n in such a way that the lowest wage w_0 in each approximation is the lowest wage in the support of G. Similarly, suppose that the highest wage w_m in each approximation is also the highest wage in the support of G.

Lemma 7.2. *Let $\overline{v}(y)$ be any point-wise limit for the equilibrium payoff to a worker of type y as n goes to infinity. Then $v(y) > 0$ for each $y \in Y$.*

Proof. Choose any wage w' such that $G(w') < 1$. Since worker y attains the same payoff no matter where he applies by Lemma 3.1, the limit of his equilibrium payoff is given by

$$\lim_{n\to\infty}\left(1-\int_{y}^{y_n^*(w')}\pi_{j^n}^n(y')dF(y')\right)^{n-1}w' \geq$$

$$\lim_{n\to\infty}\left(1-\int_{y}^{y_n^*(w')}\pi_{j^n}^n(y^*(w'))dF(y')\right)^{n-1}w' =$$

$$\lim_{n\to\infty}\left(1-\int_{y}^{y_n^*(w')}\pi_{m}^n(y^*(w'))\left(\frac{w_m}{w'}\right)^{\frac{1}{n-1}}dF(y')\right)^{n-1}w' \geq$$

$$\lim_{n\to\infty}\left(1-\frac{1}{\left(1-G_n^-(w')\right)m}\left(\frac{w_m}{w'}\right)^{\frac{1}{n-1}}\left[F\left(y^*\left(w'\right)-F\left(y\right)\right)\right]\right)^{n-1}w' =$$

$$e^{-\frac{F(y^*(w')-F(y))}{1-G^-(w')}}w' > 0$$

The first inequality follows from (21) and the fact that higher types have higher reservation wages so that they allocate their application probabilities over fewer firms. The second equality simply substitutes using (21). The third inequality comes from the fact that the sum of the application probabilities over all firms whose wage is w' or higher must be equal to one. The limit is a standard one in directed search. □

Lemma 7.3. *Let j^n be the index of w in the n^{th} approximation to G. Let $\omega(y)$ be a limit of the sequence $\omega_n(y)$ as defined in Lemma 7.1. Then for any y such that $\pi_{j^n}^n > 0$ for infinitely many n,* $\lim_{n\to\infty}\pi_{j^n}^n(y)(n-1) = \frac{\tau}{1-G^-(\omega(y))} \equiv k\left(y\right)$

Proof. Let $y < \overline{y}$. For any finite n, there is a firm m who offers the highest wage in the support of the distribution G_n. Call this highest wage w_m. This highest wage is no higher than $\overline{w}$.

We can use this to show that $\pi_j^n\left(y\right)\left(n-1\right)$ is uniformly bounded. By Lemma 3.1, every worker type y applies to the firm offering the highest wage w_m with strictly positive probability. From (19), the payoff to a worker of type y who applies to firm m is bounded above by

$$(1-\pi_m^n(y)(1-F(y)))^{n-1}\,\overline{w} \tag{24}$$

This is an upper bound for two reasons. Firm m will offer a wage that is no higher than $\overline{w}$, and workers whose types are higher than y will apply to firm m with probability at least as high as a worker of type y.

We first show that $\pi_m^n(y)(n-1)$ is uniformly (in y) bounded above by an integrable function, then work down to show the result for firms with indices lower than m. Let $b\left(y\right)$ be a measurable function from Y

into $\mathbb{R}$ and suppose that $\pi^n_m(y)(n-1)$ has a limit $b(y)$ or larger. Then the upper bound given by (24) can be no larger than

$$\lim_{n\to\infty}(1-(\frac{b(y)}{n-1})(F(\overline{y})-F(y)))^{n-1}\overline{w}$$
$$= \exp\left\{\lim_{n\to\infty}\frac{\log\left(1-\frac{1}{n-1}b(y)(1-F(y))\right)}{\frac{1}{n-1}}\right\}\overline{w}$$
$$= e^{-b(y)(F(\overline{y})-F(y))}\overline{w}$$

where the last result follows by L'Hopital's rule. Since the upper bound on the worker's payoff must be at least $\overline{v}(y)$, then $\lim_{n\to\infty}\pi^n_m(y)(n-1)$ must have an upper bound $b(y)$ that satisfies $e^{-b(y)(F(\overline{y})-F(y))}\overline{w}\geq\overline{v}(y)$, for all y, or $b(y)(F(\overline{y})-F(y))\leq-\log\left(\frac{\overline{v}(y)}{\overline{w}}\right)$. Now observe that

$$\int_y^{\overline{y}} b(y')dF(y')\leq\int_y^{\overline{y}} b(y')(1-F(y'))\frac{F'(y')}{1-F(y')}dy'\leq-\log\left(\frac{\overline{v}(y)}{\overline{w}}\right)\cdot B\cdot(\overline{y}-y)$$

where B is the uniform bound on the ratio $\frac{F'(y')}{1-F(y')}$. By Lemma 7.2, $\log\left(\frac{\overline{v}(y)}{\overline{w}}\right)$ is bounded. Thus the bound $b(y)$ is an F-integrable function of y that uniformly bounds $\pi^n_m(y)(n-1)$ for all n large enough. Define $\overline{k}(y)=\lim_{n\to\infty}\pi^n_m(y)(n-1)$.

Now we extend this uniform upper bound to firms with indices below m. From (21)

$$\pi^n_j(y)(n-1)=\left(\frac{w_m}{w}\right)^{\frac{1}{n-1}}\pi^n_m(y)(n-1) \tag{25}$$

for each j such that $\pi^n_j(y)>0$. From the previous result, the right hand side of this equation is uniformly bounded by the F-integrable function $\frac{w_m}{w}b(y)$, so the left hand side is also uniformly bounded. Furthermore, taking limits in (25) with respect to n gives

$$\lim_{n\to\infty}\pi^n_{j^n}(y)(n-1)=\overline{k}(y)$$

Recall that $\omega_n(y)$ is the lowest wage to which a worker of type y applies with positive probability in the continuation equilibrium with n workers. From (25)

$$\sum_{j:w_j\geq\omega_n(y)}\pi^n_j(y)(n-1)=\pi^n_m(y)(n-1)\sum_{j:w_j\geq\omega_n(y)}\left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} \tag{26}$$

The sum on the left hand side of this last equation is $n-1$ since the application probabilities sum to one. On the right hand side, observe that

$$\sum_{j:w_j \geq \omega_n(y)} 1 \leq \sum_{j:w_j \geq \omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} \leq \left(\frac{w_m}{w_{j_n^*}}\right)^{\frac{1}{n-1}} \sum_{j:w_j \geq \omega_n(y)} 1$$

Dividing this by m gives

$$(1-G_n^-(\omega_n(y))) \leq \sum_{j:w_j \geq \omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} /m \leq \left(\frac{w_m}{w_{j_n^*}}\right)^{\frac{1}{n-1}} (1-G_n^-(\omega_n(y)))$$

This implies that

$$\lim_{n\to\infty} \sum_{j:w_j \geq \omega_n(y)} \left(\frac{w_m}{w_j}\right)^{\frac{1}{n-1}} /m = 1 - G^-(\omega(y))$$

where $\omega(y)$. Then from (26)

(27)

$$\lim_{n\to\infty} \pi_{j^n}^n(y)(n-1) = \lim_{n\to\infty} \frac{n-1}{1 - G_n^-(\omega_n(y)) \cdot m} = \frac{\tau}{1 - G^-(\omega(y))} = k(y)$$

which gives the result.

Recall the convention that each finite approximation to G assigns the lowest wage w_0 to be equal to the lowest wage in the support of G. For any wage $w \geq w_0$, let w^- be the highest wage in the support of G that is less than or equal to w. Note that this means that $w^- = w$ whenever w lies in the support of G. □

Lemma 7.4. *Let j^n be the index of w in the n^{th} approximation to G. Let $\omega(y)$ be a limit of the sequence $\omega_n(y)$ as defined in Lemma 7.1. Then*

$$\lim_{n\to\infty} \left(1 - \int_y^{y_n^*(w)} \pi_{j^n}^n(y')dF(y')\right)^{n-1} =$$

$$\frac{w^-}{w} e^{-\int_y^{y^*(w^-)} \frac{\tau}{1-G^-(\omega(y'))} dF(y')}$$

when $w \geq w_0$, and

$$\min\left[1, \frac{w^0}{w} e^{-\int_y^{y^*(w_0)} \frac{\tau}{1-G^-(\omega(y'))} dF(y')}\right]$$

otherwise.

Proof. Suppose first that $w \geq w_0$. Then

$$\lim_{n\to\infty} \left(1 - \int_y^{y_n^*(w)} \pi_{j^n}^n(y')dF(y')\right)^{n-1} =$$

(28)
$$\lim_{n\to\infty}\left(1-\int_{y}^{y_n^*(w_{j^n-1})}\pi_{j^n}^n(y')dF(y')-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)}\pi_{j^n}^n(y')dF(y')\right)^{n-1}$$

By the definition of $y_n^*(w_{j^n-1})$, a worker of this type who applies to the firm offering wage w will be hired with probability

$$\left(1-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)}\pi_{j^n}^n(y')dF(y')\right)^{n-1}$$

He will be hired for sure if he applies to the firm offering w_{j^n-1}. So

$$\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)}\pi_{j^n}^n(y')dF(y')=1-\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}}$$

Substitute this into (28) above to get

$$\lim_{n\to\infty}\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}}-\int_{y}^{y_n^*(w_{j^n-1})}\pi_{j^n}^n(y')dF(y')\right)^{n-1}=$$

$$\lim_{n\to\infty}\exp\left\{(n-1)\log\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}}-\frac{1}{n-1}\int_{y}^{y_n^*(w_{j^n-1})}\pi_{j^n}^n(y')(n-1)dF(y')\right)\right\}$$

Since the exponential function is continuous, the limit can moved inside the first bracket. So we compute

$$\text{(29)}\quad \lim_{n\to\infty}\frac{\log\left(\left(\frac{w_{j^n-1}}{w}\right)^{\frac{1}{n-1}}-\frac{1}{n-1}\left\{\int_{y}^{y_n^*(w_{j^n-1})}\pi_{j^n}^n(y')(n-1)dF(y')\right\}\right)}{\frac{1}{n-1}}$$

which can be written as

$$\lim_{x\to 0,t\to\gamma,z\to\zeta}\frac{\log\left(t^x-xz\right)}{x}$$

where $\gamma=\lim_{n\to\infty}\frac{w_{j^n-1}}{w}$ is 1 if w is in the support of G, and is equal to $\frac{w^s}{w}$ otherwise. The value of the constant ζ follows from the bounded convergence theorem and Lemma 7.3. ζ is equal to $\int_{y}^{y^*(w)}k(y')dF(y')$ when w is in the support of G and to $\int_{y}^{y^*(w^s)}k(y')dF(y')$ when w lies above the support of G. Now apply L'Hopital's rule to get the limit of (29) as

$$\frac{w^-}{w}e^{-\int_{y}^{y^*(w^-)}k(y')dF(y')}$$

when w is above the support of G, and to $e^{-\int_{y}^{y^*(w)}k(y')dF(y')}$ when w is in the support of G.

When $w < w_0$, the argument is similar. The limit of interest is

$$\lim_{n\to\infty}\left(1-\int_{y}^{y_n^*(w)}\pi_{j^n}^n(y')dF(y')\right)^{n-1}=$$

$$\lim_{n\to\infty}\min\left[1,\frac{w_0}{w}\left(1-\int_{y}^{y_n^*(w_0)}\pi_1^n(y')dF(y')\right)^{n-1}\right]=$$

$$\min\left[1,\frac{w_0}{w}\lim_{n\to\infty}\left(1-\frac{1}{n-1}\int_{y}^{y_n^*(w_0)}(n-1)\,\pi_1^n(y')dF(y')\right)^{n-1}\right]$$

The equality follows from the fact that for any worker who applies at both wages w and $w_0$with positive probability,

$$\left(1-\int_{y}^{y_n^*(w)}\pi_{j^n}^n(y')dF(y')\right)^{n-1}w=\left(1-\int_{y}^{y_n^*(w_0)}\pi_1^n(y')dF(y')\right)^{n-1}w_0$$

The min operator appears because types close to $y_n^*(w_0)$ apply at wage w with probability zero, so every such type would be hired with probability 1 if they did apply. Now evaluating the limit as above gives

$$\min\left[1,\frac{w_0}{w}e^{-\int_y^{y^*(w_0)}k(y')dF(y')}\right]$$

□

7.4. **Proof of Theorem 4.1.**

Proof. The proof of Theorem 4.1 now follows from Lemmas 7.4 and 7.1. A firm of type x who offers wage w has profit

$$\int_{\underline{y}}^{\overline{y}}v(w,y,x)d\phi_{j_n}^n(y)$$

The argument now depends on whether $w \geq w_1$ (i.e., whether or not there is a wage below w in the support of G). Suppose first that $w \geq w_1$ and let j_n be the index of the wage w in the distribution G_n associated with the n^{th} approximation. Substituting for ϕ gives

$$\int_{\underline{y}}^{y_n^*(w)}v(w,y,x)d\left[1-\int_{y}^{y_n^*(w)}\pi_{j_n}^n(y')dF(y')\right]^n=$$

$$\int_{\underline{y}}^{y_n^*(w_{j^n-1})}v(w,y,x)d\left[1-\int_{y}^{y_n^*(w)}\pi_{j_n}^n(y')dF(y')\right]^n+$$

$$\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} v(w,y,x) d\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n =$$

$$\int_{\underline{y}}^{y_n^*(w_{j^n-1})} v(w,y,x) n\pi_{j_n}^n(y)\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^{n-1} F'(y)dy +$$

$$v(w,y_n^*(w),x) - v(w,y_n^*(w_{j^n-1}),x)\left[1-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n +$$

$$\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)}\left[1-\int_y^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^n \frac{\partial v(w,y,x)}{\partial y}dy$$

That last two terms in this expression are derived by integrating by parts. Now observe that a worker of type $y_n^*(w_{j^n-1})$ is just indifferent between applying at the wage w_{j^n-1} and being hired for sure, or applying at wage w and being hired with probability

$$\left[1-\int_{y_n^*(w_{j^n-1})}^{y_n^*(w)} \pi_{j_n}^n(y')dF(y')\right]^{n-1}$$

So substitute $\frac{w_{j^n-1}}{w}$ for this probability in the second term, and take limits using the results of Lemma 7.4 to get

$$\int_{\underline{y}}^{y^*(w^-)} v(w,y,x)k(y)\frac{w^-}{w}e^{-\int_y^{y^*(w^-)} k(y')dF(y')} +$$

$$v(w,y^*(w),x)\left(1-\frac{w^-}{w}\right)$$

The first term follows from the bounded convergence theorem and Lemma 7.3. The second term follows from the substitution made above, and from the fact that $y_n^*(w)-y_n^*(w_{j^n-1})$ converges to zero with n (if not, the probability of being hired at wage w for traders between $y_n^*(w)$ and $y_n^*(w_{j^n-1})$ goes to zero. The convergence of $y_n^*(w)-y_n^*(w_{j^n-1})$ to zero also reduces the last term in the expansion to zero because the derivative of v with respect to y is bounded (and the term multiplying it is less than 1).

Now consider the case where $w < w_0$. The firm's profit is

$$\int_{\underline{y}}^{y_n^*(w)} v(w,y,x) d\left[1-\int_y^{y_n^*(w)} \pi_1^n(y')dF(y')\right]^n =$$

$$\int_{\underline{y}}^{y_n^*(w)} v(w,y,x) n\pi_1^n(y)\left[1-\int_y^{y_n^*(w)} \pi_1^n(y')dF(y')\right]^{n-1} F'(y)dy =$$

$$\frac{n}{n-1}\int_{\underline{y}}^{y_n^*(w)} v(w,y,x)\,(n-1)\,\pi_1^n(y)\frac{w_0}{w}\left[1-\int_y^{y_n^*(w_0)}\pi_1^n(y')dF(y')\right]^{n-1}F'(y)dy$$

Now apply Lemmas 7.3 and 7.4 and use the bounded convergence theorem to take limits of this expression, yielding

$$\int_{\underline{y}}^{y(w)} v\left(w,y,x\right)\tau\frac{w_1}{w}e^{-\int_y^{y^*(w_1)}\tau dF(y')}F'\left(y\right)dy$$

where $y\left(w\right)$ is either $\underline{y}$ or the solution to

$$\frac{w_0}{w}e^{-\int_y^{y^*(w_0)}\tau dF(y')}=1$$

whichever is higher.

The last part of the argument is to show that

$$y^*(w)=\sup\{y:\omega(y)\leq w\}$$

Suppose the contrary that for some w, $y^*(w)>\sup\{y:\omega_n(y)\leq w\}=y_n^*(w)$ for all large n. Observe that for each n, $\omega_n(y_n^*(w))\geq w$. Furthermore, note that a worker of type $y_n^*(w)$ has a type that is at least as high as any other worker who applies at wage w. So such a worker is hired for sure at wage w. Let $y_0=\lim_{n\to\infty}\sup\{y:\omega_n(y)\leq w\}<y^*(w)$.

At the other extreme, if $y^*(w)$ is not a continuity point of ω, then since the latter function is right continuous and non-decreasing, there is a point $y_0<y_1<y^*(w)$ at which ω is continuous (and $\omega(y_1)\leq w$). For large n, it must be that $\omega_n(y_1)>w$ since otherwise $y_n^*(w)$ would be at least as large as y_1. Yet since y_1 is a continuity point of ω and ω_n converges weakly to ω, $\omega_n(y_1)\to\omega(y_1)$.

Then using Lemma 7.3, the payoff to a worker of type $y_n^*(w)$ who applies at the wage $\omega_n(y)$ is converging to

$$we^{-\int_{y_0}^{y_1}k(y')dF(y')}<w$$

This contradicts the property that workers should receive the same expected payoff by applying to all wages that are at least as large as their reservation wage.

A similar argument establishes a contradiction when $y_0=\lim_{n\to\infty}\sup\{y:\omega_n(y)\leq w\}>y^*(w)$. □

7.5. **Example.** This appendix works out an example of a dispersed wage equilibrium when all firms are identical. Suppose profit functions given by

$$v\left(w,y,x\right)=y-w$$

It is assumed that $F\left(y\right)=y\left(2-y\right)$ on an interval $[0,1]$. Then $F'\left(y\right)=2-2y$.

Then (14) becomes

$$y_0 - w_0 = \int_{\underline{y}}^{y_0} y\tau e^{-\tau[F(y_0)-F(y')]}dy'.$$

Integrating the second term by parts and using the fact that the lower bound of the support is 0 gives the condition

$$w_0 = \int_{\underline{y}}^{y_0} e^{-\tau(F(y_0)-F(y'))}dy' \tag{30}$$

which then represents the locus SS from Figure 4 in closed form. The derivative of the right hand side of this condition is

$$1 - \tau\,(2 - 2y_0) \int_{\underline{y}}^{y_0} e^{-\tau(F(y_0)-F(y'))}dy'$$

$$= 1 - \tau\,(2 - 2y_0)\, w_0 \tag{31}$$

by (30). We return to this condition momentarily.

Now (12) can be written

$$v\left[\omega(y), y, h(y)\right] = -\int_{\underline{y}}^{y} \left[v_w\left(\omega(y), y', h(y)\right) - \frac{v\left(\omega(y), y', h(y)\right)}{\omega(y)}\right]\omega'(y')\,dy'$$

$$y - \omega(y) = -\int_{\underline{y}}^{y} \left[-1 - \frac{y' - \omega(y)}{\omega(y)}\right]\omega'(y')\,dy'$$

or

$$(y - \omega(y))\,\omega(y) = \int_{\underline{y}}^{y} y'\omega'(y')\,dy'$$

Using the fact that this has to hold uniformly, the derivatives of both sides of this equation must be the same, so

$$(y - \omega(y))\,\omega'(y) + \omega(y)\,(1 - \omega'(y)) = y\omega'(y)$$

This gives the simple condition $\omega'(y) = \frac{1}{2}$. In terms of the ideas expressed in Figure 4, this means that all the potential solutions for (13) are straight lines with slope equal to $\frac{1}{2}$. So $\omega(y) = w_0 + \frac{1}{2}(y - y_0)$.

The condition that $\omega'(y_0) = \omega(y_0)\,\tau F'(y_0)$ at the starting point for the equilibrium distribution then reduces to the requirement that $w_0\tau\,(2 - 2y_0) = \frac{1}{2}$. The locus of solutions to this equation (TT in Figure 4) is upward sloping, has value $\frac{1}{4\tau}$ at $y_0 = 0$ and derivative at this point equal to $\frac{1}{8\tau}$. This ensures that by taking τ large enough, we can guarantee that this locus intersects the locus of solutions to (30) at least once. It tends to infinity as y_0 approaches 1. Since the locus of solutions to (30) gives a finite wage when $y_0 = 1$, there is a rightmost intersection point of the two loci TT and SS. This is the one used to compute the solution.

Finally, note that at any intersection (y_0, w_0) of TT and SS, $w_0\tau(2-2y_0) = \frac{1}{2}$. So the derivative of the locus SS must be equal to $\frac{1}{2}$ by (31) derived above. Now focus on the rightmost intersection of the two curves. At any point (y, w) on SS to the right of the intersection with TT, $w\tau(2-2y) < \frac{1}{2}$ since TT lies above SS. That means that the derivative of SS is larger than $\frac{1}{2}$ at every point to the right of this intersection. At the rightmost intersection of TT and SS, the locus TT must then cut SS from below. So the locus TT lies everywhere above the solution curve $\omega(y) = w_0 + \frac{1}{2}(y - y_0)$. Since ω has slope $\frac{1}{2}$ and we have just established that the slope of SS is larger than $\frac{1}{2}$ to the right of such an intersection, it follows that the solution ω lies everywhere below both SS and TT to the right of this intersection. So consider

$$
\begin{aligned}
G(w) &= 1 - 2w\tau F'(2(w - w_0) + y_0) \\
&= 1 - 2w\tau(2 - 4(w - w_0) - 2y_0) \qquad (32)
\end{aligned}
$$

Each pair $(2 - 4(w - w_0) - 2y_0, w)$ lies on the locus $\omega(y) = w_0 + \frac{1}{2}(y - y_0)$ to the right of y_0. This locus is a straight line that lies every where below the locus of solutions to $w\tau(\overline{y} - y) = \frac{1}{2}$. Since this latter locus is convex, it follows that $w_2 > w_1$ implies

$$\frac{1}{2} - w_2\tau(2 - 4(w_2 - w_0) - 2y_0) > \frac{1}{2} - w_1\tau(2 - 4(w_1 - w_0) - 2y_0)$$

or

$$w_1\tau(2 - 4(w_1 - w_0) - 2y_0) > w_2\tau(2 - 4(w_2 - w_0) - 2y_0)$$

It follows that the function (32) is increasing as required.

Finally, it is straightforward to verify that the function $\omega(y)$ below y_0 is a convex function, while firms iso expected profit lines to the left of y_0 are concave. This ensures that no deviation below w_0 is profitable for firms. The firm's profit function for wages above $\omega(\overline{y})$ is readily shown to be concave for large enough τ, so global optimality conditions can be satisfied for this case.

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